"Vasile Alecsandri" University of Bacău
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SOME CLASSES OF *GL*-METRICS ON *TM*

VALER NIMINEȚ

Abstract. In the present paper we continue the investigations of GL-space and study the geometry of such GL-metrics on TM by means the nonlinear connection.

1. INTRODUCTION

A *GL*-space is defined as a pair, which consists of a smooth *n*-dim manifold and a *d*-tensor field $g_{ij}(x, y)$ of rank *n* symmetric and of constant signature.

The generalized Lagrange spaces $GL^n = (M, g_{ij}(x, y))$ shortly *GL*-space with the fundamental tensor

$$(1.1)\ g_{ij}(x, y) = \gamma_{ij}(x) + \left(1 - \frac{1}{n^2(x, y)}\right) y_i y_j,\ y_i = \gamma_{ij}(x) y^j$$

where $\gamma_{ij}(x)$ is a Lorentz metric tensor and $n(x, y) > 1$ were studied by R. Miron and M. Anastasiei in [1]. The notion was suggested by the fact that for many properties of the Lagrange space, only the metric tensor $g_{ij}(x, y)$ is involved.

A more general case i.e. of the Finsler-Synge metric obtained from (1.1) by replacing $\gamma_{ij}(x)$ with $\gamma_{ij}(x, y)$, the fundamental tensor of a Finsler space, was studied by author in the papers [4].

One can see that *GL*-metric (1.1) is obtained by a deformation of the Riemannian metric γ_{ij}.

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The geometry of generalized Lagrange spaces is developed with the same devices as the geometry of Lagrange spaces. That is, if we associate to a generalized Lagrange space a nonlinear connection then, a canonical metric connection is found.

The structure of *GL*-space is lifted to an almost Hermitian structure on *TM* and its canonical metric *N*-linear connection appears as an almost Hermitian connection on *TM*.

Definition 1.1. *A d-tensor field $g_{ij}(x, y)$ on TM is said to be a GL-metric if*

i) $g_{ij}(x, y) = g_{ji}(x, y)$

ii) $\det(g_{ij}(x, y)) \neq 0$

iii) the quadratic form $g_{ij}(x, y)\xi^i \xi^j$, $\xi^i \in R^n$ *has a constant signature.*

Then, the pair $GL^n = (M, g_{ij}(x, y))$ is called a *GL*-space.

Definition 1.2. *A GL-metric $g_{ij}(x, y)$ is said to be reducible to a Lagrangian metric if there exists a smooth function $L : TM \to R$ such that*

(1.2) $$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j}(x, y)$$

Then, the pair $L^n = (M, L)$ is a Lagrange space, where L is a regular Lagrangian.

2. METRIC CLASSES ON TM

The space $GL^n = (M, g_{ij})$ with the fundamental tensor g_{ij} in (1.1) has the reciprocal g^{ij}:

(2.1) $$g^{ij}(x, y) = \gamma^{ij}(x) - \frac{1}{a(x, y)}\left(1 - \frac{1}{n^2(x, y)}\right) y^i y^j$$

where $$a(x, y) = 1 + \left(1 - \frac{1}{n^2}\right)\|y\|^2, \|y\|^2 = y_i y^i > 0$$

The *GL*-metric (1.1) is not reducible neither to a Lagrangian metric.

The nonlinear connection *N* with the coefficients

(2.2) $$N^i_j = \gamma^i_{jk}(x) y^k - \beta F^i_j,$$

where $\beta = b_i(x)y^i$ is defined by $b_i(x), i, j = \overline{1, n}$ and F_j^i is the electromagnetic tensor in the pseudo-Riemannian space R^n has the tensor of torsion

$$t_{jk}^i = \frac{\partial N_j^i}{\partial y^k} - \frac{\partial N_k^i}{\partial y^i}$$

and the curvature

$$R_{jk}^i = \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j} \tag{2.3}$$

The coefficients N_j^i of *N* can be written in the form

$$N_j^i = y^k B_{kj}^i(x)$$

where

$$B_{kj}^i(x) = \frac{\partial N_j^i}{\partial y^k} = \gamma_{kj}^i(x) - b_k F_j^i(x)$$

are the coefficients of the Berwald connection $B\Gamma(N)$.

The curvature tensor of N, R_{jk}^i is as follows

$$R_{jk}^i = y^s B_{sjk}^i(x) \tag{2.4}$$

where B_{sjk}^i is the curvature tensor of the Berwald connection $B\Gamma(N)$ of the nonlinear connection *N*.

We have

Theorem 2.1. *The torsion t_{jk}^i of N is the torsion of Berwald connection $B\Gamma(N)$ and the curvature R_{jk}^i of N is given by (2.4) where B_{sjk}^i is the curvature of Berwald connection.*

Definition 2.1. *A GL-metric $g_{ij}(x, y)$ on $\widetilde{T}\widetilde{M}$ is reducible to a Finslerian metric, if there exists a function $F : TM \to R_+$ connections on the null section, which is positively homogeneous of order 1 in y^i and satisfies*

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)$$

Here *F* is a fundamental Finsler function and $F^n = (M, F)$ is a Finsler space.

A *GL*-metric $g_{ij}(x, y)$ on *TM* is said to be reducible to a Riemannian metric if it does not depend on y.

Definition 2.2. *A GL-metric* $g_{ij}(x, y)$ *is called a locally Minkowski metric if there exists a system of coordinates on TM in which the components* g_{ij} *of the metric depend on y only.*

Let $F:\widetilde{T}\widetilde{M} \to R_+$, $F(y) = \left((y^1)^m + (y^n)^m\right)^{1/m}$, $n \in N$, $m \geq 3$, then $\gamma_{ij}(y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is a locally Minkowski *F*-metric and $g_{ij}(x, y) = e^{u(y)}\gamma_{ij}(y)$, with u a smooth function on *TM* is an example of a locally Minkowski *GL*-metric.

Now, we consider the case when the space $R^n = (M, \gamma_{ij})$ is locally Euclidian and around every point $(x, y) \in TM \setminus \{0\}$ there exists a local system of coordinates in which the metric tensor γ_i does not depend on (x^i).

Thus, the fundamental tensor g_{ij} of the space $GL^n = (M, g_{ij}(y))$ is

$$g_{ij}(y) = \gamma_{ij} + \left(1 - \frac{1}{n^2(y)}\right) y_i y_j,\ y_i = \gamma_{ij} y^j \tag{2.5}$$

where

$$\frac{\partial \gamma_{ij}}{\partial x^k} = 0, \frac{\partial \gamma_{ij}}{\partial y^k} = 0 .$$

The nonlinear connection *N* has the coefficients

$$N^i_j = -\beta F^i_j$$

The autoparallel curves of *N* are given by

$$\frac{dx^i}{dt} = y^i; \frac{d^2 x^i}{dt^2} = \left(b_j \frac{dx^j}{dt}\right) F^i_k \frac{dx^k}{dt} \tag{2.6}$$

The Berwald connection is $B\Gamma(N) = (B^i_{jk}, 0)$ with

$$B^i_{kj} = -b_k F^i_j \tag{2.7}$$

and

$$N^i_j = y^k B^i_{kj}$$

Proposition 2.1. *The Lorentz nonlinear connection N of the locally Euclidian space* (M, γ_{ij}) *is integrable if and only if* $B^i_{sjk} = 0$.

The *GL*-space with the fundamental tensor

$$g_{ij}(y) = \gamma_{ij} + \left(1 - \frac{1}{n^2(y)}\right) y_i y_j,\ y_i = \gamma_{ij} y^i \tag{2.8}$$

which is a particular case of (1.1), endowed with the nonlinear connection *N* (2.2) has a natural almost symplectic structure on the *TM*:

(2.9) $$\theta = g_{ij}(y)\delta y^i \Lambda dx^j$$

If we consider the Berwald connection determined by nonlinear connection *N*,

$$B\Gamma(N) = \left(B^i_{jk}, 0\right), B^i_{jk} = \frac{\partial N^i_j}{\partial y^k}$$

we have

$$B^i_{jk} = -F^i_j b_k, \ N^i_j = -B^i_{jk} y^k$$

The curvature R^i_{jk} of *N* is given by

$$R^i_{jk} = y^s B^i_{sjk}(x)$$

Then we have

Theorem 2.2. *The almost symplectic structure θ of the GL-space is a* symplectic *structure if and only if the following equations hold:*

$$g_{ij\|k} - g_{ik\|j} + g_{is} t^s_{jk} = 0$$

$$g_{is} R^s_{jk} + g_{jk} R^s_{ki} + g_{ks} R^s_{ij} = 0$$

$$g_{ij\|k} - g_{ij\|j} = 0$$

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Department of Mathematics and Informatics,
“Vasile Alecsandri” University of Bacău,
600114 - Bacău,
Romania
E-mail: valern@ub.ro