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SOME GENERAL FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE D-MAPPINGS

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Abstract. In this paper some general fixed point theorems for owc D-mappings satisfying an implicit relation are proved by generalizing some results in [1], [4] and [6].

1. Introduction and preliminaries

Let (X, d) be a metric space and $B(X)$ the set of all nonempty bounded sets of X. As in [7] and [8] we define the functions $\delta(A, B)$ and $D(A, B)$, where $A, B \in B(X)$, by

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$$
$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.$$

If A consists of a single point a, we write $\delta(A, B) = \delta(a, B)$. If B consists also of a single point b, we write $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of δ that

$$\delta(A, B) = \delta(B, A) \geq 0, \forall A, B \in B(X),$$
$$\delta(A, B) \leq \delta(A, C) + \delta(C, B), \forall A, B, C \in B(X).$$

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Definition 1.1. *A sequence* $\{A_n\}$ *of nonempty subsets of* X *is said to be convergent to a set* A *of* X *(*[7], [8]*) if:*

(i) *each point* $a \in A$ *is the limit of a convergent sequence* a_n, *where* $a_n \in A_n$ *for all* $n \in \mathbb{N}$

(ii) *for an arbitrary* $\varepsilon > 0$, *there exists an integer* $m > 0$ *such that* $A_n \subset A_\varepsilon$ *for* $n > m$, *where* A_ε *denotes the set of all points* $x \in X$ *for which there exists a point* $a \in A$, *depending on* x, *such that* $d(x, a) < \varepsilon$.

A *is said to be the limit of the sequence* $\{A_n\}$.

Lemma 1.1. *(Fisher (*[7]*))* *If* $\{A_n\}$ *and* $\{B_n\}$ *are sequences in* $B(X)$ *converging to* A *and* B *respectively, then the sequence* $\{\delta(A_n, B_n)\}$ *converges to* $\delta(A, B)$.

Lemma 1.2. *(Fisher and Sessa* [8]*)* *Let* $\{A_n\}$ *be a sequence in* $B(X)$ *and* $y \in X$ *such that* $\delta(A_n, y) \to 0$, *then the sequence* $\{A_n\}$ *converges to* $\{y\}$ *in* $B(X)$.

Let A and S be self-mappings of a metric space (X, d). Jungck [9] defined A and S to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = t$ for some $t \in X$.

A point $x \in X$ is a coincidence point of A and S if $Ax = Sx$.

We denote by $C(A, S)$ the set of all coincidence points of A and S. In [14], Pant defined A and S to be pointwise R-weakly commuting mappings if for all $x \in X$, there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$. It has been proved in [15] that pointwise R-weakly commuting is equivalent to commuting at coincidence points.

Definition 1.2. A *and* S *are said to be weakly compatible* [10] *if* $ASu = SAu$ *for* $u \in C(A, S)$.

Definition 1.3. A *and* S *are said to be occasionally weakly compatible (owc)* [3] *if* $ASu = SAu$ *for some* $u \in C(A, S)$.

Remark 1.1. *If* A *and* S *are weakly compatible and* $C(A, S) \neq \varnothing$, *then* A *and* S *are owc, but the converse is not true (Example,* [3]*).*

Some fixed point theorems for owc mappings have been proved in [13] and other papers.

Definition 1.4. *Let $f : X \longrightarrow X$ and $F : X \longrightarrow B(X)$. Then*

(1) *a point $x \in X$ is said to be a coincidence point of f and F if $fx \in Fx$. We denote by $C(f, F)$ the set of all coincidence points of f and F.*
(2) *a point $x \in X$ is said to be a strict coincidence point of f and F if $\{fx\} = Fx$.*
(3) *a point $x \in X$ is a fixed point of F if $x \in Fx$.*
(4) *a point $x \in X$ is a strict fixed point of F if $\{x\} = Fx$.*

Definition 1.5. *The mappings $f : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ are said to be δ-compatible* [11] *if $\lim_{n\to\infty} \delta(fFx_n, Ffx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $fFx_n \in B(X), fx_n \to t, Fx_n \to \{t\}$ for some $t \in X$.*

Definition 1.6. *The hybrid pair $(f, F), f : X \longrightarrow X$, and $F : X \longrightarrow B(X)$ is weakly compatible* [12] *if for all $x \in C(f, F), fFx = Ffx$.*

If the pair (f, F) is δ-compatible, then it is weakly compatible, but the converse is not true in general [12].

Definition 1.7. *Let S and T be two single-valued self-mappings of a metric space (X, d). We say that S and T satisfy property (E.A)* [1] *if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.*

Remark 1.2. *It is clear that two self-mappings S and T of a metric space (X, d) will be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that either $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or does not exist. Therefore, two noncompatible self-mappings of a metric space (X, d) satisfy property (E.A).*

Recently, Djoudi and Khemis [6] have introduced a generalization of a pair of mappings satisfying property (E.A), named D-mappings.

Definition 1.8. *The mappings $f : X \longrightarrow X$ and $F : X \longrightarrow B(X)$ are said to be D-mappings* [6] *if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$.*

Obviously, two mappings which are not δ-compatible are D-mappings.

Definition 1.9. *The hybrid pair* $f : X \longrightarrow X$ *and* $F : X \longrightarrow B(X)$ *is owc* [2] *if there exists* $x \in C(f,F)$ *such that* $fFx = Ffx$.

Remark 1.3. *If the pair* (f,F) *is weakly compatible and* $C(f,F) \neq \varnothing$ *then the pair* (f,F) *is owc. owc pairs which are not weakly compatible do exist (Example 1.13* [2]*).*

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying the following conditions:

(φ_1) φ is continuous,

(φ_2) φ is nondecreasing on $\mathbb{R}_+$,

(φ_3) $0 < \varphi(t) < t$ for each $t > 0$.

The following theorem has been proved in [1].

Theorem 1.1. *Let* A, B, S *and* T *be self-mappings of a metric space* (X,d) *such that*

(1.1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

(1.2) $d(Ax, By) \leq \varphi(\max\{d(Sx,Ty), d(Sx,By), d(Ty,By)\})$ *for all* $x, y \in X$,

(1.3) (A,S) *and* (B,T) *are weakly compatible*

(1.4) (A,S) *or* (B,T) *satisfy property (E.A).*

If one of $A(X), B(X), S(X), T(X)$ *is a closed set of* X, *then* A, B, S *and* T *have a unique common fixed point.*

For D-mappings the following theorems have been recently proved:

Theorem 1.2. [6] *Let* $I, J : X \longrightarrow X$ *and* $F, G : X \longrightarrow B(X)$ *such that*

(1.5) $G(X) \subset I(X)$ *and* $F(X) \subset J(X)$,

(1.6) $\delta(Fx, Gy) < \max\{c\,d(Ix,Jy), c\,\delta(Ix,Fx), c\,\delta(Jy,Gy), aD(Ix,Gy) + bD(Jy,Fx)\}$

for all $x, y \in X$, *where* $0 \leq c < 1, 0 \leq a + b < 1$, *hold whenever the right hand side of* (1.6) *is positive.*

If the pairs of mappings $\{F, I\}$ and $\{G, J\}$ are weakly compatible and D-mappings and either $F(X)$ or $G(X)$ (or $I(X)$ or $J(X)$, respectively) is closed, then F, G, I and J have a unique common fixed point in X.

Theorem 1.3. [4] *Let* (X,d) *be a metric space. and* $I, J : X \longrightarrow X$ *and* $F, G : X \longrightarrow B(X)$ *satisfying the following conditions:*

(1.7) $F(X) \subset J(X)$ *and* $G(X) \subset I(X)$,

(1.8) $\delta(Fx,Gy) < \alpha \max\{d(Ix,Jy), \delta(Ix,Fx), \delta(Jy,Gy)\} + $
$+(1-\alpha)[aD(Ix,Gy) + bD(Jy,Fx)]$

for all $x, y \in X$, *where* $0 \leq \alpha < 1, a \geq 0, b \geq 0, a+b < 1$, *whenever the right hand side of* (1.8) *is positive.*

If either

(1.9) *F and I are weakly compatible D-mappings, G and J are weakly compatible and $F(X)$ or $I(X)$ is closed, or*

(1.9') *G and J are weakly compatible D-mappings, F and I are weakly compatible and $G(X)$ or $J(X)$ is closed, then there is a unique common fixed point t in X such that $Ft = Gt = \{t\} = \{It\} = \{Jt\}$.*

In [16] and [17], the study of fixed points for mappings satisfying implicit relations was introduced and the study of hybrid pairs of mappings satisfying implicit relations was initiated in [18].

In this paper some general fixed point theorems for owc D-mappings satisfying an implicit realtion are proved which generalize the results in Theorems 1.1-1.3.

2. **Implicit relations**

Definition 2.1. *Let $\mathcal{F}_D$ be the set of all real continuous functions $\phi(t_1,\ldots,t_6) : \mathbb{R}_+^6 \longrightarrow \mathbb{R}$ satisfying the following conditions:*

(ϕ_1) *ϕ is nonincreasing in variables t_5 and t_6,*
(ϕ_2) $\phi(t,0,0,t,t,0) \leq 0$ *or* $\phi(t,0,t,0,0,t) \leq 0$ *implies* $t = 0$,
(ϕ_3) $\phi(t,t,0,0,t,t) \geq 0, \forall t > 0$.

Example 2.1. $\phi(t_1,\ldots,t_6) = t_1 - \alpha\max\{t_2,t_3,t_4\} - (1-\alpha)[at_5 + bt_6]$, *where* $0 \leq \alpha < 1, a \geq 0, b \geq 0$ *and* $a+b < 1$.

(ϕ_1) : *Obviously.*
(ϕ_2) : $\phi(t,0,0,t,t,0) = (1-\alpha)(1-a)t \leq 0$ *implies* $t = 0$,
$\phi(t,0,t,0,0,t) = (1-\alpha)(1-b)t \leq 0$ *implies* $t = 0$.
(ϕ_3) : $\phi(t,t,0,0,t,t) = t(1-\alpha)[1-(a+b)] \geq 0, \forall t > 0$.

Example 2.2. $\phi(t_1,\ldots,t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $0 \leq c < 1, a \geq 0, b \geq 0$ *and* $a+b < 1$.

(ϕ_1) : *Obviously.*
(ϕ_2) : $\phi(t,0,0,t,t,0) = t(1-\max\{a,c\}) \leq 0$ *implies* $t = 0$,
$\phi(t,0,t,0,0,t) = t(1-\max\{b,c\}) \leq 0$ *implies* $t = 0$.

$(\phi_3): \phi(t,t,0,0,t,t) = t(1-\max\{c,a+b\}) \geq 0, \forall t > 0.$

Example 2.3. $\phi(t_1,\ldots,t_6) = t_1 - at_2 - b(t_3+t_4) - c(t_5+t_6)$, *where* $a \geq 0, b \geq 0, c \geq 0, b+c < 1$ *and* $a+2c < 1$.

Example 2.4. $\phi(t_1,\ldots,t_6) = t_1 - at_2 - b(t_3+t_4) - c\min\{t_5,t_6\}$, *where* $a,b,c \geq 0, b < 1, a+c < 1$.

Example 2.5. $\phi(t_1,\ldots,t_6) = t_1 - \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{k(t_5+t_6)}{2}\right\}$, *where* $0 \leq k < 1$.

Example 2.6. $\phi(t_1,\ldots,t_6) = t_1 - \max\left\{k_1 t_2, \frac{k_2(t_3+t_4)}{2}, \frac{t_5+t_6}{2}\right\}$, *where* $0 \leq k_1 < 1, 1 \leq k_2 < 2$.

Example 2.7. $\phi(t_1,\ldots,t_6) = t_1 - \max\left\{k_1(t_2+t_3+t_4), \frac{k_2(t_5+t_6)}{2}\right\}$ *where* $0 \leq k_1 < 1, 0 \leq k_2 < 1$.

Example 2.8. $\phi(t_1,\ldots,t_6) = t_1 - h\max\{t_2,t_3,t_4,t_5,t_6\}$, *where* $0 \leq h < 1$.

Example 2.9. $\phi(t_1,\ldots,t_6) = t_1^2 - at_2^2 - t_3t_4 - bt_5^2 - ct_6^2$, *where* $a,b,c \geq 0$ *and* $a+b+c < 1$.

Example 2.10. $\phi(t_1,\ldots,t_6) = t_1^3 - k(t_2^3+t_3^3+t_4^3+t_5^3+t_6^3)$, *where* $0 \leq k < \frac{1}{3}$.

Example 2.11. $\phi(t_1,\ldots,t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$, *where* $a,b,c,d \geq 0$ *and* $a+c+d < 1$.

Example 2.12. $\phi(t_1,\ldots,t_6) = t_1^3 - \frac{t_3^2t_4^2+t_5^2t_6^2}{1+t_2+t_3+t_4}$.

Example 2.13. $\phi(t_1,\ldots t_6) = t_1 - \varphi\left(\max\left\{t_2,t_3,t_4,t_5,\frac{1}{2}t_6\right\}\right)$.

Example 2.14. $\phi(t_1,\ldots,t_6) = t_1 - \varphi\left(\max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}\right)$.

Example 2.15. $\phi(t_1,\ldots,t_6) = t_1 - \varphi\left(\max\left\{t_2,t_3,t_4,\frac{k(t_5+t_6)}{2}\right\}\right)$, *where* $0 \leq k < 1$.

3. Main results

Theorem 3.1. *Let* $I:(X,d) \longrightarrow (X,d)$ *and* $F: X \longrightarrow B(X)$ *be owc self-mappings. If* I *and* F *have a unique point of strict coincidence* $\{z\} = \{Ix\} = Fx$, *then* z *is the unique common fixed point of* I *and* F *which is a strict fixed point of* F.

Proof. Since I and F are owc, there exists $x \in X$ such that $\{z\} = \{Ix\} = Fx$ implies $IFx = FIx$. Then $\{Iz\} = \{IIx\} = IFx = FIx = Fz = \{u\}$, hence u is a point of strict coincidence of I and F. By hypothesis $u = z$.

Hence $\{z\} = \{Iz\} = Fz$ and z is a common fixed point of I and F which is a strict fixed point of F. Suppose that $v \neq z$ is another common fixed point of I and F. which is a strict fixed point of F. Hence $\{v\} = \{Iv\} = Fv$.

Therefore v is a point of strict coincidence of I and F and by hypothesis $v = z$. ∎

Theorem 3.2. *Let $I, J : (X, d) \longrightarrow (X, d)$ and $F, G : (X, d) \longrightarrow B(X)$ such that the inequality*

$$\phi\left(\delta\left(Fx, Gy\right), d\left(Ix, Jy\right), \delta\left(Ix, Fx\right), \delta\left(Jy, Gy\right), D\left(Ix, Gy\right), D\left(Jy, Fx\right)\right) < 0 \tag{3.1}$$

holds for all $x, y \in X$ and ϕ satisfying property (ϕ_3).

Suppose that there exist $x, y \in X$ such that $\{u\} = \{Ix\} = Fx$ and $\{v\} = \{Jy\} = Gy$. Then u is the unique point of strict coincidence of I and F and v is the unique point of strict coincidence of J and G.

Proof. First we prove that $Ix = Jy$. Suppose $Ix \neq Jy$. Then by (3.1) we obtain

$$\phi\left(d\left(Ix, Jy\right), d\left(Ix, Jy\right), 0, 0, d\left(Ix, Jy\right), d\left(Ix, Jy\right)\right) < 0,$$

a contradiction of (ϕ_3), hence $Ix = Jy$. Thus $\{u\} = \{Ix\} = Fx = Gy = \{Jy\}$. Suppose that there exists $z \in X$ such that $\{w\} = \{Iz\} = Fz$ with $w \neq u$. Then by (3.1) we have

$$\phi\left(d\left(Iz, Jy\right), d\left(Iz, Jy\right), 0, 0, d\left(Iz, Jy\right), d\left(Iz, Jy\right)\right) < 0,$$

$$\phi\left(d\left(u, w\right), d\left(u, w\right), 0, 0, d\left(u, w\right), d\left(u, w\right)\right) < 0,$$

a contradiction of (ϕ_3), hence $\{w\} = \{Iz\} = \{Jy\} = Gy = Fx = \{Ix\} = \{u\}$. Hence $u = w$ and u is the unique point of strict coincidence of I and F. Similarly, v is the unique point of strict coincidence of J and G. ∎

Theorem 3.3. *Let (X, d) be a metric space, $I, J : X \longrightarrow X$ and $F, G : X \longrightarrow B(X)$ satisfying the following conditions:*

The inequality (3.1) holds for all $x, y \in X$, where $\phi \in \mathcal{F}_D$.

$$F(X) \subset J(X) \text{ and } G(X) \subset I(X) \tag{3.2}$$

If the pair (F, I) or (G, J) is a D-mapping and $F(X)$ or $G(X)$ (or $J(X)$ or $I(X)$, respectively) is a closed set of X then

$$F \text{ and } I \text{ have a strict point of coincidence,} \tag{3.3}$$

$$G \text{ and } J \text{ have a strict point of coincidence.} \tag{3.4}$$

Moreover, if the pairs (I, F) and (J, G) are owc, then I, J, F and G have a unique common fixed point which is a strict fixed point of F and G.

Proof. Since the pair (F, I) is a D-mapping, there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ix_n = t$ and $\lim_{n\to\infty} Fx_n = \{t\}$ for some $t \in X$. Since $F(X)$ is closed, there exists $u \in X$ such that $t = Ju$. By (3.1) we have

$$\phi\left(\begin{array}{c} \delta(Fx_n, Gu), d(Ix_n, Ju), \delta(Ix_n, Fx_n), \\ \delta(Ju, Gu), D(Ix_n, Gu), D(Ju, Fx_n) \end{array}\right) < 0.$$

Letting n tend to infinity we obtain

$$\phi(\delta(Ju, Gu), 0, 0, \delta(Ju, Gu), \delta(Ju, Gu), 0) \leq 0.$$

By (ϕ_2) it follows that $\delta(Ju, Gu) = 0$, i.e. $\{Ju\} = Gu$ and u is a strict coincidence point of J and G.

Since $G(X) \subset I(X)$, there is a point $v \in X$ such that $\{Iv\} = Gu$. Then, by (3.1) we have successively

$$\phi\left(\begin{array}{c} \delta(Fv, Gu), d(Iv, Ju), \delta(Iv, Fv), \\ \delta(Ju, Gu), D(Iv, Gu), D(Ju, Fv) \end{array}\right) < 0,$$

$$\phi(\delta(Fv, Gu), 0, \delta(Gu, Fv), 0, 0, \delta(Fv, Gu)) < 0.$$

By (ϕ_2) it follows that $\delta(Fv, Gu) = 0$, i.e. $Fv = Gu$. Hence

$$\{t\} = \{Ju\} = Gu = Fv = \{Iv\}. \tag{3.5}$$

Hence v is a strict coincidence point of I and F. By Theorem 3.2, z is the unique point of coincidence of I and F, and also z is the unique point of strict coincidence of J and G. By Theorem 3.1, z is the unique common fixed point of I and F which is a strict fixed point for F. Similarly z is the unique common fixed point of J and G which is a strict fixed point of G. Hence, z is the unique common fixed point of I, J, F and G which is a strict fixed point for F and G. ∎

By Theorem 3.3 and Example 2.2 we obtain the following generalization of Theorem 1.2:

Corollary 3.1. *Let* (X, d) *be a metric space and* $I, J : X \longrightarrow X$, $F, G : X \longrightarrow B(X)$ *satisfying the conditions (1.5) and (1.6). If the pairs* (F, I) *or* (G, J) *are* D*-mappings and* $F(X)$ *or* $G(X)$ *(or* $J(X)$ *or* $I(X)$*, respectively) is a closed set of* X*, then*

(1) F *and* I *have a strict coincidence point*
(2) G *and* J *have a strict coincidence point.*

Moreover, if the pairs (I, F) *and* (J, G) *are owc, then* I, J, F *and* G *have a unique common fixed point which is a strict fixed point for* F *and* G.

By Theorem 3.3 and Example 2.1 we obtain the following generalization of Theorem 1.3:

Corollary 3.2. *Let* (X, d) *be a metric space and* $I, J : X \longrightarrow X$, $F, G : X \longrightarrow B(X)$ *satisfying the conditions (1.7) and (1.8). If the pairs* (F, I) *or* (G, J) *are* D*-mappings and* $F(X)$ *or* $G(X)$ *(or* $J(X)$ *or* $I(X)$*, respectively) is a closed set of* X*, then*

(1) F *and* I *have a strict coincidence point*
(2) G *and* J *have a strict coincidence point.*

Moreover, if the pairs (F, I) *and* (G, J) *are owc, then* I, J, F *and* G *have a unique common fixed point which is a strict fixed point for* F *and* G.

Also, for single-valued functions we obtain by Theorem 3.3 the following theorem.

Theorem 3.4. *Let* A, B, S *and* T *be self-mappings of a metric space* (X, d) *such that*

(1) $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,
(2)

$$\phi\left(\begin{array}{c} d(Ax, By), d(Sx, Ty), d(Sx, Ax), \\ d(Ty, By), d(Sx, By), d(Ty, Ax) \end{array}\right) < 0$$

for all $x, y \in X$*, where* $\phi \in \mathcal{F}_D$.

If the pairs (A, S) *and* (G, T) *satisfy property (E.A) and if one of* $A(X)$*,* $B(X)$*,* $S(X)$*,* $T(X)$ *is a closed subset of* X*, then*

(3) A *and* S *have a coincidence point*

(4) B *and* T *have a coincidence point.*

Moreover, if (A, S) *and* (B, T) *are owc then* A, B, S *and* T *have a unique common fixed point.*

Remark 3.1. *By Theorem 3.4 and Example 2.13 we obtain a generalization of Theorem 1.1 because by (1.2) we obtain*

$$d(A, B) \leq \varphi\left(\max\left\{d(Sx, Ty), d(Ax, Sx), d(Ty, By), d(Sx, By), \frac{1}{2}d(Ty, Ax)\right\}\right).$$

Let $f : (X, d) \longrightarrow (X, d)$ and $F : (X, d) \longrightarrow B(X)$. We denote $Fix(f) = \{x \in X : x = fx\}$ and $SFix(F) = \{x \in X : \{x\} = Fx\}$.

Theorem 3.5. *Let* $I, J : X \longrightarrow X$ *and* $F, G : X \longrightarrow B(X)$ *such that the inequality (3.1) holds for all* $x, y \in X$ *and* $\phi \in \mathcal{F}_D$*. Then*

$$[Fix(I) \cap Fix(J)] \cap SFix(F) = [Fix(I) \cap Fix(J)] \cap SFix(G).$$

Proof. Let $u \in [Fix(I) \cap Fix(J)] \cap SFix(F)$. Then $\{u\} = \{Iu\} = \{Ju\} = Fu$. Then by (3.1) we obtain

$$\phi(\delta(u, Gu), 0, 0, \delta(u, Gu), \delta(u, Gu), 0) < 0$$

which implies by (ϕ_2) that $\delta(u, Gu) = 0$, i.e. $\{u\} = Gu$, hence $u \in SFix(G)$ and $[Fix(I) \cap Fix(J)] \cap SFix(F) \subset [Fix(I) \cap Fix(J)] \cap SFix(G)$. Similarly, $[Fix(I) \cap Fix(J)] \cap SFix(G) \subset [Fix(I) \cap Fix(J)] \cap SFix(F)$.

Theorem 3.3 and Theorem 3.5 imply the next one. ∎

Theorem 3.6. *Let* I, J *be self-mappings of a metric space* (X, d) *and* $F_n : X \longrightarrow B(X), n \in \mathbb{N}^*$ *a sequence of set valued mappings such that*

(1) $F_2(X) \subset I(X)$ *and* $F_1(X) \subset J(X)$.

(2) *either* $F_2(X)$ *or* $F_1(X)$ *(*$I(X)$ *or* $J(X)$*, respectively) is a closed subset of* X*.*

(3) *The inequality*

$$\phi\left(\begin{array}{c}\delta\left(F_n x, F_{n+1} y\right), d\left(Ix, Jy\right), \delta\left(Ix, F_n x\right), \\ \delta\left(Jy, F_{n+1} y\right), D\left(Ix, F_{n+1} y\right), D\left(Jy, F_n x\right)\end{array}\right) < 0$$

holds for all $x, y \in X, \phi \in \mathcal{F}_D$ *and* $n \in \mathbb{N}^*$.

(4) (F_1, I) *is a* D*-mapping and* (F_1, I) *and* (F_2, J) *are owc, or* (F_2, J) *is a* D*-mapping and* (F_1, I) *and* (F_2, J) *are owc.*

Then, there exists a unique common fixed point of $I, J, \{F_n\}_{n\in\mathbb{N}^*}$ *which is a strict fixed point for* $\{F_n\}_{n\in\mathbb{N}^*}$.

Remark 3.2. *By Theorem 3.6 and Example 2.1 we obtain a generalization of Theorem 3.5* [4].

References

[1] A. Aamri and D. El-Moutawakil, **Some new common fixed point theorems under strict contractive conditions**, *Math. Anal.Appl.* **270**(2002), 181-188.

[2] A. Aliouche and V.Popa, **General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications** (submitted)

[3] M.A. Al-Thagafi and N. Shahzad **Generalized I-nonexpansive self-maps and invariant approximations**, *Acta Math. Sinica* **24**(5) (2008) 867-876.

[4] H. Bouhadjera and A. Djoudi, **Common fixed point theorems for single and set-valued maps without continuity**, *Anal. St.Univ.Ovidius, Constanţa* **16**(1) (2008) 49-58.

[5] H. Bouhadjera and A. Djoudi, **Common fixed point theorems for single-valued and multivalued D-maps satisfying an integral type contractive condition**, *Annales Mathematicae et Informaticae* **35**(2008) 43-59.

[6] A. Djoudi and R.Khemis, **Fixed points for set and single-valued maps without continuity**, *Demonstratio Math.* **38**,3(2005), 739-751.

[7] B. Fisher, **Common fixed points of mappings and set-valued mappings**, *Rostok Math. Kolloq.* **18**(1981), 69-77.

[8] B. Fisher and S. Sessa, **Two common fixed point theorems for weakly commuting mappings**, *Periodica Math. Hungar.* **20**(3) (1989), 207-218.

[9] G. Jungck, **Compatible mappings and common fixed points**, *Int. J.Math.Math.Sci.* **9**(1986), 771-779.

[10] G. Jungck, **Common fixed points for noncontinuous nonself maps on non-metric spaces**, *Far East J. Math. Sci.* **4**(2) (1996), 199-215.

[11] G. Jungck and B.E. Rhoades, **Some fixed point theorems for compatible maps**, *Int.J.Math.Math.Sci.* **16**(1993), 417-428.

[12] G. Jungck and B.E. Rhoades, **Fixed point theorems for set-valued functions without continuity**, *Indian J. Pure Appl. Math.* **29**(3) (1998), 227-238.

[13] G. Jungck and B.E. Rhoades, **Fixed point theorems for occasionally weakly compatible mappings**, *Fixed Point Theory*, **7**,2 (2006), 287-296.

[14] R.P. Pant, **Common fixed points of noncommuting mappings**, *J. Math. Anal. Appl.*, **188**(1994), 436-440.

[15] R.P.Pant, **Common fixed points of four mappings**, *Bull. Cal.Math.Soc*, **9**(1998), 281-286.

[16] V. Popa, **Fixed point theorems for implicit contractive mappings**, *Stud. Cerc. Şt. Ser. Mat. Univ. Bacău* **7**(1997), 127-133.

[17] V. Popa, **Some fixed point theorems for compatible mappings satisfying an implicit relation**, *Demonstratio Math.* **32**(1999), 157-163.

[18] V.Popa, "A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation", *Demonstratio Math.* **33**,1 (2000), 159-164.

[19] S. Sharma, B. Deshpande and R.Pathak, **Common fixed point theorems for hybrid pairs of mappings with some weaker conditions**, *Fasciculi Mathematici* **39**(2008), 71-86.

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