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NUMBER OF JUMPS FOR SAMPLE FUNCTIONS OF LEVY PROCESSES

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Abstract. The structure of jumps of a Levy process is determined by its Levy (or characteristic) measure. For an n-dimensional Levy process, the Levy measure of $D \subset R^n$ is given by the expected number, per unit time, of jumps whose size belongs to D.

1. INTRODUCTION

Levy processes are popular mathematical tools in Engineering, Physics and Mathematical finance [4]. These processes have been the subject of intense research and applications in recent years, because their paths can be decomposed into a Brownian motion with drift plus an independent superposition of jumps of all possible size. This decomposition of Levy processes makes them suited for modelling random phenomena which manifest jumps [1], [2]. Levy processes have some important features such that these processes have paths consist of continuous motion interspersed with jump discontinuities of random size.

The paper is organized as follows. Section 2 reviews well known properties of Levy process. In Section 3 we describe probabilistic structure and path properties of Levy processes. In Section 4, we introduce independent processes. Finally, Section 5 includes conclusions.

2. LEVY PROCESSES

In this section we introduce definition and some basic properties of Levy processes. The name Levy processes honour the work of the French mathematician Paul Levy (1886-1971). General references on Levy processes are [1], [2], [11].

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An R^n – valued stochastic process $\{X_t : t \geq 0\}$ is a family of R^n – valued random variables $X_t(\omega)$ with parameter $t \in [0, \infty)$ defined on a probability space $(\Omega, \mathcal{F}, P)$.

An R^n – valued stochastic process $\{X_t : t \geq 0\}$ is called a Levy process on R^n or n-dimensional Levy process, if the following five conditions are satisfied:

(1) it has independent increments, that is, for any choice of $n \geq 1$ and $0 \leq t_0 < t_1 < ... < t_n$ the random variables $X_{t_0}, X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent;

(2) it starts at the origin, $X_0 = 0$ a.s.(almost surely);

(3) it is time homogeneous, that is, the distribution of $X_{s+t} - X_s : t \geq 0$ does not depend on s;

(4) it is stochastically continuous, that is, for any $\varepsilon > 0, P[|X_{s+t} - X_s| > \varepsilon] \to 0 \ \ as\, t \to 0$;

(5) as a function of t, $X_t(\omega)$ is right-continuous with left limits a.s.

Note that the (4) condition does not imply that the path of Levy processes are continuous. It only requires that for a given time t, the probability of seeing a jump at t is zero, i. e. jumps occur at random times.

Here we say that a property for ω holds a.s., if there is $B \in \mathcal{F}$ with $P(B) = 1$ such that the property holds for every $\omega \in B$.

Next, we try to understand the structure of Levy processes.

Let $X_t(\omega)$ be a Levy process. The sample function of X will be denoted by $X(\cdot, \omega)$. We will assume, that $X = X_t(\omega) \in D$ for every ω. The set of all $t > 0$ for which

$$\left|X_t(\omega) - X_{t_-}(\omega)\right| > 0$$

is denoted by $I(\omega)$. This set $I(\omega)$ consists of all discontinuity points of $X(\cdot, \omega)$. We also consider the 2-dimensional set

$$J = J(\omega) = \left\{ \left(t, X_t(\omega) - X_{t_-}(\omega)\right) : t \in I(\omega) \right\}$$

Then $J(\omega)$ is a countable subset of $T_0 \times R_0$, where

$$T_0 = T - \{0\}, \qquad R_0 = R^1 - \{0\}$$

because $X(\cdot, \omega) \in D$.

Let $\mathcal{B}(T_0 \times R_0)$ denote the class of all Borel subsets of $T_0 \times R_0$ and let $\mathcal{B}^*(T_0 \times R_0)$ denote the class of all $A \in \mathcal{B}(T_0 \times R_0)$, such that

$$A \subset (0,a) \times \left\{ u : |u| > \frac{1}{a} \right\} \text{ for some } a > 0.$$

3. NUMBER OF JUMPS

By virtue of $X(\cdot,\omega) \in D$ $A \cap J(\omega)$ is a finite set depending on ω as long as $A \in \mathcal{B}^*(T_0 \times R_0)$. The number of points in this set $A \cap J(\omega)$ will be denoted by $N(A,\omega)$.

Remark 1. That $N(A,\omega)$ is measurable in ω and so is a random variable.

More precisely we have:

Theorem 1. If $A \in \mathcal{B}^*(T_0 \times R_0)$ and $A \subset (s,t] \times R_0$, then *N(A)* is measurable $\overline{\mathcal{B}}_{st}[dX]$.

Proof. Denote $E_{s,t,a} = (s,t] \times (a,\infty) \subset T_0 \times R_0$ the set where $0 \le s < t < \infty$ and $a > 0$ and $Q \subset (s,t]$ be a countable dense subset of *(s, t]* including the right end point t. It is easy to see that

$$\{N(E_{s,t,a}) \ge 1\} = \{X(\tau) - X(\tau_-) > a \quad \textit{for} \quad \textit{some } \tau \in (s,t]\}$$

$$\bigcup_p \bigcap_q \bigcap_{\substack{s+1/p \le r < r' \le t \\ r,r' \in Q, r'-r < 1/q}} \{X(r') - X(r) \ge a + 1/p\} \in \overline{\mathcal{B}}_{st}[dX]$$

Observing

$$\{N(E_{s,t,a}) \ge k+1\} = \bigcup_{r \in Q \cap (s,t)} \{N(E_{s,t,a}) \ge k\} \cap \{N(E_{s,t,a}) \ge 1\}$$

we get

$$\{N(E_{s,t,a}) \ge k\} \in \overline{\mathcal{B}}_{st}[dX], \qquad k = 1,2,\dots,,$$

which proves that $N(E_{s,t,a})$ is measurable $(\overline{\mathcal{B}}_{st}[dX])$.

Remark 2. Writing $E'_{s,t,a}$ for $(s,t] \times (-\infty, -a)$ $(a > 0)$ and using the same argument as above, we can see that $N(E'_{s,t,a})$ is also measurable $(\overline{\mathcal{B}}_{st}[dX])$.

If $A \in \mathcal{B}^*(T_0 \times R_0)$ and $A \subset (s,t] \times R_0$, we have

$$A = (A \cap E_{s,t,a}) \cup (A \cap E'_{s,t,a})$$

and so

$$N(A) = N(A \cap E_{s,t,a}) + N(A \cap E'_{s,t,a})$$

by taking *a>0* sufficiently small. Therefore *N(A)* is measurable $(\overline{\mathcal{B}}_{st}[dX])$.

Theorem 2.
(a) If $A \in \mathcal{B}(T_0 \times R_0)$ then *N(A)* is Poisson distributed with infinite parameter.
(b) If $A \in \mathcal{B}^*(T_0 \times R_0)$, then *N(A)* is Poisson distributed with finite parameter.

Proof. (b) First we will discuss the case $A \in \mathcal{B}^*(T_0 \times R_0)$. Write *A(t)* for the intersection $A \cap (0,t] \times R_0$ and consider the stochastic process

$$N(t) = N_t(\omega) = N_{A(t)}(\omega)$$

It is obvious that $N_t(\omega)$ is a right continuous step function in *t* increasing with jumps = 1. Since

$$N(t) - N(s) = N(A(t) - A(s)) = N[A \cap ((s,t] \times R_0) \quad for \;\; s < t]$$

it is easy to see by Theorem 1 that *N(t)* is an additive process. Since, for every t fixed,

$$P(N(t) - N(t_-) \neq 0) \leq P(X_t(\omega) - X_{t_-}(\omega) \neq 0) = 0$$

N(t) is continuous i.p.* Therefore *N(t)* is a Levy processes of Poisson type [7]. Since $A \in \mathcal{B}^*(T_0 \times R_0)$, we have *A* = *A(t)* and so *N(A)* ≡ *N(A(t))* for sufficiently large t. Therefore *N(A)* is Poisson distributed with finite parameter.

(a) Let $A \in \mathcal{B}(T_0 \times R_0)$. Then we have an increasing sequence $A_n \in \mathcal{B}^*(T_0 \times R_0)$ $n = 1, 2, \ldots$, such that $A_n \uparrow A$. Then

$$N(A) = \lim_{n \to \infty} N(A_n).$$

Each $N(A_n)$ is Poisson distributed with the parameter $\lambda_n \equiv E(N(A_n))$ and λ_n is increasing. If $\lambda = \lim_n \lambda_n < \infty$, then *N(A)* is Poisson distributed with parameter λ. If $\lambda = \infty$, then

$$P(N(A) \leq k) \leq P(N(A_n) \leq k) = e^{-\lambda_n} \sum_{j=0}^{k} \frac{\lambda_n{}^j}{j!} \to 0$$

as n $\to \infty$ for k fixed. This proves $P(N(A) = \infty) = 1$, completing the proof.

It is obvious that $N_A(\omega)$ is a measure in $A \in \mathcal{B}(T_0 \times R_0)$, which takes the values 0, 1, 2, … and ∞. Therefore

$$\begin{aligned} n(A) &= E(N(A)) \\ &= the\ parameter\ of\ the\ Poisson\ variable\ N(A) \end{aligned}$$

is also a measure in $A \in \mathcal{B}(T_0 \times R_0)$, which may take the value ∞.

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* i.p.= in probability

Remark 3. In Theorem 2 we saw that *n(A)* $< \infty$ if $A \in \mathcal{B}^*(T_0 \times R_0)$. The measure *n(A)* is called the Levy measure of the Levy process *X*.

4. INDEPENDENT PROCESSES

For $A \in \mathcal{B}^*(T_0 \times R_0)$ we can consider

$$S(A) = S_A(\omega) = \sum_{(t, X_t(\omega) - X_{t\_}(\omega)) \in A} (X_t(\omega) - X_{t\_}(\omega)) = \sum_{(t,u) \in A \cap J(\omega)} u$$

Since $A \cap J(\omega)$ is a finite set for such A, the sum is a finite sum and so there is no problem of convergence. This $S_A(\omega)$ is also measurable in ω and therefore a random variable, because

$$S_A(\omega) = \lim_{n \to \infty} \sum_k \frac{k}{n} N\left(A \cap \left(T_0 \times \left(\frac{k-1}{n}, \frac{k}{n} \right] \right), \omega \right) \qquad (1)$$

This relation is also expressed as

$$S_A(\omega) = \iint_{(t,u) \in A} uN(dtdu, \omega) \qquad (1')$$

Theorem 3. Let $A_1, A_2, ..., A_n \in \mathcal{B}(T_0 \times R_0)$ be disjoint. Then $N(A_1), N(A_2), ..., N(A_n)$ are independent.

Proof. Since every $A \in \mathcal{B}(T_0 \times R_0)$ can be expressed as the limit of an increasing sequence of sets in $\mathcal{B}^*(T_0 \times R_0)$, we can assume that

$$A \in \mathcal{B}^*(T_0 \times R_0),\ i = 1, 2, ..., n$$

Then we can take t_0 so large that $A_i = A_i(t_0)$ for every i. Therefore $N(A_i)$, *i=1, 2, ..., n* are independent [5].

For $A \in \mathcal{B}^*(T_0 \times R_0)$ we set

$$A_{mk} = \left\{ (s,u) \in A : \frac{k}{m} < u \le \frac{k+1}{m} \right\}$$

Then $S_A(\omega) = \iint_A uN(dsdu, \omega) = \lim_{m \to \infty} \sum_k \frac{k}{m} N_{A_{mk}}(\omega)$

Theorem 4. If $A \in \mathcal{B}^*(T_0 \times R_0)$, then

$$E\left[e^{izS(A)}\right] = \exp\left\{ \iint_A \left(e^{izu} - 1\right) n(dsdu) \right\}$$

Proof. Since $\{A_{mk}\}$ are disjoint for each *m*, $\{N(A_{mk})\}_k$ are independent. Since $N(A_{mk})$ is Poisson distributed with parameter $n(A_{mk})$, we have

$$E\left[e^{iz\left(\frac{k}{m}\right)N(A_{mk})}\right] = \exp\left\{\left(e^{\frac{izk}{m}} - 1\right)n(A_{mk})\right\}$$

Therefore we have:

$$E\left[e^{izS(A)}\right] = \lim_{m\to\infty} E\left[e^{iz\sum_k\left(\frac{k}{m}\right)N(A_{mk})}\right]$$

$$= \lim_{m\to\infty}\prod_k E\left[e^{iz\left(\frac{k}{m}\right)N(A_{mk})}\right]$$

$$= \lim_{m\to\infty}\prod_k \exp\left\{\left(e^{\frac{izk}{m}} - 1\right)n(A_{mk})\right\}$$

$$= \exp\left\{\iint_A \left(e^{izu} - 1\right)n(dsdu)\right\}$$

noticing that $n(A) < \infty$.

Similarly we have:

Theorem 5. If $A \in \mathcal{B}^*(T_0 \times R_0)$ is included in $\{(s,u): |u| < m\}$ for some m $< \infty$, then

$$E[S(A)] = \iint_A un(dsdu)$$

$$V[S(A)] = \iint_A u^2 n(dsdu)$$

Since $(0,t]\times\{u:|u|>\varepsilon\}$ $(\varepsilon > 0)$ belongs to $\mathcal{B}^*(T_0 \times R_0)$, we have

$$\iint_{\substack{0<s<t\\|u|>\varepsilon}} n(dsdu) < \infty$$

Example. Let *X(t)* be a Levy process of Gauss type such that $E(X(t)) = 0$ and $V(X(t)) = V(t)$.

Then $S_n \equiv \sum_{k=1}^{n}\left(X\left(\frac{k}{n}t\right) - X\left(\frac{k-1}{n}t\right)\right)^2 \to V(t)$ *i.p.*.

For fixed *t* and *n*, denote $t_k = \left(\frac{k}{n}\right)t$ for k=0, ..., n. Since $X(t_k) - X(t_{k-1})$ is Gauss distributed with mean 0 and variance $V(t_k) - V(t_{k-1})$ we have

$$E\left[(X(t_k)-X(t_{k-1}))^2\right]=V(t_k)-V(t_{k-1})$$
$$E\left[(X(t_k)-X(t_{k-1}))^4\right]=3[V(t_k)-V(t_{k-1})]^2$$

Hence $E(S_n)=V(t)$ and

$$E\left(S_n^2\right)=\sum_{k,j=1}^{n}E\left[(X(t_k)-X(t_{k-1}))^2\left(X(t_j)-X(t_{j-1})\right)^2\right]$$
$$=\sum_k E\left[(X(t_k)-X(t_{k-1}))^4\right]$$
$$+\sum_{k\neq j}E\left[(X(t_k)-X(t_{k-1}))^2\right]E\left[\left(X(t_j)-X(t_{j-1})\right)^2\right]$$
$$=3\sum_k[V(t_k)-V(t_{k-1})]+\sum_{k\neq j}[V(t_k)-V(t_{k-1})]\left[V(t_j)-V(t_{j-1})\right]$$
$$=\left(\sum_k[V(t_k)-V(t_{k-1})]\right)^2+2\sum_k[V(t_k)-V(t_{k-1})]^2$$

It follows that

$$Var(S_n)=E\left(S_n^2\right)-[E(S_n)]^2=2\sum_k[V(t_k)-V(t_{k-1})]^2$$
$$\leq 2\max_k|V(t_k)-V(t_{k-1})|V(t)$$

which tends to 0 as $n\to\infty$ since *V(s)* is uniformly continuous on [0, t]. Hence by Chebyshev's inequality

$$P(|S_n-V(t)|>\varepsilon)\leq Var(S_n)/\varepsilon^2\to 0 \quad as\ n\to\infty$$

V. CONCLUSIONS

The law of a Levy process X_t is completely determined by the law of X_1. The class of Levy models is include the Brownian model as a special case but contrar to the brownian model, allow us to modelling jumps. Levy processes are suitable for modelling market price fluctuations because include jumps. Jumps are useful to capture unexpected changes in the market.

References

[1] Applebaum, D., Levy **Processes and Stochastic Calculus,** Cambridge University Pres, (2004).

[2] Bertoin, J., **Levy Processes**, Cambridge Univ. Press, Cambridge, 1996.

[3] Billingsley, P., **Convergence of Probability Measures**, Wiley, New York, 1968.

[4] Kyprianou, A.E., **Introductory Lectures on Fluctuations of Levy Processes with Applications**, Springer, (2006).

[5] Kunita, H., **Stochastic Flows and Stochastic Differential Equations,** Cambridge Univ. Press, Cambridge, 1990.

[6] Orman, V.G., **Capitole de matematici aplicate**, Ed. Albastră, Cluj-Napoca, 1999.

[7] Ito. K., **Stochastic Processes**, Springer Verlag Berlin Heidelberg New York, 2003.

[8] Orman, V.G., **Măsură şi procese aleatoare**, Editura Universității Transilvania Braşov, 2000.

[9] Orman, V.G., **Handbook of limit theorems and stochastic approximation**, Editura Universității Transilvania Braşov, 2003.

[10] Papapantoleon, A., **An Introduction to Levy Processes with Applications in Finance**, Lecture Notes, (2006).

[11] Sato. K., **Levy Processes and Infinitely Divisible Distributions, Cambridge Univ. Press**, Cambridge, 1999.

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