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THE NORMAL GRAPH CONJECTURE IS TRUE FOR MINIMAL UNBREAKABLE GRAPHS

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Abstract. A graph is normal if there exists a cross-intersecting pair of set families one of which consists of cliques while the other one consists of stable sets, and furthermore every vertex is obtained as one of these intersections. It is known that perfect graphs are normal. Korner and de Simone observed that C_5, C_7 and $\overline{C}_7$ are minimal not normal and conjectured, as generalization of the Strong Perfect Graph Theorem, that every $(C_5, C_7, \overline{C}_7)$-free graph is normal (Normal Graph Conjecture). In this paper we prove this conjecture for the class of minimal unbreakable graphs.

As it turns out, unbreakable graphs find natural applications to wireless network security. These applications, detailed in ([15], [22], [24]), range from increasing network resilience, to enhancing confidentiality, to reducing interference. An intriguing new development, documented in ([11], [12], [14], [18]), show direct relevance of unbreakable graphs to social and peer-to-peer networks.

1. Introduction

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph ([1]), without loops and multiple edges, having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G.

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If $U \subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use $x \sim y$, and $x \not\sim y$ whenever x, y are not adjacent in G. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote by $A \sim B$, while by $A \not\sim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say A and B are *non-adjacent*.

The *neighbourhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we denote $N(v)$ and $N[v]$, when G appears clearly from the context. The *degree* of v in G is $d_G(v) = |N_G(v)|$. The neighbourhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighbourhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A graph is complete if every pair of distinct vertices is adjacent. A *clique* is a subset Q of V with the property that $G(Q)$ is complete. The *clique number* of G, denoted by $\omega(G)$, is the size of the maximum clique. A clique cover is a partition of the vertices set such that each part is a clique. $\theta(G)$ is the size of a smallest possible clique cover of G; it is called the *clique cover number* of G. A stable set is a subset X of vertices where every two vertices are not adjacent. $\alpha(G)$ is the number of vertices is a stable set o maximum cardinality; it is called the *stability number* of G. $\chi(G) = \omega(\overline{G})$ and it is called *chromatic number*.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

Let $\mathcal{F}$ denote a family of graphs. A graph G is called *$\mathcal{F}$-free* if none of its subgraphs is in $\mathcal{F}$. The *Zykov sum* of the graphs G_1, G_2 is the graph $G = G_1 + G_2$ having:

$$V(G) = V(G_1) \cup V(G_2),$$
$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}.$$

The class of graphs containing no induced odd cycle of length ≥ 5 or the complement of such a cycle is called *Berge*.

A graph is called *triangulated* if it does not contain chordless cycles having the length greater or equal four.

A *circulant* C_n^k is a graph with nodes $1, \ldots, n$ where ij is an edge if i and j differ by at most $k(mod\ n)$ and $i \neq j$.

When searching for recognition algorithms, frequently appears a type of partition for the set of vertices in three classes A, B, C, which we call a *weakly decomposition*, such that: A induces a connected subgraph, C is totally adjacent to B, while C and A are totally non-adjacent.

In ([14]) J. Kleinberg considers a model for monitoring the connectivity of a network subject to node or edge failures. They interest to discover $(e, k) - failures$: event in which an advesary deletes up to k network elements (nodes or edges), after which there are two sets of nodes A and B, each at least an e fraction of the network, that are disconnected from one another. They say that a set D of nodes is an $(e, k) - detection$ set if, for any (e,k)-failure of the network, some two nodes in D are no longer able to communicate. Recent results show that for any graph G, there is an (e,k)-detection set of size bounded by a polynomial in k and e, independent of the size of G.

Security represents one of the most important issues in communication network. The starting point of this paper is a particular security problem arising in message passing distributed systems. A communication system is $confidential$ if it is possible to exchange a message between every pair of nodes such that every other specified node can't intercept this massage. If between two nodes there is a direct link in the network, it is clear that they can communicate in a confidential way. However, if there is no direct link between the two vertices then the message exchanged must follow a path having in its set of internal nodes neither the specified node nor one of its neighbors. Usually, the topology of the communication network is modeled as a (simple, undirected) graph.

The above type of communication suggests a new type of graph connectivity (see Definition 4), which is interesting by itself and is closely related to some well known concepts in the theory of perfect graphs (a graph G is confidentially connected iff G does not have a star cutset ([21])).

In ([24]) is showed the importance of the notion cut-set in network interdependency security risk assessment: The invention is directed to providing threat and risk analysis for a network that has a high degree of interrelationships and interdependencies among the assets comprising it, using a "cut set" enumeration method. The identified cut sets are used as the basis to the threat and risk analysis, since each cut set may affect the traffic between two dependent assets in the network, and thereby affect the security state of the dependent assets themselves. The affected security state may be confidentiality, integrity, availability, or other network or security relevant parameter.

In ([15]) is showed the importance of the notion cutset in Wireless Networks.

In ([22]), are listed some metrics from the networking literature and some graph-theoretic metrics that have plausible networking interpretations of which we mention:
resilience, the size of a cut set for a balanced bipartition;
distortion, or the minimum communication cost spanning tree.

In ([12]), in social network analysis is showed the confidential connection between the actors ([18], [11]).

The structure of the paper is the following. In Section 2 we present a characterization of the unbreakable graphs and prove that the Normal Graph Conjecture is true for the minimal unbreakable graphs.

2. The Normal Graph Conjecture for minimal unbreakable graphs

In this section we recall that the class of (α, ω)-partitionable class is a subclass of those without star cutset, give a new characterization of the unbreakable graphs and show that the Normal Graph Conjecture is true for the minimal unbreakable graphs. We conclude that the minimal unbreakable graphs are easy to recognize and the stability number, clique number, chromatic number and the minimum clique covering number are also easy to determine.

Definition 1. ([5]) *A graph $G = (V, E)$ is called unbreakable if it has at least three vertices and neither G nor $\overline{G}$ has a star cutset. The subset $A \subset V$ is called a cutset if $G - A$ is not connected. If, in addition, some $v \in A$ is adjacent to every vertex in $A - \{v\}$, then A is called a star cutset and v is called the center of A.*

Theorem 1. ([13]) *In an unbreakable graph, every vertex belongs either to some C_k or to $\overline{C}_k$,* where $k \geq 5$.

Definition 2. *A graph $G = (V, E)$ with at least three vertices is confidentially connected if for any three distinct vertices $v, x, y \in V$, there exists a path P_{xy} in G such that $N_G[v] \cap V(P_{xy}) \subseteq \{x, y\}$.*

Recall, a graph G is confidentially connected iff G does not have a star cutset ([21]).

A graph G is called (α, ω)-partitionable if for every $v \in V(G)$, $G - v$ admits a partition in α ω-cliques and a partition of ω α-stable sets.

Star Cutset Lemma. ([5]) *No minimal imperfect graph has a star cutset.*

In ([21]) we proved that every (α, ω)-partitionable graphs is confidentially connected. Thus, a graph G is unbreakable iff G and $\overline{G}$ are confidentially connected. This means that an important class of graphs, as the one of (α, ω)-partitionable graphs, is included in that of confidentially connected graphs.

We give, in Theorem 2 below, a necessary and sufficient condition for a connected and non-complete graph to be unbreakable. A similar result is stated in ([8]) and proved in ([9]) and ([21]), but for confidentially connected graphs.

Theorem 2. *A connected and non-complete graph $G = (V, E)$ is unbreakable if and only if $\{\overline{N}_G(v) | v \in V\}$ is the family of the weakly components of G, while $\{\overline{N}_{\overline{G}}(v) | v \in V\}$ is the family of the weakly components of $\overline{G}$.*

Theorem 2 provides the following recognition algorithm for unbreakable graphs.

Input: A connected non-complete graph $G = (V, E)$.
Output: An answer to the question: "Is G unbreakable" ?
begin

1. Generate L_G, the family of the weakly components of G as follows:

$L_G \leftarrow \emptyset$
while $V \neq \emptyset$ *do*
determine the weakly component A with the weakly decomposition algorithm
$L \leftarrow L \cup \{A\}$
$V \leftarrow V - A$

Generate L'_G, the family of the weakly components of $\overline{G}$

2. Determine $\overline{N}_G(v)$, $\forall v \in V$
Determine $\overline{N}_{\overline{G}}(v)$, $\forall v \in V$
3. *If* $\exists A \in L_G$ such that $A \neq \overline{N}_G(v)$, $\forall v \in V$
then Return: "G is not unbreakable"

else

if $\exists B \in L_G^{'}$ such as $B \neq \overline{N}_{\overline{G}}(v)$, $\forall v \in V$

then Return: "G is not unbreakable"

else Return: "G is unbreakable"

end.

As in [21] we give an $O(n+m)$ algorithm for building a weakly decomposition for a non-complete and connected graph, it follows that step 1 of the algorithm above is $O(n \cdot (n+m))$. Because steps 2 and 3 perform in smaller time, it follows that the complexity of the recognition algorithm for unbreakable graphs is $O(n \cdot (n+m))$.

Definition 3. *A graph G is called normal if G admits a clique cover $\mathcal{C}$ and a stable set cover $\mathcal{S}$ such that every clique in $\mathcal{C}$ intersects every stable set in $\mathcal{S}$.*

Normal graphs form an superclass of perfect graphs and can be seen as closure of perfect graphs by means of co-normal products ([16]) and graph entropy ([10]). Perfect graphs have been characterized as those graphs without odd holes and odd antiholes (**Strong Perfect Graph Theory** ([4])). By analogy, Korner and de Simone stated that the non-existence of the three smallest odd holes and odd antiholes implies normality ([17]) and formulated the following conjecture.

Normal Graph Conjecture. Graphs without any C_5, C_7 or $\overline{C}_7$ as induced subgraphs are normal.

Proving or disproving the Normal Graph Conjecture is subject of investigations.

Wagler proved that Normal Graph Conjecture is true for the circulant graphs ([24]).

In this paper we show that Normal Graph Conjecture is true for the minimal unbreakable graphs.

We denote:

$$P(G,v) : \begin{cases} i)\overline{G}(N_G(v))\, and\, G(N_{\overline{G}}(v))\, are\, connected\, graphs\, , \forall v \in V(G) \\ ii) N_G(N_{\overline{G}}(v)) = N_G(v)\, si\, N_{\overline{G}}(N_G(v)) = N_{\overline{G}}(v), \forall v \in V(G). \end{cases}$$

Corollary 1. *A graph G is unbreakable if and only if $P(G,v)$ holds for every $v \in V(G)$.*

Definition 4. *A graph G is called minimal unbreakable if G is unbreakable and none of its proper induced subgraphs is unbreakable.*

Remark 1. *G is minimally unbreakable if and only if $\overline{G}$ is minimally unbreakable.*

Remark 2. *If a graph G is normal then its complement $\overline{G}$ is a normal graph.*

Theorem 3. *If G is a minimal unbreakable graph with at least eight vertices then G is a normal graph.*

Proof. Let G be a minimal unbreakable graph. We show that G is C_k or $\overline{C}_k$ for some $k \geq 5$. Let $v \in V(G)$ and $\mathcal{H} = \{H|H$ is a subgraph induced of G, containing v such that $P(H, v)$ holds $\}$. Then $\mathcal{H} \neq \emptyset$, because $G \in \mathcal{H}$, from the hypothesis and from Theorem 4. Let H be a minimal member of $\mathcal{H}$. Let $A = \{a|a \in N_{\overline{H}}(v),\ H(N_{\overline{H}}(v)) - a$ is connected $\}$.

We have that $|N_{\overline{H}}(v)| \geq 2$. Indeed, for every $y_1 \in N_{\overline{H}}(v)$ it follows that there exists $x_1 \in N_H(v)$ such that $x_1y_1 \notin E(H)$ (because $N_{\overline{H}}(N_H(v)) = N_{\overline{H}}(v)$, according to *ii*) in $P(H, v)$). For $x_1 \in N_H(v)$, it follows that there exists $y_2 \in N_{\overline{H}}(v)$ such that $x_1y_2 \in E(H)$ (because $N_H(N_{\overline{H}}(v)) = N_H(v)$, according to *ii*) in $P(H, v)$). Obviously, $y_2 \neq y_1$ because $y_1x_1 \notin E(H)$ and $y_2x_1 \in E(H)$. Therefore, we have that $|N_{\overline{H}}(v) \geq 2$.

We conclude that A has at least two vertices.

For every $a \in A$ there exists $b \in N_H(v)$ such that a is the unique neighbour of b in A. Indeed, at first we prove that for every $a \in A$ there exists at least a $b \in N_H(v)$ such that $ab \in E(H)$. If an $a \in A$ would exist such that for every $b \in N_H(v)$ we would have $ab \notin E(H)$, then $\{a\} \not\sim N_H[v]$. So, for every $b \in N_H(v)$ there exists $y \in N_{\overline{H}}(v) - \{a\}$ such that $by \in E(H)$, which means that *ii*) is fulfilled for $N_{\overline{H}}(v) - \{a\}$. As *i*) is also fulfilled for $N_{\overline{H}}(v) - \{a\}$, it follows that $P(H - \{a\}, v)$ holds, contradicting the minimality of H.

Now we prove that for every $a \in A$ no more than a vertex exists in $N_H(v)$, such that a is its neighbour in A. If we suppose that $a', a" \in A$ exists ($a' \neq a"$) with $ba', ba" \in E(H)$ then $H' = H - \{a'\}$ contains v, *i*) is fulfilled for H' and v (because $H'(N_{\overline{H'}}(v))$ is connected), *ii*) is also fulfilled for H' and v (because for every neighbour s of v, including b, there exists a non-neighbour t of v ($a"$ for b), such that $st \in E(H')$). Then $H' \in \mathcal{H}$, contradicting the minimality of H.

Similarly, let $B = \{b|b \in N_H(v),\ \overline{H}(N_H(v)) - b$ is connected $\}$.

As above, $|B| \geq 2$ holds.

If $|A| \geq 3$ then for every $b \in B$, $H" = H - \{b\} \in \mathcal{H}$ holds, contradicting the minimality of H. Indeed, because $\overline{H}(N_H(v)) - \{b\}$ is connected, *i*) follows for $H"$ and v. Because $H \in \mathcal{H}$, it follows that

$N_{\overline{H}}(v) \subseteq N_{\overline{H}}(N_H(v))$. We prove that $N_{\overline{H"}}(v) \subseteq N_{\overline{H"}}(N_H"(v))$. For every $x \in A$ there exists $y \in N_H(v)$ such that x is the only neighbour of y in A. If for some $t_b \in A$, b it self has t_b as unique neighbour in A then, as $|A| \geq 3$, there exists $t' \in A$ non-adjacent to b. But in this case there exists $b' \in N_H(v)$ such that t' is the only neighbour of b' in A, so $b't_b \notin E(H)$. We conclude that for this vertex t_b in $N_{\overline{H}}(v)$, there exists a neighbour b' of v such that t_b and b' are not-adjacent, as well. As $N_H(v) \subseteq N_H(N_{\overline{H}}(v))$ we have that $N_H"(v) \subseteq N_H"(N_{\overline{H"}}(v))$, because $N_H"(v) = N_H(v) - \{b\}$ and $N_{\overline{H"}}(v) = N_{\overline{H}}(v)$. So $P(H", v)$ holds, contradicting the minimality of H.

Similarly, if $|B| \geq 3$ then for every $a \in A$, $H - \{a\} \in \mathcal{H}$ holds, contradicting the minimality of H.

Therefore, $|A| = |B| = 2$, which means that $H(N_{\overline{H}}(v))$ and $\overline{H}(N(H(v))$ are isomorphic to P_k and P_l respectively, with $k, l \geq 2$. Let $P_k = [a_1, \ldots, a_k]$, $P_l = [b_1, \ldots, b_l]$, such that a_1 is adjacent only to b_1, a_k is adjacent only to b_l. If $k \geq 3$ and $l \geq 3$ simultaneously, then $C = [b_1, a_1, \ldots, a_k, b_l]$ is a cycle induced in G, with at least five vertices, which means it is unbreakable and because $v \notin C$ we have that G is not minimal unbreakable. We conclude that either $l = 2$ or $k = 2$. For $l = 2$ it follows that $G = C_{k+3} = [v, b_1, a_1, \ldots, a_k, b_l]$. For $k = 2$ it follows that $G = C_{l+3}$.

If $G = C_{2p}$ then, taking $S = \{\{1, 3, \ldots, 2p-1\}, \{2, 4, \ldots, 2p\}\}$, a covering with stable sets and $C = \{\{1, 2\}, \{3, 4\}, \ldots, \{2p-1, 2p\}\}$, a clique covering, we conclude that C_{2p} is a normal graph.

It is well known that $G = C_{2p+1}$ is a normal graph for $p \geq 4$ ([17]). It follows that C_s $(s \geq 8)$ is a normal graph. As $\overline{C}_s$ $(s \geq 8)$ is a normal graph, it follows that G is a normal graph.

The fact that holes and antiholes are normal graphs comes from C_n^1 is a hole and from ([24]), where Wagler proves that Normal Graph Conjecture is true for circulants and their complements.

From Theorem 5 and the fact that C_k and $\overline{C}_k$ (for some $k \geq 5$) are minimal unbreakable we obtain the following result.

Corollary 2. *G is minimal unbreakable if and only if G is C_k or $\overline{C_k}$ for some $k \geq 5$.*

With respect to the hole-free graphs, we recall that the problem of testing if a graph contains an even hole can be solved in polynomial time. There are two known algorithms. One is due to Conforti, Cornuejols, Kapoor, and Vuskovic ([6]), and the other to Chudnovsky, Kawarabayshi, and Seymour ([3]). In ([2]) is given a polynomial time algorithm for testing if a graph is Berge (and therefore perfect). Another polynomial algorithm for recognizing perfect graphs is given in ([7]). An interesting result was stated by Olariu, that proves that the strong conjecture of perfect graphs is true for a class of graphs that is described through forbidden configurations ([20]). Among the heuristic algorithms we mention ([19]).

Consequence 1. *If G is minimal unbreakable then $\alpha(G)$, $\omega(G)$, $\chi(G)$ and $\theta(G)$ are easy to determine. These graphs are also easy to recognize, namely when the degree of every edge is 2 or $n-2$.*

3. Conclusions and future work

In this paper we proved that Normal Graph Conjecture is true for minimal unbreakable graphs. Our future work concerns the verification of Normal Graph Conjecture for other classes of graphs, such as O-graphs, which is a particular class of (α, ω)-partitionable graphs.

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