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WELL-POSEDNESS OF A FIXED POINT PROBLEM USING G-FUNCTIONS

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Abstract. We study the well-posedness of the fixed point problem for asymptotically regular self-mappings of a metric space (X, d) which satisfy a contractive condition (see inequality (2.1)) defined by a G-type function (see [5]). So, in particular, our result provides some improvements to a result of [5].

1. Introduction

In 1974, Ćirić ([3]) has first introduced orbitally continuous mappings and orbitally complete metric spaces.

Definition 1.1. *Let T be a self-mapping on a metric space (X, d). If for any $x \in X$, every Cauchy sequence of the orbit $O_T(x) := \{x, Tx, T^2x, \ldots\}$ is convergent in X, then the metric space is said to be $T-$orbitally complete.*

Remark 1. *Every complete metric space is $T-$orbitally complete for any T. An orbitally complete space may not be complete metric space (see* [6]*, Example and* [14]*, Example 1).*

Browder and Petryshyn (see [2]) defined the following notion.

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Definition 1.2. *A selfmapping T on a metric space (X, d) is said to be asymptotically regular at a point x in X, if*

$$\lim_{n\to\infty} d(T^n x, T^n T x) = 0, \tag{1.1}$$

where $T^n x$ denotes the n-th iterate of T at x.

In [5], the following class of functions was introduced.

Definition 1.3. *A function $g : [0,\infty)^5 \to [0,\infty)$ is called a G-function, if it satisfies the following three condition:*

(i) g is continuous.

(ii) g is nondecreasing in each variable.

(iii) If $h(r) = g(r, r, r, r, r)$, then the function $r \to r - h(r)$ is strictly increasing and positive in $(0,\infty)$.

Examples of G-type functions are given in [5].

By using these functions, the following result was proved in [5].

Theorem 1.1. *([5]). Let (X, d) be a metric space and $A : X \to X$ be a self-mapping satisfying the inequality*

$$d(Ax, Ay) \le g(d(x, y), d(Ax, x), d(Ay, y), d(Ax, y), k.d(Ay, x)) \tag{1.2}$$

where g belongs to the class of G-type functions and $0 < k \le \frac{1}{2}$.

Then for any $x \in X$, the sequence $\{A^n x\}$ is such that

$$\lim_{n\to\infty} d(A^n x, A^{n+1} x) = 0. \tag{1.3}$$

Further, if $\{A^n x\}$ is convergent then it converges to the unique fixed point of A. Also in that case any other sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} d(x_n, Ax_n) = 0 \tag{1.4}$$

will also converge to the unique fixed point of A.

The aim of this paper is to study the well-posedness (see Definition 1.4 below) of the fixed point problem for a self-mapping T of a metric space (X, d) which satisfies the contractive condition (1.2).

The notion of well-posednes of a fixed point problem has evoked much interest to a several mathematicians, for examples, F.S. De Blassi and J. Myjak (see [1]), S. Reich and A. J. Zaslavski (see [12]), B.K. Lahiri and P. Das (see [6]) and V. Popa (see [10] and [11]).

Definition 1.4. *Let (X,d) be a metric space and $T:(X,d)\to(X,d)$ a mapping. The fixed point problem of T is said to be well posed if:*

(a) T has a unique fixed point z in X;

(b) for any sequence $\{x_n\}$ of points in X such that $\lim_{n\to\infty} d(Tx_n,x_n)=0$, we have $\lim_{n\to\infty} d(x_n,z)=0$.

2. Main result

The main result of this paper is the following.

Theorem 2.1. *Let (X,d) be a metric space and $T:X\to X$ be a self-mapping satisfying the inequality*

$$d(Tx,Ty)\le g\left(d(x,y),d(Tx,x),d(Ty,y),d(Tx,y),kd(Ty,x)\right) \quad (2.1)$$

for all $x,y\in X$, where g belongs to the class of G-type functions and k is a given number in $(0,\frac{1}{2}]$.

Suppose that (X,d) is $T-$orbitally complete. Then T has a unique fixed point z in X and the fixed point problem of T is well-posed. Moreover, T is continuous at its unique fixed point.

Proof. 1) Let x_0 be a point of X. Then according to Theorem 1.1, T is asymptotically regular at x_0. We show that $\{x_n\}$ is a Cauchy sequence, where $x_n=T^nx_0$. To simplify notations, we denote

$$d_n := d(x_n,x_{n+1}). \quad (2.2)$$

Let ϵ be a given positive real number. We choose a real number δ such that

$$0<\delta<\frac{\epsilon-h(\epsilon)}{3}, \quad (2.3)$$

where (as before) $h(t)=g(t,t,t,t,t)$ for every $t\in[0,\infty)$. Since $\lim_{n\to\infty} d_n=0$, then there exists an positive integer N_δ such that

$$\max\{d_n,d_m\}<\delta, \quad \text{for all integers } n,m\ge N_\delta. \quad (2.4)$$

Using the triangle inequality, from (2.1) and (ii) of Definition 1.3, we have

$$\begin{aligned} d(x_n,x_m) &\le d_n + d(Tx_n,Tx_m)+d_m \\ &\le d_n+d_m+g(d(x_n,x_m),d_n,d_m,d(Tx_n,x_m),kd(Tx_m,x_n)) \\ &\le d_n+d_m+g(d(x_n,x_m),d_n,d_m,d(Tx_n,x_m),d(Tx_m,x_n)). \end{aligned}$$

Using (2.4) and the fact that g is non-decreasing in each variable, we have

$$d(x_n,x_m)\le 2\delta+g(d(x_n,x_m),\delta,\delta,\delta+d(x_n,x_m),\delta+d(x_n,x_m))$$

$$\leq 2\delta + h(\delta + d(x_n, x_m)).$$

We deduce that

$$d(x_n, x_m) + \delta \leq 3\delta + h(\delta + d(x_n, x_m)),$$

which implies (by using (2.4)) that

$$(d(x_n, x_m) + \delta) - h(\delta + d(x_n, x_m)) \leq 3\delta \leq \epsilon - h(\epsilon). \tag{2.5}$$

Since the function $t \mapsto t - h(t)$ is strictly increasing and positive in $(0, \infty)$ the inequality (2.5) implies

$$d(x_n, x_m) + \delta < \epsilon, \qquad \text{for all integers } n, m \geq N_\delta. \tag{2.6}$$

From (2.6), we deduce that the sequence $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a $T-$orbitally complete metric space, there is some z in X such that

$$\lim_{n\to\infty} x_n = z. \tag{2.7}$$

2) Now we show that z is a fixed point of T. Sppose that $d(z, Tz) > 0$. From (2.1) we have

$$d(Tz, x_{n+1}) = d(Tz, Tx_n) \leq g\left(d(z, x_n), d(Tz, z), d(x_{n+1}, x_n), d(Tz, x_n), d(x_{n+1}, z)\right). \tag{2.8}$$

Making $n \to \infty$ and noting that g is continuous, we obtain from (2.8) that

$$d(Tz, z) \leq g\left(0, d(Tz, z), 0, d(Tz, z), 0\right) \leq h(d(Tz, z))$$

which implies that

$$d(Tz, z) - h(d(Tz, z)) \leq 0.$$

This is a contradiction with the property (iii) of h in Definition 1.3. It follows that $d(Tz, z) = 0$, or equivalently, that z is a fixed point of T.

3) To prove the uniqueness of z, let us suppose that u and v are two different fixed points of T. From (2.1), we have

$$\begin{aligned} d(u, v) = d(Tu, Tv) &\leq g\left(d(u, v), d(Tu, u), d(Tv, v), d(Tu, v), d(Tv, u)\right) \\ &= g\left(d(u, v), 0, 0, d(u, v), d(u, v)\right) \leq h(d(u, v)) \end{aligned}$$

which implies that

$$d(u, v) - h(d(u, v)) \leq 0.$$

Using the property (iii) of h in Definition 1.3, it follows that

$$d(u,v)=0,$$

or equivalently, that $u=v$ which is a contradiction. Thus z is the unique fixed point of T.

4) Let $\{y_n\}$ be any arbitrary sequence of points in X such that

$$\lim_{n\to\infty} d(Ty_n,y_n)=0. \tag{2.9}$$

Let us show that the sequence $\{y_n\}$ converges to the unique fixed point z of T. Let $\epsilon>0$ be a given number. Choose a real number δ such that

$$0<\delta<\frac{\epsilon-h(\epsilon)}{2}. \tag{2.10}$$

where $h(t)=g(t,t,t,t,t)$. By assumption (2.9), we can find a nonnegative integer M_δ such that

$$\forall n\in\mathbb{N},\ n\geq M_\delta \Longrightarrow d(Ty_n,y_n)\leq\delta. \tag{2.11}$$

Using the triangle inequality, from (2.1) and the condition (ii) of Definition 1.3, we have

$$\begin{aligned} d(y_n,z) &\leq d(y_n,Ty_n)+d(Ty_n,Tz)\\ &\leq d(y_n,Ty_n)+g(d(y_n,z),d(y_n,Ty_n),0,d(Ty_n,z),d(z,y_n))\\ &\leq d(y_n,Ty_n)+g(d(y_n,z),d(y_n,Ty_n),0,d(Ty_n,y_n)+d(y_n,z),d(z,y_n)). \end{aligned}$$

Using (2.4) and the fact that g is non-decreasing in each variable, we have

$$\begin{aligned} d(y_n,z) &\leq \delta+g(d(y_n,z),\delta,\delta,d(z,y_n)+\delta,d(z,y_n))\\ &\leq \delta+h(\delta+d(z,y_n)). \end{aligned}$$

We deduce that

$$d(y_n,z)+\delta\leq 2\delta+h(\delta+d(z,y_n)),$$

which implies (by using the condition (2.10)) that

$$(d(z,y_n)+\delta)-h(\delta+d(z,y_n))\leq 2\delta\leq\epsilon-h(\epsilon). \tag{2.12}$$

Since the function $t\mapsto t-h(t)$ is strictly increasing and positive on the set $(0,\infty)$, then the inequality (2.12) implies

$$d(z,y_n)+\delta<\epsilon, \qquad \text{for all integers } n\geq M_\delta. \tag{2.13}$$

From (2.13), we deduce that the sequence $\{y_n\}$ converges to z. This proves that the fixed point problem of T is well-posed.

5) To prove that T is continuous at z, suppose that $z_n \to z = Tz$ and suppose that the sequence $\{Tz_n\}$ does not converge to $Tz = z$. Then we can find a positive number $\eta > 0$ and a subsequence $\{w_n\}$ of $\{z_n\}$ such that

$$d(Tw_n, z) \geq 2\eta, \quad \forall n \geq 0. \tag{2.14}$$

Since $\lim_{n\to\infty} d(w_n, z) = 0$, then we can find a positive integer N_η such that

$$n \geq N_\eta \Longrightarrow d(w_n, z) \leq \eta - h(\eta). \tag{2.15}$$

Then from (2.1), we have

$$\begin{aligned} d(Tw_n, z) = d(Tw_n, Tz) &\leq g\left(d(w_n, z), d(Tw_n, w_n), d(Tz, z), d(Tw_n, z), d(Tz, w_n)\right) \\ &= g\left(d(w_n, z), d(Tw_n, z) + d(z, w_n), 0, d(Tw_n, z), d(z, w_n)\right) \\ &\leq h(d(Tw_n, z) + d(z, w_n)) \end{aligned} \tag{2.16}$$

From (2.16), we obtain that

$$d(Tw_n, z) + d(z, w_n) - h(d(Tw_n, z) + d(z, w_n)) \leq d(z, w_n) \leq \eta - h(\eta). \tag{2.17}$$

Since $t - h(t)$ is strictly increasing on $(0, \infty)$, from (2.17), we obtain that

$$d(Tw_n, z) + d(z, w_n) \leq \eta, \quad \text{for all integer} \quad n \geq N_\delta. \tag{2.18}$$

making $n \to \infty$ in (2.18), we obtain that $2\eta \leq \eta$, which is a contradiction. We conclude that T is continuous at its unique fixed point z, and this ends the proof. □

As a consequence, we have the following result.

Theorem 2.2. *Let (X, d) be a metric space and $T : X \to X$ be a self-mapping satisfying the inequality*

$$d(Tx, Ty) \leq g\left(d(x, y), d(Tx, x), d(Ty, y), d(Tx, y), d(Ty, x)\right) \tag{2.19}$$

for all $x, y \in X$, where g belongs to the class of G-type functions.

Suppose that T is asymptotically regular at some $x_0 \in X$ and that (X, d) is $T-$orbitally complete. Then T has a unique fixed point z in X and the fixed point problem of T is well-posed. Moreover, T is continuous at its unique fixed point.

Proof. The result can be deduced from the proof given for Theorem 2.1. So we omit the details. □

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