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Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 20 (2010), No. 1, 61 - 82

SEMIVARIATION AND EXHAUSTIVITY OF SET MULTIFUNCTIONS

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Abstract. In this paper we study exhaustivity and the properties of semivariation for set multifunctions. An extension theorem by preserving the properties is obtained and several results concerning fuzzy set multifunctions are given.

1. Introduction

Exhaustivity is an important property in many problems of measure theory. For instance, it is well-known that for every exhaustive submeasure can be indicated a control measure (see [4]).

In recent years, in the context of fuzzy measures, people began to study exhaustivity and another important property, autocontinuity, which is more general than subadditivity. Properties like autocontinuity, exhaustivity, increasing convergence, decreasing convergence and o-continuity are very popular in the literature of non-additive measures. We mention here the contributions of Denneberg [2], Jiang and Suzuki [13,14], Jiang, Suzuki, Wang and Klir [12], Asahina, Uchino and Murofushi [1], Tan and Zhang [17], Pap [16], Zhang [20], Wang and Klir [18] and many others.

Keywords and phrases: exhaustive, semivariation, fuzzy set multifunction, uniformly autocontinuity, multisubmeasure.
(2000)Mathematics Subject Classification: 26E25, 28B20.

On the other hand, due to its numerous applications in Mathematics Economics, Theory of Control, Decision Theory and other fields, a set-valued fuzzy measures theory became to develop. Guo and Zhang [10], Zhang, Guo and Liu [21], Zhang and Wang [22] generalized different problems of fuzzy measures and integrals theory to the set-valued case. In three recent papers, Gavrilut [7], Gavriluţ and Croitoru [8,9] studied regularity, non-atomicity, Darboux property and other problems for fuzzy set multifunctions with respect to the Hausdorff topology.

In this paper we study exhaustivity and the properties of semivariation for set multifunctions taking values in the family of non-void, closed subsets of a real normed space, endowed with the Hausdorff pseudometric h. Several results concerning fuzzy set multifunctions are also obtained and an extension theorem by preserving the properties (exhaustivity, autocontinuity and increasing convergence) is established for monotone set multifunctions taking values in the family of non-void, closed, bounded subsets of a Banach space.

Let T be an abstract, nonvoid set, $\mathcal{C}$ a ring of subsets of T, X a real normed space, $\mathcal{P}_0(X)$ the family of all nonvoid subsets of $X, \mathcal{P}_f(X)$ the family of all nonvoid, closed subsets of X, $\mathcal{P}_{bf}(X)$ the family of all nonvoid, closed, bounded subsets of X and h the Hausdorff pseudometric on $\mathcal{P}_f(X)$, which becomes a metric on $\mathcal{P}_{bf}(X)$.

It is known that $h(M,N) = \max\{e(M,N), e(N,M)\}$, where
$e(M,N) = \sup\limits_{x\in M} d(x,N)$, for every $M, N \in \mathcal{P}_f(X)$ and
$d(x,N)$ is the distance from x to N induced by the norm of X.

We denote $|M| = h(M,\{0\})$, for every $M \in \mathcal{P}_f(X)$, where 0 is the origin of X. If X is complete, then the same is $\mathcal{P}_{bf}(X)$ (see [11], Theorem 1.6).

On $\mathcal{P}_0(X)$ we introduce the Minkowski addition " $\overset{\bullet}{+}$ ", defined by:

$$M \overset{\bullet}{+} N = \overline{M+N}, \text{ for every } M, N \in \mathcal{P}_0(X),$$

where $M+N = \{x+y; x\in M, y\in N\}$ and $\overline{M+N}$ is the closure of $M+N$ with respect to the topology induced by the norm of X.

First, we recall the following classical notions:

Definition 1.1. Let $m : \mathcal{C} \to \mathbb{R}_+$ be a set function. m is said to be:

I) *exhaustive* if $\lim\limits_{n\to\infty} m(A_n) = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

II) *increasing convergent* if $\lim\limits_{n\to\infty} m(A_n) = m(A)$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A$ (that is, $A_n \subset A_{n+1}$, for every $n \in \mathbb{N}^*$) and $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

III) *decreasing convergent* if $\lim\limits_{n\to\infty} m(A_n) = m(A)$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow A$ (that is, $A_n \supset A_{n+1}$, for every $n \in \mathbb{N}^*$) and $\bigcap\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

IV) *monotone* if $m(A) \leq m(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$.

V) *fuzzy* if $m(\emptyset) = 0$ and m is monotone, increasing convergent and decreasing convergent.

VI) a *submeasure* (in the sense of Drewnowski [3]) if $m(\emptyset) = 0$, m is monotone and *subadditive*, that is, $m(A \cup B) \leq m(A) + m(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

VII) *uniformly autocontinuous* if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that for every $A, B \in \mathcal{C}$, with $m(B) < \delta$, we have $m(A\cup B) < m(A)+\varepsilon$.

We shall need the following notions in the set valued case:

Definition 1.2. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$.

I) We call the *total variation of* μ, the real extended valued set function $\overline{\mu}$ defined by:

$$\overline{\mu}(A) = \sup\{\sum_{i=1}^{n} |\mu(A_i)|\}, \text{ for every } A \subset T,$$

where supremum is extended over all finite families $(A_i)_{i=\overline{1,n}}$ of pairwise disjoint subsets of $\mathcal{C}$, contained in A.

II) We say that μ is of *finite variation* if $\overline{\mu}(A) < \infty$, for every $A \subset T$.

III) We call the *semivariation of* μ, the real extended valued set function $\widehat{\mu}$ defined by:

$$\widehat{\mu}(A) = \sup\{|\mu(B)|; B \subset A, B \in \mathcal{C}\}, \text{for every } A \subset T.$$

By $|\mu|$ we mean the real extended valued set function defined by $|\mu|(A) = |\mu(A)|$, for every $A \in \mathcal{C}$.

Definition 1.3. A set multifunction $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is said to be:

I) a *multisubmeasure* ([5], [6]) if

a) $\mu(\emptyset) = \{0\}$,

b) μ is *monotone* (ie. $\mu(A) \subseteq \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \subseteq B$) and

c) μ is *subadditive* (ie. $\mu(A \cup B) \subseteq \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$)

(or, equivalently, for every $A, B \in \mathcal{C}$);

II) a *multimeasure* if $\mu(A \cup B) = \mu(A) \overset{\bullet}{+} \mu(B)$, for every $A, B \in \mathcal{C}$, with $A \cap B = \emptyset$.

Definition 1.4. Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction. μ is said to be:

I) *increasing convergent* (with respect to h) if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$.

II) *decreasing convergent* (with respect to h) if $\lim\limits_{n\to\infty} h(\mu(A_n), \mu(A)) = 0$, for every decreasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow A \in \mathcal{C}$.

III) *fuzzy* if it is monotone, increasing convergent, decreasing convergent and $\mu(\emptyset) = \{0\}$.

IV) *exhaustive* (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$.

V) *uniformly autocontinuous* if for every $\varepsilon > 0$, there is $\delta(\varepsilon) > 0$ so that for every $A, B \in \mathcal{C}$, with $|\mu(B)| < \delta$, we have $h(\mu(A \cup B), \mu(A)) < \varepsilon$.

VI) *o-continuous* (with respect to h) if $\lim\limits_{n\to\infty} |\mu(A_n)| = 0$, for every sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$.

VII) *$h-\sigma$-subadditive* if $|\mu(\bigcup\limits_{n=1}^{\infty} A_n)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{C}$.

Remark 1.5. i) If μ is a multisubmeasure, then μ is uniformly autocontinuous. Indeed, if $A, B \in \mathcal{C}$, then $e(\mu(A), \mu(A \cup B)) = 0$ and $e(\mu(A \cup B), \mu(A)) \leq |\mu(B)|$, hence the statement follows.

There are uniformly autocontinuous set multifunctions, which are not multisubmeasures:

Let $\mathcal{C} = \mathcal{P}(\mathbb{N})$ and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined for every $A \subset \mathbb{N}$ by

$$\mu(A) = \begin{cases} \{0\}, \text{ if } A \text{ is finite} \\ [1, \infty), \text{ if } A \text{ is countable infinite.} \end{cases}$$

Then μ is uniformly autocontinuous and it is not a multisubmeasure.

ii) If μ is uniformly autocontinuous, then for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ so that for every $A, B \in \mathcal{C}$, with $|\mu(A\Delta B)| < \delta$, we have $h(\mu(A), \mu(B)) < \varepsilon$.

Indeed, by the uniformly autocontinuity of μ, we get that for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ so that for every $A, B \in \mathcal{C}$, with $|\mu(A\Delta B)| < \delta$, we have $h(\mu(A), \mu(A \cup (A\Delta B))) = h(\mu(A), \mu(A \cup B)) < \frac{\varepsilon}{2}$. Analogously, $h(\mu(B), \mu(A \cup B)) < \frac{\varepsilon}{2}$. This yields $h(\mu(A), \mu(B)) < \varepsilon$, as claimed.

iii) Definitions 1.4 I)-VII) generalize the classical ones mentioned in Definition 1.1. Indeed, one can easily check that, if $m : \mathcal{C} \to \mathbb{R}_+$ is a set function and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ is defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$, then μ is fuzzy (respectively, increasing convergent, decreasing convergent, exhaustive, o-continuous, monotone uniformly autocontinuous) if and only if the same is m. So, in this direction, our definitions generalize those well-known from the classical case.

Also, we observe that if $m : \mathcal{C} \to \mathbb{R}_+$ is a set function and $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$ is defined by $\mu(A) = \{m(A)\}$, for every $A \in \mathcal{C}$, then μ is increasing convergent, decreasing convergent, exhaustive, o-continuous if and only if the same is m. Let us note that the monotonicity of μ implies that $\mu(A) = \{0\}$, for every $A \in \mathcal{C}$.

The statements follow since $|\mu(A)| = m(A)$, for every $A \in \mathcal{C}$ and $h([0, a], [0, b]) = |a - b|$, for every $a, b \in \mathbb{R}_+$.

iv) If $\mathcal{C}$ is finite, then any set multifunction, with $\mu(\emptyset) = \{0\}$ is increasing convergent, decreasing convergent, exhaustive and o-continuous.

v) If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is a multisubmeasure, then $|\mu|$ is a submeasure in the sense of Drewnowski [3], $\overline{\mu}$ is finitely additive on $\mathcal{C}$ and $\overline{\mu}(A) \geq |\mu(A)|$, for every $A \in \mathcal{C}$.

For the special case of uniformly autocontinuous set multifunctions, there are some immediate implications among increasing convergence, o-continuity and decreasing convergence:

Theorem 1.6. *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be an uniformly autocontinuous set multifunction, with $\mu(\emptyset) = \{0\}$. Then:*

i) If μ is o-continuous, then μ is increasing convergent;

ii) μ is o-continuous if and only if it is decreasing convergent.

iii) If μ is monotone, then μ is o-continuous if and only if it is fuzzy.

Proof. i) Let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A \in \mathcal{C}$. Then $A\backslash A_n \searrow \emptyset$, so, by the o-continuity of μ, $\lim\limits_{n\to\infty} |\mu(\ A\Delta A_n)| = \lim\limits_{n\to\infty} |\mu(\ A\backslash A_n)| = 0$. According to Remark 1.5 ii), $\lim\limits_{n\to\infty} h(\mu(A), \mu(A_n)) = 0$, so μ is increasing convergent.

ii) The *if part* follows as in i).

The *only if part* is an immediate consequence of definitions.

iii) The statement easily follows by definitions, i) and ii).

2. Exhaustive fuzzy and non-fuzzy set multifunctions

In this section we establish different results concerning exhaustive set multifunctions. We also generalize several known results from single-valued fuzzy measures theory.

Theorem 2.1. *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction, with $\mu(\emptyset) = \{0\}$. Then:*

i) μ is exhaustive if and only if every monotone sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ is Cauchy with respect to μ, that is, $\lim\limits_{\substack{n\to\infty\\ m\to\infty}} |\mu(A_n\Delta A_m)| = 0$.

ii) If μ is exhaustive and uniformly autocontinuous, then $\lim\limits_{\substack{n\to\infty\\ m\to\infty}} h(\mu(A_n), \mu(A_m)) = 0$.

Proof. i) For the *if part*, suppose without any loss of generality that $(A_n)_{n\in\mathbb{N}}$ is increasing. Let us suppose, by the contrary, that it is not a Cauchy one. Then there exist $\varepsilon_0 > 0$ and an increasing sequence $(n_k)_k \subset \mathbb{N}^*$ so that $|\mu(A_{n_k}\Delta A_{n_{k+1}})| \geq \varepsilon_0$, for every $k \in \mathbb{N}^*$.

Let $B_{n_k} = A_{n_{k+1}}\backslash A_{n_k}$, for every $k \in \mathbb{N}^*$. Then

$$|\mu(B_{n_k})| = |\mu(A_{n_k}\Delta A_{n_{k+1}})| \geq \varepsilon_0, \text{ for every } k \in \mathbb{N}^*,$$

which is false because B_{n_k} are all pairwise disjoint and μ is exhaustive.

For the *only if part*, let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ be pairwise disjoint and consider $B_n = \overset{n}{\underset{i=1}{\cup}} A_i$, for every $n \in \mathbb{N}^*$. Then $(B_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ is increasing, so

$$\lim_{n\to\infty} |\mu(A_n)| = \lim_{n\to\infty} |\mu(B_{n+1}\backslash B_n)| = \lim_{n\to\infty} |\mu(B_{n+1}\Delta B_n)| = 0,$$

as claimed.

ii) The conclusion immediately follows according to Remark 1.5 ii).

By Remark 1.5 iii), Theorem 2.1 i) generalizes Proposition 1 of [12].

In what follows, we generalize Proposition 3 of [13]. Note that this proposition also appears in a different form in [17], Proposition 2.1.

Theorem 2.2. *If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is an exhaustive increasing convergent set multifunction, then μ is o-continuous.*

Proof. Suppose, by the contrary, that there exist $\varepsilon_0 > 0$ and $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \searrow \emptyset$ and $|\mu(A_n)| > \varepsilon_0$, for every $n \in \mathbb{N}^*$. Since for every $k \in \mathbb{N}^*$ arbitrary, but fixed, $A_k\backslash A_n \nearrow A_k$ and μ is increasing convergent, then $\lim\limits_{n\to\infty} h(\mu(A_1\backslash A_n), \mu(A_1)) = 0$. Also, $|\mu(A_1)| > \varepsilon_0$.

Since $h(M, N) \geq ||M| - |N||$, for every $M, N \in \mathcal{P}_f(X)$, then

$$h(\mu(A_1\backslash A_n), \mu(A_1)) \geq ||\mu(A_1)| - |\mu(A_1\backslash A_n)||, \text{ for every } n \in \mathbb{N}^*,$$

hence

$$\lim_{n\to\infty} |\mu(A_1\backslash A_n)| \geq |\mu(A_1)| > \varepsilon_0.$$

Then there is $n_1 \in \mathbb{N}^*$ so that $|\mu(A_1\backslash A_{n_1})| > \varepsilon_0$.

The same as before, there exists $n_2 > n_1$ such that $|\mu(A_{n_1}\backslash A_{n_2})| > \varepsilon_0$. Continuing this way, we find an increasing sequence $(n_k)_k \subset \mathbb{N}^*$ so that $|\mu(A_{n_k}\backslash A_{n_{k+1}})| > \varepsilon_0$, a contradiction, because μ is exhaustive.

A converse of the above theorem is valid for monotone set multifunctions:

Theorem 2.3. *Let $\mathcal{C}$ be a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ an o-continuous monotone set multifunction. Then μ is exhaustive.*

Proof. Let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ be pairwise disjoint and consider $B_n = \overset{\infty}{\underset{k=n}{\cup}} A_k$, for every $n \in \mathbb{N}^*$. Then $(B_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ and $B_n \searrow \emptyset$, so, by the o-continuity of μ, $\lim_{n\to\infty}|\mu(B_n)| = 0$. Since $A_n \subset B_n$, for every $n \in \mathbb{N}^*$, then $\lim_{n\to\infty}|\mu(A_n)| = 0$, so μ is exhaustive.

Note that a theorem of this type appears in fuzzy measures theory in different forms: Proposition 3 of [12], Proposition 1 of [13] (cited as Proposition 2 of [19]). It is also known for submeasures (see [3]).

Corollary 2.4. Let $\mathcal{C}$ be a σ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a monotone increasing convergent set multifunction. Then μ is exhaustive if and only if it is o-continuous.

In what follows, we give some sufficient conditions for the exhaustivity of a set multifunction:

Theorem 2.5. *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a set multifunction.*
i) *If μ is h-σ-subadditive and of finite variation, then μ is exhaustive.*
ii) *If $\mathcal{C}$ is the σ-ring generated by a δ-ring $\mathcal{C}_1$ and if μ is an increasing convergent multisubmeasure of finite variation, then μ is exhaustive.*

Proof. i) Let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ be pairwise disjoint. By the definition of h-σ-subadditivity, since μ is of finite variation, then $\lim_{n\to\infty}|\mu(A_n)| = 0$, so μ is exhaustive.
ii) It is suficient to prove that μ is h-σ-subadditive. Let $\varepsilon > 0$ and $(A_n)_n \subset \mathcal{C}$ pairwise disjoint so that $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{C}$.

Since $(A_n)_n \subset \mathcal{C}$ and $\mathcal{C}$ is the σ-ring generated by a δ-ring $\mathcal{C}_1$, then for every $n \in \mathbb{N}^*$ there exists $(B_k^n)_k \subset \mathcal{C}_1$ so that $B_k^n \nearrow A_n$. So, $A_n = \bigcup_{k=1}^{\infty} B_k^n$, for every $n \in \mathbb{N}^*$.

Denote $C_k = \bigcup_{n=1}^{\infty} B_k^n$, for every $k \in \mathbb{N}^*$. Obviously, $(C_k)_k \subset \mathcal{C}$ and $C_k \nearrow A$, so there is $k_0(\varepsilon) \in \mathbb{N}^*$ such that $h(\mu(C_k), \mu(A)) < \frac{\varepsilon}{2}$, for every $k \geq k_0$.

Because $\bigcup_{i=1}^{n} B_{k_0}^i \nearrow C_{k_0}$ and μ is increasing convergent, there exists $n_0(\varepsilon) \in \mathbb{N}^*$ so that $h(\mu(C_{k_0}), \mu(\bigcup_{i=1}^{n} B_{k_0}^i)) < \frac{\varepsilon}{2}$, for every $n \geq n_0$.

On the other hand, for every $n \geq n_0$,

$$|\mu(\overset{n}{\underset{i=1}{\cup}} B^i_{k_0})| \leq \sum_{i=1}^{n} |\mu(B^i_{k_0})| \leq \sum_{n=1}^{\infty} |\mu(B^n_{k_0})| \leq \sum_{n=1}^{\infty} |\mu(A_n)|.$$

Then, for every $n \geq n_0$,

$$|\mu(C_{k_0})| \leq h(\mu(C_{k_0}), \mu(\overset{n}{\underset{i=1}{\cup}} B^i_{k_0})) + |\mu(\overset{n}{\underset{i=1}{\cup}} B^i_{k_0})| < \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} |\mu(A_n)|.$$

Consequently,

$$|\mu(A)| \leq h(\mu(C_{k_0}), \mu(A)) + |\mu(C_{k_0})| < \sum_{n=1}^{\infty} |\mu(A_n)| + \varepsilon,$$

for every $\varepsilon > 0$, so $|\mu(A)| \leq \sum\limits_{n=1}^{\infty} |\mu(A_n)|$. By i), the conclusion follows.

We note that, generally, not even for a multisubmeasure, exhaustivity does not imply increasing convergence and the converse is also not valid, as we observe from the following examples:

Example 2.6. I) There are increasing convergent multisubmeasures which are not exhaustive:

I) Let $\mathcal{C}$ =$\{A \subset \mathbb{R},\ A$ is finite$\}$ be a ring of subsets of $T = \mathbb{R}$ and $m : \mathcal{C} \to \mathbb{R}_+$ be the set function defined for every $A \in \mathcal{C}$ by:

$$m(A) = \begin{cases} 0, & A = \emptyset \\ 1 + cardA, & A \neq \emptyset,\ A \subset \mathbb{R},\ A \text{ finite} \end{cases}$$

(where $card\, A$ represents the number of elements of A).

One can easily check that the multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined by $\mu(A) = [0, m(A)]$, for every $A \in \mathcal{C}$ is o-continuous, hence increasing convergent, but it is not exhaustive.

II) There are exhaustive multisubmeasures which are not increasing convergent:

Let $\mathcal{C}$ be the algebra $\{A \subset T,\ A$ is finite or cA is finite$\}$ of subsets of an infinite, countable set T and the multisubmeasure $\mu : \mathcal{C} \to \mathcal{P}_f(\mathbb{R})$, defined for every $A \in \mathcal{C}$ by:

$$\mu(A) = \begin{cases} \{0\}, & A \text{ is finite} \\ \{0, 1\}, & cA \text{ is finite} \end{cases}.$$

Then μ is exhaustive but it is not increasing convergent. Indeed, let be the sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, defined by $A_n = \{x_1, x_2, \ldots, x_n\}$, for every $n \in \mathbb{N}^*$. Obviously, $A_n \nearrow T = \{x_1, x_2, \ldots, x_n, \ldots\} \in \mathcal{C}$, $\mu(A_n) = \{0\}$, for every $n \in \mathbb{N}^*$ and $\mu(T) = \{0,1\}$, hence $\lim_{n\to\infty} h(\mu(A_n), \mu(T)) = |\{0,1\}| = 1 \neq 0$, so μ is not increasing convergent.

Now, we prove that μ is exhaustive. Let $(B_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ pairwise disjoint. Since $cB_n \cup cB_m = T$, for every $m \neq n$, then there can exist only one set, for instance $B_{n_0} \in \mathcal{C}$ so that cB_{n_0} is finite. Then B_n is finite for every $n > n_0$, so $\lim_{n\to\infty} |\mu(B_n)| = 0$.

3. The semivariation of set multifunctions

In this section we study if the semivariation $\widehat{\mu}$ of a set multifunction μ preserves the properties of μ. Also, an extension theorem by preserving the properties (autocontinuity, exhaustivity and increasing convergence) for monotone set multifunctions from a δ-ring to the generated σ-ring is established. Note that different types of extensions in fuzzy measures theory were studied by Denneberg [2], Murofushi [15], Pap [16], Wang and Klir [18] etc.

Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be an arbitrary set multifunction.

Remark 3.1. i) $\widehat{\mu}$ is monotone on $\mathcal{P}(T)$.

ii) If μ is a multisubmeasure, one can easily check that $\widehat{\mu}$ is a submeasure on $\mathcal{C}$ in the sense of Drewnowski [3]. Also, in this case $\widehat{\mu}(A) \leq \widehat{\mu}(A \backslash B) + \widehat{\mu}(B)$, for every $A \subset T$ and every $B \in \mathcal{C}$, with $B \subset A$.

In what follows, let be $\sigma(\mathcal{C})$, the σ-ring generated by a ring $\mathcal{C}$ and $\mathcal{C}_\sigma = \{A \subset T$; there exists an increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ with $A = \bigcup_{n=1}^{\infty} A_n\}$.

It is easy to verify that $\mathcal{C} \subset \mathcal{C}_\sigma, (\mathcal{C}_\sigma)_\sigma = \mathcal{C}_\sigma$ and if $\mathcal{C}$ is a δ-ring, then $\mathcal{C}_\sigma = \sigma(\mathcal{C})$.

First, we establish a result concerning the semivariation of increasing convergent set multifunctions:

Theorem 3.2. *Let $\mathcal{C}$ be a δ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ a set multifunction. Then:*

i) If μ is increasing convergent, then $\widehat{\mu}$ is also increasing convergent on $\mathcal{C}_\sigma$;

ii) If μ is monotone, increasing convergent and uniformly autocontinuous, then the same is $\widehat{\mu}$ on $\mathcal{C}_\sigma$;

iii) If, moreover, μ is a multisubmeasure, then $\widehat{\mu}$ is a submeasure on $\mathcal{C}_\sigma$.

Proof. i) Let $\varepsilon > 0$ and $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}_\sigma$ be an increasing sequence of sets. Then $A = \overset{\infty}{\underset{n=1}{\cup}} A_n \in \mathcal{C}_\sigma$. Since $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}_\sigma$, then for every $n \in \mathbb{N}^*$, there exists an increasing sequence of sets $(B_k^n)_k \subset \mathcal{C}$ so that $B_k^n \nearrow A_n$.

Consider $C_n = \underset{i,k\le n}{\cup} B_k^i$, for every $n \in \mathbb{N}^*$. Then $C_n \in \mathcal{C}, C_n \subset A_n$ for every $n \in \mathbb{N}^*$ and $C_n \nearrow A$.

Let $B \in \mathcal{C}$, with $B \subset A$. Because $(C_n \cap B)_n \subset \mathcal{C}, C_n \cap B \nearrow B$ and μ is increasing convergent, there exists $n_0(\varepsilon) \in \mathbb{N}^*$ such that

$$h(\mu(B), \mu(C_n \cap B)) < \varepsilon, \text{ for every } n \ge n_0.$$

Consequently,

$$|\mu(B)| \le h(\mu(B), \mu(C_n\cap B))+|\mu(C_n\cap B)| < \varepsilon+|\mu(C_n\cap B)| \le \varepsilon+\widehat{\mu}(A_n),$$

which yields $\widehat{\mu}(A) \le \varepsilon + \widehat{\mu}(A_n)$, for every $n \ge n_0$.

Since $\widehat{\mu}(A_n) \le \widehat{\mu}(A)$, for every $n \in \mathbb{N}^*$, then $\lim\limits_{n\to\infty} \widehat{\mu}(A_n) = \widehat{\mu}(A)$, that is, $\widehat{\mu}$ is increasing convergent on $\mathcal{C}_\sigma$.

ii) Let $\varepsilon > 0$ and $A \in \mathcal{C}_\sigma$. There is an increasing sequence $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ so that $A = \overset{\infty}{\underset{n=1}{\cup}} A_n$. Since μ is uniformly autocontinous, then for every $n \in \mathbb{N}^*$ there is $\delta(\varepsilon) > 0$ so that for every $C \in \mathcal{C}$, with $|\mu(C)| < \delta$, we have $h(\mu(A_n), \mu(A_n \cup C)) < \frac{\varepsilon}{2}$. Let $B \in \mathcal{C}_\sigma$, with $\widehat{\mu}(B) < \delta$. There is an increasing sequence $(B_k)_{k\in\mathbb{N}^*} \subset \mathcal{C}$ so that $B = \overset{\infty}{\underset{k=1}{\cup}} B_k$. Obviously, $|\mu(B_k)| < \delta$, for every $k \in \mathbb{N}^*$, so, $h(\mu(A_n), \mu(A_n \cup B_k)) < \frac{\varepsilon}{2}$, for every $n \in \mathbb{N}^*$ and every $k \in \mathbb{N}^*$. Then

$$|\mu(A_n \cup B_k)| \le h(\mu(A_n), \mu(A_n \cup B_k)) + |\mu(A_n| < \frac{\varepsilon}{2} + \widehat{\mu}(A),$$

for every $n \in \mathbb{N}^*$ and every $k \in \mathbb{N}^*$, which, particularly, implies $|\mu(A_n\cup B_n)| < \frac{\varepsilon}{2} + \widehat{\mu}(A)$, for every $n \in \mathbb{N}^*$. By the increasing convergence of

$\widehat{\mu}$ on $\mathcal{C}_\sigma$, there exists $n_0 \in \mathbb{N}^*$ so that $\widehat{\mu}(A \cup B) < |\mu(A_n \cup B_n)| + \frac{\varepsilon}{2}$, for every $n \geq n_0$. Then $\widehat{\mu}(A \cup B) < \varepsilon + \widehat{\mu}(A)$, that is, $\widehat{\mu}$ is uniformly autocontinuous on $\mathcal{C}_\sigma$.

iii) Suppose μ is a multisubmeasure. Obviously, $\widehat{\mu}(\emptyset) = 0$ and $\widehat{\mu}$ is monotone on $\mathcal{C}_\sigma$.

In order to prove that $\widehat{\mu}$ is finitely subadditive on $\mathcal{C}_\sigma$, we demonstrate that, moreover, $\widehat{\mu}$ is σ-subadditive on $\mathcal{C}_\sigma$, that is, $\widehat{\mu}(A) \leq \sum\limits_{n=1}^{\infty} \widehat{\mu}(A_n)$, for every pairwise disjoint sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}_\sigma$, with $A = \mathop{\cup}\limits_{n=1}^{\infty} A_n$. Indeed, let $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}_\sigma$ and $A = \mathop{\cup}\limits_{n=1}^{\infty} A_n$. Then $A \in \mathcal{C}_\sigma$.

On the other hand, for every $n \in \mathbb{N}^*$, there exists an increasing sequence $(B^n_k)_k \subset \mathcal{C}$ so that $A_n = \mathop{\cup}\limits_{k=1}^{\infty} B^n_k$. If $C_k = \mathop{\cup}\limits_{n=1}^{\infty} B^n_k$, for every $k \in \mathbb{N}^*$, then $C_k \in \mathcal{C}_\sigma$, for every $k \in \mathbb{N}^*$ and $C_k \nearrow A$.

Since $\widehat{\mu}$ is a submeasure on $\mathcal{C}$, we get that

$$\widehat{\mu}(\mathop{\cup}\limits_{i=1}^{n} B^i_k) \leq \sum_{i=1}^{n} \widehat{\mu}(B^i_k) \leq \sum_{n=1}^{\infty} \widehat{\mu}(B^n_k) \leq \sum_{n=1}^{\infty} \widehat{\mu}(A_n),$$

for every $n \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$.

Because the sequence $(\mathop{\cup}\limits_{i=1}^{n} B^i_k)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ is increasing, $\mathop{\cup}\limits_{i=1}^{\infty} B^i_k = C_k \in \mathcal{C}_\sigma$ and $\widehat{\mu}$ is increasing convergent on $\mathcal{C}_\sigma$, then $\lim\limits_{n\to\infty} \widehat{\mu}(\mathop{\cup}\limits_{i=1}^{n} B^i_k) = \widehat{\mu}(C_k)$. Consequently, $\widehat{\mu}(C_k) \leq \sum\limits_{n=1}^{\infty} \widehat{\mu}(A_n)$, for every $k \in \mathbb{N}^*$.

On the other hand, $\widehat{\mu}$ is increasing convergent on $\mathcal{C}_\sigma, (C_k)_k \subset \mathcal{C}_\sigma$ and $C_k \nearrow A$. So, $\lim\limits_{n\to\infty} \widehat{\mu}(C_k) = \widehat{\mu}(A)$, which yields $\widehat{\mu}(A) \leq \sum\limits_{n=1}^{\infty} \widehat{\mu}(A_n)$, as claimed.

Theorem 3.3. *Let $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ be a monotone set multifunction. Then μ is exhaustive on $\mathcal{C}$ if and only if $\widehat{\mu}$ is exhaustive on $\mathcal{P}(T)$.*

Proof. If $\widehat{\mu}$ is exhaustive on $\mathcal{P}(T)$, then $\widehat{\mu}|_{\mathcal{C}} = |\mu|$ is also exhaustive and the same is true for μ on $\mathcal{C}$.

Conversely, let $\varepsilon > 0$ and $(A_n)_{n\in\mathbb{N}^*} \subset T$ be pairwise disjoint.

By the definition of $\widehat{\mu}$, for every $n \in \mathbb{N}^*$, there exists $B_n \subset A_n$, $B_n \in \mathcal{C}$, so that

$$|\mu(B_n)| \leq \widehat{\mu}(A_n) \text{ and } |\mu(B_n)| > \widehat{\mu}(A_n) - \frac{1}{n}, \text{ for every } n \in \mathbb{N}^*.$$

Because μ is exhaustive and $(B_n)_n$ is also pairwise disjoint, then $\lim\limits_{n\to\infty} |\mu(B_n)| = 0$ and by the last inequality we get $\lim\limits_{n\to\infty} \widehat{\mu}(A_n) = 0$, hence $\widehat{\mu}$ is exhaustive on $\mathcal{P}(T)$.

Corollary 3.4. If $\mathcal{C}$ is a δ-ring and $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is an exhaustive, fuzzy multisubmeasure on $\mathcal{C}$, then $\widehat{\mu}$ is an exhaustive fuzzy submeasure on $\mathcal{C}_\sigma$.

Proof. Since μ is a fuzzy multisubmeasure, then it is increasing convergent, so, by Theorem 3.2, $\widehat{\mu}$ is increasing convergent on $\mathcal{C}_\sigma$. Now, because $\widehat{\mu}$ is also exhaustive by Theorem 3.3, then $\widehat{\mu}$ is o-continuous on $\mathcal{C}_\sigma$. Consequently, by Remark 3.1, $\widehat{\mu}$ is an exhaustive fuzzy submeasure on $\mathcal{C}_\sigma$.

Note that $\widehat{\mu} : \mathcal{C}_\sigma \to \mathbb{R}_+$ because of its exhaustivity.

In the sequel, we shall prove that exhaustivity allows for any set A of T, the approach by a set B of $\mathcal{C}$, with the aid of semivariation.

Theorem 3.5. *If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is an exhaustive set multifunction, then for every $A \subset T$ and every $\varepsilon > 0$, there exists $B \in \mathcal{C}, B \subset A$, so that $\widehat{\mu}(A \backslash B) < \varepsilon$.*

Proof. Suppose that, on the contrary, there exist $A \subset T$ and $\varepsilon_0 > 0$ such that $\widehat{\mu}(A\backslash B) \geq \varepsilon_0$, for every $B \in \mathcal{C}$, with $B \subset A$.

Let $B = \emptyset$. Then $\widehat{\mu}(A) \geq \varepsilon_0$ and, by the definition of $\widehat{\mu}$, there is $B_1 \in \mathcal{C}$, $B_1 \subset A$ so that $|\mu(B_1)| > \frac{\varepsilon_0}{2}$.

Let $B = B_1$. Then $\widehat{\mu}(A\backslash B_1) \geq \varepsilon_0$, which implies the existence of a set $B_2 \in \mathcal{C}$, such that $B_2 \subset A\backslash B_1$ and $|\mu(B_2)| > \frac{\varepsilon_0}{2}$.

Let $B = B_1 \cup B_2 \in \mathcal{C}$. Then $\widehat{\mu}(A\backslash(B_1 \cup B_2)) \geq \varepsilon_0$, so there is a set $B_3 \in \mathcal{C}$, $B_3 \subset A\backslash(B_1 \cup B_2)$ such that $|\mu(B_3)| > \frac{\varepsilon_0}{2}$.

By induction we obtain a pairwise disjoint sequence of sets $(B_n)_{n\in N^*} \subset \mathcal{C}$ with $|\mu(B_n)| > \frac{\varepsilon_0}{2}$, for every $n \in \mathbb{N}^*$.

Consequently, $\lim\limits_{n\to\infty} |\mu(B_n)| \neq 0$, which is false because μ is exhaustive.

Corollary 3.6. If $\mu : \mathcal{C} \to \mathcal{P}_f(X)$ is an exhaustive multisubmeasure, then:

i) $\widehat{\mu}(T) < \infty$;

ii) for every $A \subset T$, there exists $B \in \mathcal{C}_\sigma, B \subset A$ so that $\widehat{\mu}(A) = \widehat{\mu}(B)$ and $\widehat{\mu}(A \backslash B) = 0$;

iii) for every $A \subset T$, there exists an increasing sequence of sets $(B_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$ so that $B_n \subset A$, for every $n \in \mathbb{N}^*$ and $\widehat{\mu}(A) = \lim\limits_{n\to\infty} \widehat{\mu}(B_n) = \widehat{\mu}(B)$, where $B = \bigcup\limits_{n=1}^{\infty} B_n$.

Proof. i) Let $\varepsilon > 0$. By Theorem 3.5, there exists for T a subset $B \in \mathcal{C}$, so that $\widehat{\mu}(T \backslash B) < \varepsilon$. By the definition of $\widehat{\mu}$ we get that

$$\widehat{\mu}(T) \leq \widehat{\mu}(T \backslash B) + \widehat{\mu}(B) < \varepsilon + \widehat{\mu}(B) = \varepsilon + |\mu(B)|.$$

Because μ is exhaustive, it is also bounded, so there is $M > 0$ such that $|\mu(B)| \leq M$. Consequently, $\widehat{\mu}(T) < \varepsilon + M < \infty$.

ii) Let $A \subset T$. Since μ is exhaustive, by Theorem 3.5 we get that for every $n \in \mathbb{N}^*$, there exists $B_n \in \mathcal{C}, B_n \subset A$ such that $\widehat{\mu}(A \backslash B_n) < \frac{1}{n}$.

Consider $B = \bigcup\limits_{n=1}^{\infty} B_n$. Then $B \in \mathcal{C}_\sigma, B \subset A$ and

$$0 \leq \widehat{\mu}(A \backslash B) \leq \widehat{\mu}(A \backslash B_n) < \frac{1}{n}, \text{ for every } n \in \mathbb{N}^*,$$

which implies $\widehat{\mu}(A \backslash B) = 0$.

On the other hand,

$$\widehat{\mu}(A) \leq \widehat{\mu}(A \backslash B_n) + \widehat{\mu}(B_n), \text{ for every } n \in \mathbb{N}^*.$$

So,

$$\widehat{\mu}(A) < \frac{1}{n} + \widehat{\mu}(B), \text{ for every } n \in \mathbb{N}^*,$$

and, because $\widehat{\mu}(B) \leq \widehat{\mu}(A)$, we finally get $\widehat{\mu}(A) = \widehat{\mu}(B)$.

iii) Let $A \subset T$. Because μ is exhaustive, there exists a sequence $(B_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $B_n \subset A$, for every $n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \widehat{\mu}(A \backslash B_n) = 0$.

Without any loss of generality, we may suppose that $(B_n)_{n\in\mathbb{N}^*}$ is an increasing one (if not, considering $B_n^1 = \bigcup\limits_{i=1}^{n} B_i$, for $n \in \mathbb{N}^*$, then $B_n^1 \subset A, (B_n^1)_{n\in N^*} \subset \mathcal{C}$ and $\widehat{\mu}(A \backslash B_n^1) \leq \widehat{\mu}(A \backslash B_n)$, for every $n \in \mathbb{N}^*$, which implies $\lim\limits_{n\to\infty} \widehat{\mu}(A \backslash B_n^1) = 0$).

Then

$$\widehat{\mu}(B_n) \leq \widehat{\mu}(A) \leq \widehat{\mu}(A \backslash B_n) + \widehat{\mu}(B_n), \text{ for every } n \in \mathbb{N}^*$$

and, consequently,

$$\widehat{\mu}(A) = \lim_{n\to\infty} \widehat{\mu}(B_n) = \lim_{n\to\infty} |\mu(B_n)|.$$

On the other hand, from the proof of ii), it follows $\widehat{\mu}(A) = \widehat{\mu}(B)$, where $B = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} B_n^1$.

Consequently, $\lim_{n\to\infty} \widehat{\mu}(B_n) = \widehat{\mu}(B)$. This equality shows that if a multisubmeasure is exhaustive, then there is a sequence of sets on which its semivariation is increasing convergent.

The following result shows that, if μ_1 and μ_2 are two exhaustive multisubmeasures, then for the equality of $\widehat{\mu}_1$ and $\widehat{\mu}_2$ on $\mathcal{P}(T)$, it is sufficient they are equal on $\mathcal{C}_\sigma \subset \mathcal{P}(T)$.

Theorem 3.7. Let $\mu_1, \mu_2 : \mathcal{C} \to \mathcal{P}_f(X)$ be two exhaustive multisubmeasures. If $\widehat{\mu}_1 = \widehat{\mu}_2$ on $\mathcal{C}_\sigma$, then $\widehat{\mu}_1 = \widehat{\mu}_2$ on $\mathcal{P}(T)$.

Proof. Let $A \subset T$ be an arbitrary set. Because μ_1 is exhaustive, then for every $n \in \mathbb{N}^*$, there exists $B_n^1 \in \mathcal{C}, B_n^1 \subset A$ such that $\widehat{\mu}_1(A \backslash B_n^1) < \frac{1}{n}$.

Analogously, since μ_2 is exhaustive, for every $n \in \mathbb{N}^*$, there exists $B_n^2 \in \mathcal{C}, B_n^2 \subset A$, with $\widehat{\mu}_2(A \backslash B_n^2) < \frac{1}{n}$.

Let $B_1 = \bigcup_{n=1}^{\infty} B_n^1$ and $B_2 = \bigcup_{n=1}^{\infty} B_n^2$. Then $B_1 \subset A, B_2 \subset A$ and $B_1, B_2 \in \mathcal{C}_\sigma$.

Using the same arguments as in the proof of Corollary 3.6 ii), it also easily follows that

$$\widehat{\mu}_1(B_1) = \widehat{\mu}_1(A), \widehat{\mu}_1(A\backslash B_1) = 0, \widehat{\mu}_2(B_2) = \widehat{\mu}_2(A) \text{ and } \widehat{\mu}_2(A\backslash B_2) = 0.$$

Because $A\backslash(B_1 \cup B_2) \subset A\backslash B_1$ and $A\backslash(B_1 \cup B_2) \subset A\backslash B_2$, then

$$\widehat{\mu}_1(A\backslash(B_1 \cup B_2)) = 0 \text{ and } \widehat{\mu}_2(A\backslash(B_1 \cup B_2)) = 0.$$

Let us prove now that $\widehat{\mu}_1(A) = \widehat{\mu}_1(B_1 \cup B_2)$.

Indeed,

$$\begin{aligned}\widehat{\mu}_1(B_1 \cup B_2) \le \widehat{\mu}_1(A) \le \widehat{\mu}_1(A\backslash(B_n^1 \cup B_n^2)) + \widehat{\mu}_1(B_n^1 \cup B_n^2) \le \\ \le \widehat{\mu}_1(A\backslash B_n^1) + \widehat{\mu}_1(B_1 \cup B_2) < \tfrac{1}{n} + \widehat{\mu}_1(B_1 \cup B_2),\end{aligned}$$

for every $n \in \mathbb{N}^*$, which implies that $\widehat{\mu}_1(A) = \widehat{\mu}_1(B_1 \cup B_2)$.

Analogously, $\widehat{\mu}_2(A) = \widehat{\mu}_2(B_1 \cup B_2)$, and, since $\widehat{\mu}_1 = \widehat{\mu}_2$ on $\mathcal{C}_\sigma$ and $B_1 \cup B_2 \in \mathcal{C}_\sigma$, then $\widehat{\mu}_1(A) = \widehat{\mu}_2(A)$. So, $\widehat{\mu}_1 = \widehat{\mu}_2$ on $\mathcal{P}(T)$.

We are now able to prove a theorem concerning the properties of the extension of μ from $\mathcal{C}$ to $\mathcal{C}_\sigma$.

Theorem 3.8. *Let $\mathcal{C}$ be a δ-ring, X a Banach space and $\mu : \mathcal{C} \to \mathcal{P}_{bf}(X)$ an increasing convergent uniformly autocontinuous exhaustive monotone set multifunction. Then:*

i) μ uniquely extends to an increasing convergent uniformly autocontinuous exhaustive monotone set multifunction $\mu^* : \mathcal{C}_\sigma \to \mathcal{P}_{bf}(X)$, defined by:

$$\mu^*(A) = \lim_{n\to\infty} \mu(A_n) \text{ (with respect to the Hausdorff metric } h\text{)},$$
$$\text{for every } A \in \mathcal{C}_\sigma, \text{ where } (A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}, A_n \nearrow A;$$

ii) If μ is a multisubmeasure, then:

a) μ^* is a fuzzy multisubmeasure;
b) $|\mu| : \mathcal{C} \to \mathbb{R}_+$ is an increasing convergent, exhaustive submeasure, which uniquely extends to the increasing convergent, exhaustive submeasure $|\mu^*| : \mathcal{C}_\sigma \to \mathbb{R}_+$;
c) $|\mu^*(A)| = \widehat{\mu}(A) = \widehat{\mu^*}(A)$, for every $A \in \mathcal{C}_\sigma$;
d) $\widehat{\mu} = \widehat{\mu^*}$ on $\mathcal{P}(T)$.

Proof. i) Let $\varepsilon > 0$ and $A \in \mathcal{C}_\sigma$. There is an increasing sequence of sets $(A_n)_n \subset \mathcal{C}$ so that $A = \bigcup_{n=1}^{\infty} A_n$. We prove that there exists $\lim_{n\to\infty} \mu(A_n)$ (with respect to the Hausdorff metric h).

Indeed, because μ is exhaustive and uniformly autocontinuous, by Theorem 2.1, we get that $\lim_{n\to\infty} h(\mu(A_n), \mu(A_m)) = 0$, so $(\mu(A_n))_n$ is a Cauchy, hence a convergent sequence in the complete metric space $\mathcal{P}_{bf}(X)$.

Denote

$$\mu^*(A) = \lim_{n\to\infty} \mu(A_n) \text{ (with respect to } h\text{), for every } A \in \mathcal{C}_\sigma.$$

Obviously, $\mu^* : \mathcal{C}_\sigma \to \mathcal{P}_{bf}(X)$.

We prove that the limit does not depend on $(A_n)_n$, that is, if $A \in \mathcal{C}_\sigma$ and $(A_m)_m$ and $(B_n)_n$ are two increasing sequences of sets such that $A = \bigcup_{m=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ and if we denote $\mu_1^*(A) = \lim_{n\to\infty} \mu(B_n)$, then $\mu^*(A) = \mu_1^*(A)$.

Indeed, because $A_m \nearrow A$, there exists $m_0 \in \mathbb{N}^*$ so that $h(\mu(A_m), \mu^*(A)) < \frac{\varepsilon}{3}$, for every $m \geq m_0$. Particularly, $h(\mu(A_{m_0}), \mu^*(A)) < \frac{\varepsilon}{3}$.

Since $A_{m_0} \cap B_n \nearrow A_{m_0} \cap A = A_{m_0}$ and μ is increasing convergent, there exists $n_1 \in \mathbb{N}^*$ so that $h(\mu(A_{m_0}), \mu(A_{m_0} \cap B_n)) < \frac{\varepsilon}{3}$, for every $n \geq n_1$.

Also, because $B_n \nearrow A$, there exists $n_2 \in \mathbb{N}^*$ such that $h(\mu_1^*(A), \mu(B_n)) < \frac{\varepsilon}{3}$, for every $n \geq n_2$.

Consequently, if $n_3 = \max\{n_1, n_2\}$, then

$$\begin{aligned} e(\mu^*(A), \mu_1^*(A)) &\leq e(\mu^*(A), \mu(A_{m_0})) + e(\mu(A_{m_0}), \mu(A_{m_0} \cap B_{n_3})) + \\ &+ e(\mu(A_{m_0} \cap B_{n_3}), \mu(B_{n_3})) + e(\mu(B_{n_3}), \mu_1^*(A)) \leq \\ &\leq h(\mu^*(A), \mu(A_{m_0})) + h(\mu(A_{m_0}), \mu(A_{m_0} \cap B_{n_3})) + \\ &+ h(\mu(B_{n_3}), \mu_1^*(A)) < \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Analogously, $e(\mu_1^*(A), \mu^*(A)) < \varepsilon$, so $\mu^*(A) = \mu_1^*(A)$, for every $A \in \mathcal{C}_\sigma$.

Note that, evidently, $\mu^*(\emptyset) = \{0\}$ if $\mu(\emptyset) = \{0\}$.

If $A, B \in \mathcal{C}_\sigma$ are so that $A \subset B$, then there are $(A_n)_n$ and $(B_n)_n$ in $\mathcal{C}$, with $A_n \nearrow A$ and $B_n \nearrow B$.

Consequently,

$$\begin{aligned} e(\mu^*(A), \mu^*(B)) &\leq e(\mu^*(A), \mu(A_n \cap B_n)) + e(\mu(A_n \cap B_n), \mu(B_n)) + \\ &+ e(\mu(B_n), \mu^*(B)) \leq h(\mu^*(A), \mu(A_n \cap B_n)) + h(\mu(B_n), \mu^*(B)). \end{aligned}$$

Since $A_n \cap B_n \nearrow A$ and $B_n \nearrow B$, by the definition of μ^* we immediately get that $\mu^*(A) \subseteq \mu^*(B)$, so μ^* is monotone on $\mathcal{C}_\sigma$.

Now, we prove that μ^* is increasing convergent. Let be $(A_n)_n \subset \mathcal{C}_\sigma$, with $A_n \nearrow A = \bigcup_{n=1}^{\infty} A_n$. Because $e(\mu^*(A_n), \mu^*(A)) = 0$, for every $n \in \mathbb{N}^*$, it is sufficient to prove that for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}^*$ so that $e(\mu^*(A), \mu^*(A_n)) < \varepsilon$, for every $n \geq n_0$.

Since $A_n \in \mathcal{C}_\sigma$, for every $n \in \mathbb{N}^*$, there exists an increasing sequence of sets $(A_m^n)_m \subset \mathcal{C}$, with $A_n = \bigcup_{m=1}^{\infty} A_m^n$.

Let $C_m = A_m^1 \cup A_m^2 \cup \ldots \cup A_m^m$. Then $(C_m)_m$ is increasing. Indeed,

$$C_m = A_m^1 \cup A_m^2 \cup \ldots \cup A_m^m \subset A_{m+1}^1 \cup A_{m+1}^2 \cup \ldots \cup A_{m+1}^{m+1} = C_{m+1},$$

because $(A_m^n)_m$ is increasing.

Moreover, since $A_m^1 \subset A_1, A_m^2 \subset A_2, ...A_m^m \subset A_m$, we get $A_m^n \subset C_m \subset A_m$, for every $m \geq n$.

Consequently,

$$A_n = \bigcup_{m=n}^{\infty} A_m^n \subset \bigcup_{m=n}^{\infty} C_m \subset \bigcup_{m=n}^{\infty} A_m = \bigcup_{n=1}^{\infty} A_n, \text{ for every } n \in \mathbb{N}^*;$$

hence, $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} C_n = A$.

On the other hand, because $(C_n)_n \subset \mathcal{C}$, we have

$$e(\mu^*(A), \mu^*(A_n)) \leq e(\mu^*(A), \mu(C_n)) + e(\mu(C_n), \mu^*(A_n))$$

and $\mu^*(C_n) = \mu(C_n)$, for every $n \in \mathbb{N}^*$.

Then there is $n_0 \in \mathbb{N}^*$ so that

$$\begin{aligned} e(\mu^*(A), \mu^*(A_n)) &\leq h(\mu^*(A), \mu(C_n)) + e(\mu^*(C_n), \mu^*(A_n)) = \\ &= h(\mu^*(A), \mu(C_n)) < \varepsilon, \end{aligned}$$

for every $n \geq n_0$.

We prove that μ^* is exhaustive on $\mathcal{C}_\sigma$. Indeed, let $(A_n)_n \subset \mathcal{C}_\sigma$ be pairwise disjoint. For every $n \in \mathbb{N}^*$, there exists $(A_n^k)_k \subset \mathcal{C}$ such that $A_n^k \nearrow A_n$.

Since for every $n \in \mathbb{N}^*$, $\mu^*(A_n) = \lim\limits_{k\to\infty} \mu(A_n^k)$, there is $k_0^n \in \mathbb{N}^*$ so that

$$|\mu^*(A_n)| \leq h(\mu^*(A_n), \mu(A_n^{k_0^n})) + |\mu(A_n^{k_0^n})| < \frac{\varepsilon}{2} + |\mu(A_n^{k_0^n})|.$$

Because $A_n^{k_0^n} \cap A_m^{k_0^m} \subset A_n \cap A_m = \emptyset$, then $A_n^{k_0^n} \cap A_m^{k_0^m} = \emptyset$, $m \neq n$. Since μ is exhaustive, then $\lim\limits_{n\to\infty} |\mu(A_n^{k_0^n})| = 0$.

Therefore, there is $n_0 \in \mathbb{N}^*$ such that $|\mu^*(A_n)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, for every $n \geq n_0$, hence μ^* is exhaustive on $\mathcal{C}_\sigma$.

We prove now that μ^* is uniformly autocontinuous. Let $\varepsilon > 0$ and $A \in \mathcal{C}_\sigma$. There is an increasing sequence $(A_n)_n \subset \mathcal{C}$, with $A_n \nearrow A$. Since μ is uniformly autocontinuous, then for every $n \in \mathbb{N}^*$ there exists $\delta(\varepsilon) > 0$ such that for every $C \in \mathcal{C}$, with $|\mu(C)| < \delta$, we have $h(\mu(A_n \cup C), \mu(A_n)) < \frac{\varepsilon}{3}$. Let $B \in \mathcal{C}_\sigma$, with $|\mu^*(B)| < \frac{\delta}{2}$. There is an increasing sequence $(B_k)_k \subset \mathcal{C}$, with $B_k \nearrow B$. By the definition of μ^*, $\mu^*(B) = \lim\limits_{k\to\infty} \mu(B_k)$, which implies the existence of a $k_1(\varepsilon) \in \mathbb{N}^*$ such that $h(\mu^*(B), \mu(B_k)) < \frac{\delta}{2}$, for every $k \geq k_1$. Consequently, $|\mu(B_k)| < \delta$, for every $k \geq k_1$. Then $h(\mu(A_n \cup B_k), \mu(A_n)) < \frac{\varepsilon}{3}$, for every $n \in \mathbb{N}^*$

and every $k \geq k_1$. Particularly, $h(\mu(A_k \cup B_k), \mu(A_k)) < \frac{\varepsilon}{3}$, for every $k \geq k_1$. On the other hand, we also have $h(\mu^*(A \cup B), \mu(A_k \cup B_k)) < \frac{\varepsilon}{3}$ and $h(\mu(A_k), \mu^*(A)) < \frac{\varepsilon}{3}$, for every $k \geq k_2$. Denote $k_0 = \max\{k_1, k_2\}$. Then:
$h(\mu^*(A \cup B), \mu^*(A)) \leq h(\mu^*(A \cup B), \mu(A_{k_0} \cup B_{k_0})) +$
$+ h(\mu(A_{k_0} \cup B_{k_0}), \mu(A_{k_0})) + h(\mu(A_{k_0}), \mu^*(A)) < \varepsilon$.
Therefore, μ^* is uniformly autocontinuous.

Now, we prove that the extension is unique. Suppose, by the contrary, that there is another increasing convergent, exhaustive, monotone uniformly autocontinuous set multifunction $\mu_1^* : \mathcal{C}_\sigma \to \mathcal{P}_{bf}(X)$ which extends μ. Let be $A \in \mathcal{C}_\sigma$ and $\varepsilon > 0$.

There exists an increasing sequence $(A_n)_n \subset \mathcal{C}$, with $A_n \nearrow A$.

Because μ^* and μ_1^* are both increasing convergent, then there is a common $n_0(\varepsilon) \in \mathbb{N}^*$ so that

$$h(\mu^*(A_n), \mu^*(A)) < \frac{\varepsilon}{2} \text{ and } h(\mu_1^*(A_n), \mu_1^*(A)) < \frac{\varepsilon}{2},$$

for every $n \geq n_0$.

Then, also,

$$h(\mu^*(A_{n_0}), \mu^*(A)) < \frac{\varepsilon}{2} \text{ and } h(\mu_1^*(A_{n_0}), \mu_1^*(A)) < \frac{\varepsilon}{2}.$$

Also, since $\mu(A_{n_0}) = \mu^*(A_{n_0}) = \mu_1^*(A_{n_0})$, we get that

$$\begin{aligned} h(\mu^*(A), \mu_1^*(A)) &\leq h(\mu^*(A), \mu(A_{n_0})) + h(\mu(A_{n_0}), \mu_1^*(A)) < \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So, $h(\mu^*(A), \mu_1^*(A)) < \varepsilon$, for every $\varepsilon > 0$, hence, finally, $\mu^*(A) = \mu_1^*(A)$, for every $A \in \mathcal{C}_\sigma$.

ii) a) It is easy to verify that μ^* is also a multisubmeasure. According to Theorem 1.6 and Theorem 2.2 ii), every exhaustive increasing convergent multisubmeasure is o-continuous, hence fuzzy.

b) We observe that $|\mu^*|$ extends $|\mu|$ because $|\mu^*(A)| = |\mu(A)|$, for every $A \in \mathcal{C}$. Also, by Theorem 3.2, $|\mu|$ and $|\mu^*|$ are increasing convergent submeasures on $\mathcal{C}$, respectively, on $\mathcal{C}_\sigma$.

Obviously, $|\mu|$ and $|\mu^*|$ are exhaustive on $\mathcal{C}$, respectively, on $\mathcal{C}_\sigma$, since the same are μ and μ^*. It only remains to establish the uniqueness. For this, let $\nu : \mathcal{C}_\sigma \to \mathbb{R}_+$ be another increasing convergent and exhaustive submeasure on $\mathcal{C}_\sigma$, which extends $|\mu|$. We prove that $\nu(A) = |\mu^*(A)|$, for every $A \in \mathcal{C}_\sigma$.

Let $\varepsilon > 0$ and $A \in \mathcal{C}_\sigma$ be arbitrarily. There exists an increasing sequence of sets $(A_n)_{n\in\mathbb{N}^*} \subset \mathcal{C}$, with $A_n \nearrow A$.

Because ν is increasing convergent on $\mathcal{C}_\sigma$, there exists $n_1(\varepsilon) \in \mathbb{N}^*$ such that $|\nu(A) - \nu(A_n)| < \frac{\varepsilon}{2}$, for every $n \geq n_1$.

Analogously, for $|\mu^*|$, there exists $n_2(\varepsilon) \in \mathbb{N}^*$ so that $||\mu^*(A)| - |\mu^*(A_n)|| < \frac{\varepsilon}{2}$, for every $n \geq n_2$.

Then

$$|\nu(A) - \nu(A_{n_0})| < \frac{\varepsilon}{2} \text{ and } ||\mu^*(A)| - |\mu^*(A_{n_0})|| < \frac{\varepsilon}{2},$$

where $n_0 = \max(n_1, n_2)$.

So, since $A_{n_0} \in \mathcal{C}$, we have

$$\begin{aligned}|\nu(A) - &|\mu^*(A)|| \leq |\nu(A) - \nu(A_{n_0})| + ||\mu^*(A)| - |\mu^*(A_{n_0})||+\\ &+||\mu^*(A_{n_0})| - \nu(A_{n_0})| < \varepsilon + ||\mu^*(A_{n_0})| - \nu(A_{n_0})| = \varepsilon.\end{aligned}$$

Consequently, $\nu(A) = |\mu^*(A)|$, for every $A \in \mathcal{C}_\sigma$.

c) Applying Theorem 3.2 and Theorem 3.3, $\widehat{\mu}$ is also an increasing convergent and exhaustive submeasure on $\mathcal{C}_\sigma$, which extends $|\mu|$. Consequently,

$$\widehat{\mu}(A) = |\mu^*(A)| = \widehat{\mu^*}(A), \text{ for every } A \in \mathcal{C}_\sigma.$$

d) We use c) and Theorem 3.7.

Concluding remarks. In this paper we study exhaustivity and the properties of semivariation for $\mathcal{P}_f(X)$-valued set multifunctions, where $\mathcal{P}_f(X)$ is the family of non-void, closed subsets of a real normed space X. Several results concerning fuzzy set multifunctions are obtained, some of them generalizing known results from single-valued fuzzy measures theory, and an extension theorem by preserving exhaustivity, autocontinuity and increasing convergence is established for monotone set multifunctions taking values in $\mathcal{P}_{bf}(X)$, the family of non-void, closed, bounded subsets of a Banach space X.

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