"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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ON ALMOST λ-CONTINUOUS FUNCTIONS

S. JAFARI, S. P. MOSHOKOA, K. R. NAILANA AND T. NOIRI

Abstract. In the paper we introduce a new class of functions between topological spaces, namely almost λ-continuous functions and present some properties for these functions.

1. introduction

Maki [9] introduced the notion of Λ-sets in topological spaces. A subset A of a topological space (X, τ) is called a Λ-set if it coincides with its kernel (the intersection of all open supersets of A). In [1], Arenas et al. introduced the notions of λ-open sets, and λ-closed sets and presented fundamental results for these sets. The purpose of this paper is to introduce the notion of an almost λ-continuous function and investigate some of the properties for this class of functions.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or X and Y) are always topological spaces on which no separation axioms are assumed unless explicitly stated. We refer the reader to [[1],[2]] for a basic reference on λ-open sets.

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Let A be a subset of X. Then A is said to be λ-closed [1] if $A = B \cap C$, where B is a Λ-set and C is a closed set. The complement of a λ-closed set is called a λ-open set. Let $x \in X$ and A be a subset of X. Then x is said to be a λ-interior point of A if there exists a λ-open set U containing x such that $U \subset A$. The set of all λ-interior points of A is called the λ-interior of A and is denoted by $Int_\lambda(A)$ [2]. A subset A of X is said to be regular open (respectively, regular closed) if $A = Int(Cl(A))$ (respectively, $A = Cl(Int(A))$). Let $x \in X$. Then by $O(X, x)$ we denote the set of all open sets in X that contains x. Furthermore, by $\lambda O(X, x)$ (resp. $RO(X, x)$), we denote the set of all λ-open (resp. regular open) sets that contain x. A function $f : X \to Y$ is λ-continuous [1] if $f^{-1}(V)$ is λ-closed in X for every closed set V in Y.

Observe that every open set in X is λ-open, but not conversely. Also every continuous function is λ-continuous but not conversely [1].

3. Main results

We begin with

Definition 3.1. *A function $f : X \to Y$ is almost λ-continuous (briefly, a.λ.c.)at $x \in X$ if for each $V \in RO(Y, f(x))$, there exists $U \in \lambda O(X, x)$ such that $f(U) \subset V$. If f is almost λ-continuous at every point of X, then it is called almost λ-continuous.*

Note that every regular open set is open, and thus λ-continuity implies almost λ-continuity. We provide an example of a function which is almost λ-continuous but not continuous.

Example 3.1 [1]

The classical Dirichelet function $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the usual real line with the usual topology:

$$f(x) = \begin{cases} 1 & \text{if} \qquad x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

This function f is λ-continuous but not continuous.

Example 3.2

Let $X = \{a, b\}$ with topologies $\tau =$ indiscrete topology and $\sigma = \{\emptyset, \{b\}, X\}$. Define $f : (X, \tau) \to (X, \sigma)$ by $f(a) = a$ and $f(b) = b$. This function is almost λ-continuous but it is neither λ-continuous nor continuous.

Example 3.3 Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{c\}, \{a, b, c\}\}$ and $\sigma = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$. Define $f : (X, \tau) \to (X, \sigma)$ by $f(x) = x$ for every $x \in X$. This function is almost λ-continuous but it is neither λ-continuous nor continuous.

In fact, we have the following implications:

$$\text{continuity} \Rightarrow \lambda\text{-continuity} \Rightarrow \text{almost } \lambda\text{-continuity.}$$

Recall [13] that a subset A of X is δ-open if for each $x \in A$, there exists a regular open set U such that $x \in U \subset A$. The complement of a δ-open set is said to be δ-closed. The intersection of all δ-closed sets containing A is called the δ-closure of A and it is denoted by $Cl_\delta(A)$.

The next two results characterize almost λ-continuous functions.

Theorem 3.2. *For a function $f : X \to Y$, the following are equivalent:*
(a) *f is a.λ.c.;*
(b) *for each $x \in X$ and each open set V containing $f(x)$, there exists λ-open set U containing x such that $f(U) \subset Int(Cl(V))$;*
(c) *$f^{-1}(F)$ is λ-closed in X for every regular closed set F in Y;*
(d) *$f^{-1}(V)$ is λ-open in X for every regular open set V in Y.*

Proof. The proof is obvious and thus omitted. □

Theorem 3.3. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following are equivalent:*
(a) *f is a.λ.c.;*
(b) *$f(Cl_\lambda(A)) \subset Cl_\delta(f(A))$ for every subset A of X;*
(c) *$Cl_\lambda(f^{-1}(B)) \subset f^{-1}(Cl_\delta(B))$ for every subset B of Y;*
(d) *$f^{-1}(F)$ is λ-closed in X for every δ-closed set F of Y;*
(e) *$f^{-1}(V)$ is λ-open in X for ever δ-open set V of Y.*

Proof. (a) $\Rightarrow$ (b). Let A be a subset of X. Since $Cl_\delta(f(A))$ is δ-closed in Y, it is denoted by $\cap\{F_\alpha : \alpha \in \nabla\}$, where F_α is regular closed in Y. The set $f^{-1}(Cl_\delta(f(A)))$ is λ-closed (Theorem 3.2) and contains A, also $f^{-1}(Cl_\delta(f(A))) = \cap\{f^{-1}(F_\alpha) : \alpha \in \nabla\}$. Hence $Cl_\lambda\delta(A) \subset f^{-1}(Cl_\delta(f(A)))$. Therefore we obtain $f(Cl_\lambda(A)) \subset Cl_\delta(f(A))$.

(b) $\Rightarrow$ (c) Let B be a subset of Y. We have $f(Cl_\lambda(f^{-1}(B))) \subset Cl_\delta(f(f^{-1}(B))) \subset Cl_\delta(B)$ and hence $Cl_\lambda(f^{-1}(B)) \subset f^{-1}(Cl_\delta(B))$.

(c) $\Rightarrow$ (d) Let F be any δ-closed set of Y. We have $Cl_\lambda(f^{-1}(F)) \subset f^{-1}(Cl_\delta(F)) = f^{-1}(F)$ and hence $f^{-1}(F)$ is λ-closed in X.

$(d) \Rightarrow (e)$ Let V be any δ-open set of Y. Then $Y - V$ is δ-closed. We have $f^{-1}(Y - V) = X - f^{-1}(V)$ is δ-closed in X. Hence $f^{-1}(V)$ is δ-open in X.

$(e) \Rightarrow (a)$ Let V be any regular open set in Y. Since V is δ-open in Y, we have $f^{-1}(V)$ is λ-open in X and hence by Theorem 3.2 f is $a.\lambda.c.$

□

Theorem 3.4. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is $a.\lambda.c.$, then f is $a.\lambda.c.$*

Proof. Let $x \in X$ and $V \in RO(Y, f(x))$. Then $g(x) = (x, f(x)) \in X \times V$. Observe that $X \times V \in RO(X \times Y, \tau \times \sigma)$. If g is $a.\lambda.c.$, then there exists $U \in \lambda O(X, x)$ such that $g(U) \subset X \times V$. It follows that $f(U) \subset V$, hence f is $a.\lambda.c.$ □

We recall that the space X is called a λ-space [1] if the set of all λ-open subsets form a topology on X. Clearly a space X is a λ- space if and only if the intersection of two λ-open sets is λ-open. An example of a λ-space is a $T_{\frac{1}{2}}$-space, where a space X is called $T_{\frac{1}{2}}$ [5] if every singleton is open or closed .

Theorem 3.5. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function defined by $g(x) = (x, f(x))$ for every $x \in X$. If X is a λ-space, then g is $a.\lambda.c.$ if and only if f is $a.\lambda.c.$.*

Proof. We only prove the sufficiency. Let $x \in X$ and $W \in RO(X \times Y, g(x))$. Then there exist regular open sets U_1 and V in X and Y, respectively such that $U_1 \times V \subset W$. If f is $a.\lambda.c.$, then there exists a λ-open set U_2 in X such that $x \in U_2$ and $f(U_2) \subset V$. Put $U = (U_1 \cap U_2)$. Then U is λ-open and $g(U) \subset U_1 \times V \subset W$. Thus g is $a.\lambda.c.$ This together with the proof of Theorem 3.4 completes our proof. □

Example 3.2 Let X be the set of non-negative integers with the topology τ whose open sets are those that contain 0 and have a finite complement. Let Y be the set of non-negative reals with the usual topology which we denote by σ. Then X is a λ-space [2]. If the function $f : X \to Y$ is $a.\lambda.c.$, then the graph function $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ is $a.\lambda.c.$ In particular Theorem 3.5 holds.

Definition 3.6. *A function $f : X \to Y$ is pre $-$ λ-open if the image of each λ-open set is λ-open.*

Definition 3.7. *A function $f : X \to Y$ is weakly λ-continuous (briefly, w.λ.c.)* [3] *if and only if for each point $x \in X$ and every $V \in O(Y, f(x))$ there exists $U \in \lambda O(X, x)$ such that $f(U) \subset Cl(V)$.*

Observe that $a.\lambda.c.$ implies $w.\lambda.c.$ In fact, from Theorem 3.2 we have the following: λ-continuity implies almost λ-continuity and almost λ-continuity implies weak λ-continuity.

Definition 3.8. *A function $f : X \to Y$ is called almost λ-open if the image of a λ-open set is open in Y.*

Proof. If $f : X \to Y$ is almost λ-open and $w.\lambda.c$, then f is $a.\lambda.c.$ □

Definition 3.9. [13] *Let $\mathcal{F}$ be a filter base. Then $\mathcal{F}$ is said to δ-converge to $x \in X$ if for every open set U containing x, there exists B in $\mathcal{F}$ such that $B \subset Int(Cl(U))$.*

Definition 3.10. *Let $\mathcal{F}$ be a filter base. Then $\mathcal{F}$ is said to λ-converge to a point $x \in X$ if for any $U \in \lambda O(X, x)$ there exists B in $\mathcal{F}$ such that $B \subset U$.*

Theorem 3.11. *If a function $f : X \to Y$ is a.λ.c., then for each point $x \in X$ and each filter base $\mathcal{F}$ in X that λ-converges to x, the filter $f(\mathcal{F})$ is δ-convergent to $f(x)$.*

Proof. Suppose that $x \in X$ and a filter base $\mathcal{F}$ in X that λ-converges to x. Suppose that f is $a.\lambda.c.$ Then for each $V \in RO(Y, f(x))$ there exists $U \in \lambda O(X, x)$ such that $f(U) \subset V$. Then there exists $B \subset \mathcal{F}$ such that $B \subset U$, hence $f(B) \subset f(U)$. It follows that $f(B) \subset V$. This shows that $f(\mathcal{F})$ is δ-convergent to $f(x)$. □

For a space (Y, σ) we denote by σ_s the semiregular topology of σ generated by regular open sets of (Y, σ). For ease of notation we simply denote the semiregularization of (Y, σ) by Y_s.

Corollary 3.12. *Let X be a space such that every λ-open set is open and a function $f : X \to Y$ be a.λ.c. Then for each point $x \in X$ and each filter base $\mathcal{F}$ in X we have $\mathcal{F}$ is λ-converging to x, if and only if the filter $f(\mathcal{F})$ is δ-convergent to $f(x)$.*

Corollary 3.13. *Let X be a space such that every λ-open set is open. Then a function $f : X \to Y$ is a.λ.c if and only if $f : X \to Y_s$ is continuous.*

Observe that generally $O(X) \subset \lambda O(X)$. Thus $f : X \to Y$ is $a.\lambda.c.$ if and only if f is almost continuous in the sense of Singal, whenever every λ-open set of X is open. Now let $f, g : X \to Y$ be functions. If X is such that every λ-open set is open, Y is Hausdorff and f and g are $a.\lambda.c$ functions, then the set $E = \{x \in X : f(x) = g(x)\}$ is closed in X. This is indeed Theorem 4 in [8].

Definition 3.14. [2] *A space X is said to be $\lambda - T_2$ if for every pair of distinct points x and y in X there exist disjoint λ-open sets U and V such that $x \in U$ and $y \in V$.*

Observe that $\lambda - T_2$ is equivalent with T_0 (see [7]).

Theorem 3.15. *If a function $f : X \to Y$ is an $a.\lambda.c.$ injection and Y is Hausdorff, then X is $\lambda - T_2$.*

Proof. Suppose that f is an almost λ-continuous injection. Note that Y_s is Hausdorff. Let x and y be distinct points in X, then $f(x) \neq f(y)$. Hence there exist disjoint open sets V and W of Y_s such that $f(x) \in V$ and $f(y) \in W$. Therefore we obtain $f^{-1}(V) \in \lambda O(X, x)$, $f^{-1}(W) \in \lambda O(X, y)$ and $f^{-1}(V) \cap f^{-1}(W) = \emptyset$. □

Definition 3.16. *For a function $f : X \to Y$, the graph $G(f) = \{(x, f(x)) : x \in X\}$ is said to be strongly almost λ-closed if for each $(x, y) \in X \times Y - G(f)$, there exist $U \in \lambda O(X, x)$ and $V \in RO(Y, y)$ such that $(U \times V) \cap G(f) = \emptyset$.*

Theorem 3.17. *A function $f : X \to Y$ has the strongly almost λ-closed graph if and only if for each $x \in X$ and $y \in Y$ such that $f(x) \neq y$, there exist $U \in \lambda O(X, x)$ and $V \in RO(Y, y)$ such that $f(U) \cap V = \emptyset$.*

Proof. It is an immediate consequence of the above definition. □

Definition 3.18. *A space X is called λ-compact* [2] *if every cover of X by λ-open sets has a finite subcover.*

It should be mentioned that λ-compact is called λO-compact in [6].

Definition 3.19. *Let A be a subset of X, then we say that A is λ-compact relative to X if every cover of A by λ-open sets of X has a finite subcover.*

Let A be a subset of X, we say that A is N-closed relative to X [4] if every cover of A by regular open sets of X has a finite subcover.

Furthermore, X is called **nearly compact** [12] if every regular open cover of X has a finite subcover .

Theorem 3.20. *If $f : X \to Y$ is a.λ.c. and K is λ-compact relative to X, then $f(K)$ is N-closed relative to Y.*

Proof. Let $\{G_\alpha : \alpha \in \nabla\}$ be any cover of $f(K)$ by regular open sets of Y. Then $\{f^{-1}(G_\alpha) : \alpha \in \nabla\}$ is a cover of K by λ-open sets of X. Hence there exists a finite subset ∇_0 of ∇ such that $K \subset \cup\{f^{-1}(G_\alpha) : \alpha \in \nabla_0\}$. Therefore, we obtain $f(K) \subset \{G_\alpha : \alpha \in \nabla_0\}$. This shows that $f(K)$ is N-closed relative to Y. $\square$

Corollary 3.21. *If $f : X \to Y$ is an a.λ.c. surjection and X is λ-compact, then Y is nearly compact.*

Definition 3.22. [10] *A function $f : X \to Y$ is said to be δ-continuous if for each $x \in X$ and open set V containing $f(x)$, there exists an open set U in X containing x such that $f(Int(Cl(U))) \subset Int(Cl(V))$.*

Theorem 3.23. *If $f : X \to Y$ is a.λ.c. and g is δ-continuous, then $g \circ f : X \to Y$ is a.λ.c.*

Proof. The proof is obvious and is omitted. $\square$

Theorem 3.24. *If $f : X \to Y$ is a pre-λ-open surjection and $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is a.λ.c., then g is a.λ.c.*

Proof. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let $G \in RO(Z, (g \circ f)(x))$. Then there exists $U \in \lambda O(X, x)$ such that $g(f(U)) \subset G$. Since f is pre-λ-open in Y, we have that g is *a.λ.c.* at y. $\square$

Theorem 3.25. *The set of all points x of X at which the function $f : X \to Y$ is not a.λ.c. is identical with the union of the λ-boundaries of the inverse of regular open subsets of Y containing $f(x)$.*

Proof. Suppose that f is not *a.λ.c.* at $x \in X$, then there exists $V \in RO(Y, f(x))$ such that for every $U \in \lambda O(X, x)$, $f(U) \cap (Y - V) \neq \emptyset$. This means that for every $U \in \lambda O(X, x)$, we must have $U \cap (X - f^{-1}(V) \neq \emptyset$. Hence, it follows that $x \in Cl_\lambda(X - f^{-1}(V))$. But $x \in f^{-1}(V)$ and hence $x \in Cl_\lambda(f^{-1}(V))$. Thus $x \in Fr_\lambda(f^{-1}(V))$. Suppose that f is *a.λ.c.* at x. Then there exists $U \in \lambda O(X, x)$ such that $f(U) \subset V_1$. Then , we have: $x \in U \subset f^{-1}(f(U)) \subset f^{-1}(V_1)$. This shows that x is a λ-interior point of $f^{-1}(V_1)$. Therefore, we have

$x \notin Cl_\lambda(X - f^{-1}(V_1))$ and $x \notin Fr_\lambda(f^{-1}(V_1))$. This is a contradiction. This means that f is not *a.λ.c.* □

Let A be a subset of X. Then A is said to be H-closed [13] or quasi-H-closed relative to X [11] if for every cover $\{U_i : i \in \nabla\}$ of A by open sets of X, there exists a finite subset ∇_0 of ∇ such that $A \subset \cup\{Cl(U_i) : i \in \nabla_0\}$.

Theorem 3.26. *If $f : X \to Y$ is weakly λ-continuous and K is λ-compact relative to X, then $f(K)$ is quasi-H-closed relative to Y.*

Proof. The proof is similar to that of Theorem 3.20 □

Lemma 3.27. *Let X be λ-compact. If $A \subset X$ is λ-closed, then A is λ-compact relative to X.*

Proof. Let $\{G_\alpha : \alpha \in \nabla\}$ be a cover of A by λ-open sets of X. Note that $(X - A)$ is λ-open and that the set $(X - A) \cup \{G_\alpha : \alpha \in \nabla\}$ is a cover of X by λ-open sets. Since X is λ-compact, the exists a finite subset ∇_0 of ∇ such that the set $(X - A) \cup \{G_\alpha : \alpha \in \nabla_0\}$ is a cover of X by λ-open sets in X. Hence $\{G_\alpha : \alpha \in \nabla_0\}$ is a finite cover of A by λ-open sets in X. □

Theorem 3.28. *Let $f : X \to Y$ be an a.λ.c. bijection. If X is λ-compact and Y is Hausdorff, then f is almost λ-open.*

Proof. Suppose that U is a λ-open subset of X. Then $X - U$ is λ-closed. By Lemma 3.27, $X - U$ is λ-compact relative to X. Therefore $f(X - U)$ is quasi-H-closed relative to Y. Since Y is Hausdorff, $Y - f(U)$ is closed in Y. Hence, $f(U)$ is open in Y. □

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College of Vestsjaelland, South Herrestraede, 11 4200 Slagelse, Denmark
Jafari@stofanet.dk

Department of Mathematical Sciences,
University of South
Africa, P. O. Box 392, Pretoria, 0003,
South Africa
moshosp@unisa.ac.za

P O BOX 2527, Pretoria, 0001, South Africa
kwena.nailana@phutadichaba.co.za

Department of Mathematics, Yatsushiro College of Technology, Yatsushiro, Kumamoto, 866 Japan t.noiri@nifty.com