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STRONGLY AND PERFECTLY CONTINUOUS MULTIFUNCTIONS

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Abstract. The notions of strong continuity of Levine (Amer. Math. Monthly 67(1960), 269) and perfect continuity due to Noiri (Indian J. Pure Appl. Math. 15(3) (1984), 241-250) are extended to the framework of multifunctions. Basic properties of strongly continuous and upper (lower) perfectly continuous multifunctions are studied and their place in the hierarchy of variants of continuity of multifunctions is discussed. The class of upper (lower) perfectly continuous multifunctions properly contains the class of strongly continuous multifunctions and is strictly contained in the class of upper (lower) cl-supercontinuous multifunctions (Applied Gen. Topol.)[5]. Examples are included to reflect upon the distinctiveness of the notions so introduced from the ones that already exist in the mathematical literature. In the process we extend several known results in the literature including those of Ekici, Singh and others to the realm of multifunctions.

1. Introduction

Noiri [14] introduced the concept of a perfectly continuous function.

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Properties of perfectly continuous functions are further elaborated in [11]. The class of perfectly continuous functions properly contains the class of strongly continuous functions of Levine [13] and is strictly contained in the class of cl-supercontinuous (≡clopen continuous) functions introduced by Reilly and Vamanamurthy [15] and further studied by Singh [16]. In this paper we extend the notions of strong continuity and (almost) perfect continuity of functions to the framework of multifunctions, and elaborate on their place in the hierarchy of strong variants of continuity of multifunctions that already exist in the mathematical literature. In the process we extend several known results in the literature including those of Singh [16], Ekici [2] and others to the realm of multifunctions. Section 2 is devoted to preliminaries and basic definitions, wherein we introduce the notions of strongly continuous multifunctions, upper (lower) perfectly continuous multifunctions and upper (lower) almost perfectly continuous multifunctions and discuss the interrelations that exist among them and other strong variants of continuity of multifunctions that already exist in the literature. Examples are included to reflect upon the distinctiveness of the notions so introduced from the ones that already exist in the mathematical literature. In Section 3 we discuss basic properties of strongly continuous multifunctions, and basic properties of upper (almost) perfectly continuous multifunctions are elaborated in Section 4, while lower (almost) perfectly continuous multifunctions are dealt with in Section 5.

2. Preliminaries and basic definitions

Throughout the paper we essentially follow the notations and terminology of Górniewicz [4]. Let $\varphi : X \multimap Y$ be a multifunction from a topological space X into a topological space Y. For a subset B of Y the set $\varphi_+^{-1}(B) = \{x \in X : \varphi(x) \cap B \neq \varnothing\}$ is called ***large inverse image***[4][1] of B and the set $\varphi_-^{-1}(B) = \{x \in X : \varphi(x) \subset B\}$ is called ***small inverse image*** [4] of B. The multifunction φ is called ***upper semicontinuous*** (respectively ***lower semicontinuous***) if $\varphi_-^{-1}(U)$ (respectively $\varphi_+^{-1}(U)$) is an open subset of X for every open set U in

[1] However, what we call "large inverse image $\varphi_+^{-1}(B)$" some authors call it 'lower inverse image' and denote it by $\varphi^-(B)$; and similarly they call "small inverse image $\varphi_-^{-1}(B)$" as 'upper inverse image' and employ the notation $\varphi^+(B)$ for the same.

Y. A subset A of a space X is said to be ***regular open*** if it is the interior of its closure, i.e. $A = \overline{A}^{\circ}$. The complement of a regular open set is referred to as ***regular closed***. A space X is said to have an ***(almost) partition topology*** ([17] [20]) if every (regular) open set in X is closed. A space X is said to be ***extremally disconnected*** [3] if the closure of every open set is open in X. It turns out that an almost partition topology is precisely an extremally disconnected topology.

2.1. **Definitions.** A space X is said to be
(i) ***mildly compact*** [19] if every clopen cover of X has a finite subcover. In [18] Sostak calls mildly compact spaces as clustered spaces;
(ii) ***cl-paracompact*** [5] (***cl-paraLindelöf*** [5]) if every clopen cover of X has locally finite (locally countable) open refinement which covers X; and
(iii) a ***P-space*** [3] if every G_δ set in X is open in X.

2.2. **Definition.** ([5] [16]) The graph Γ_φ of a multifunction $\varphi : X \multimap Y$ is said to be ***cl-closed with respect to*** X if for each $(x, y) \in (X \times Y) \backslash \Gamma_\varphi$ there exist a clopen set U containing x and an open set V containing y such that $(U \times V) \cap \Gamma_\varphi = \varnothing$.

2.3. **Definition.** [10] A subspace S of a space X is said to be ***δ-embedded*** in X if every regular open set in S is the intersection of a regular open set in X with S, or equivalently every regular closed set in S is the intersection of a regular closed set in X with S.

2.4. **Definitions.** A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is said to be
(i) ***upper cl-supercontinuous*** [5] (***upper z-supercontinuous*** ([1] [9])) at $x \in X$ if for each open set V with $\varphi(x) \subset V$, there exists a clopen (cozero) set U containing x such that $\varphi(U) \subset V$;
(ii) ***lower cl-supercontinuous*** [5] (***lower z-supercontinuous***([1] [9])) at $x \in X$ if for each open set V with $\varphi(x) \cap V \neq \varnothing$, there exists a clopen (cozero) set U containing x such that $\varphi(z) \cap V \neq \varnothing$ for each $z \in U$;
(iii) ***upper almost cl-supercontinuous*** ([6] [10] (***upper almost z-supercontinuous*** ([8] [12])) at $x \in X$ if for each regular open set V with $\varphi(x) \subset V$, there exists a clopen (cozero) set U containing x such that $\varphi(U) \subset V$; and

(iv) ***lower almost cl-supercontinuous*** ([6] [10]) (***lower almost z-supercontinuous*** ([8] [12])) at $x \in X$ if for each regular open set V with $\varphi(x) \cap V \neq \varnothing$, there exists a clopen (cozero) set U containing x such that $\varphi(\mathrm{z}) \cap \mathrm{V} \neq \varnothing$ for each $z \in U$.

2.5. **Definitions.** A multifunction $\varphi : X \multimap Y$ from a topological space X into a topological space Y is said to be
(i) ***strongly continuous*** if $\varphi_-^{-1}(B)$ is clopen in X, equivalently $\varphi_+^{-1}(B)$ is clopen in X for every subset B of Y;
(ii) ***upper (almost) perfectly continuous*** if $\varphi_-^{-1}(U)$ is clopen in X for every (regular) open subset U of Y;
(iii) ***lower (almost) perfectly continuous*** if $\varphi_+^{-1}(U)$ is clopen in X for every (regular) open subset U of Y;
(iv) ***upper (almost) completely continuous*** [7] if $\varphi_-^{-1}(U)$ is a regular open set in X for every (regular) open set U in Y; and
(v) ***lower (almost) completely continuous*** [7] if $\varphi_+^{-1}(U)$ is a regular open set in X for every (regular) open set U in Y.

The following diagram well illustrates the interrelations that exist among various strong variants of continuity of multifunctions defined in Definitions 2.4 and 2.5.

$$
\begin{tikzcd}
& \text{strongly continuous} \arrow[d] & \\
& \text{upper (lower) perfectly continuous} \arrow[r] \arrow[dl] \arrow[dr] & \text{upper (lower) completely continuous} \\
\text{upper (lower) cl-supercontinuous} \arrow[d] & & \text{upper (lower) almost perfectly continuous} \arrow[d] \\
\text{upper (lower) z-supercontinuous} \arrow[dr] & & \text{upper (lower) almost cl-supercontinuous} \arrow[dl] \\
& \text{upper (lower) almost z-supercontinuos} &
\end{tikzcd}
$$

FIGURE 1

However, none of the above implications is reversible as is well illustrated by the examples in the sequel.

2.6. **Example.** Let $X = \{a, b, c\}$ be endowed with the topology $\Im_X = \{\varnothing, X, \{a\}, \{b, c\}, \{c\}, \{a, c\}\}$ and let $Y = \{x, y\}$ be equipped with the topology $\Im_Y = \{\varnothing, Y, \{x\}\}$. Define a multifunction $\varphi : X \multimap Y$ by $\varphi(a) = \{x\}$, $\varphi(b) = \{y\}$, $\varphi(c) = \{x, y\}$. Clearly φ is upper perfectly continuous. But $\varphi_-^{-1}(\{y\}) = \{b\}$ is not clopen, and so the φ multifunction is not strongly continuous.

2.7. **Example.** Let $X = \{a, b, c\}$ be equipped with the topology $\Im_X = \{\varnothing, X, \{b\}, \{b, c\}, \{c\}, \{a, b\}\}$ and let $Y = \{x, y\}$ be endowed with the topology $\Im_Y = \{\varnothing, Y, \{x\}\}$. Define a multifunction $\varphi : X \multimap Y$ by $\varphi(a) = \{x\}$, $\varphi(b) = \{x, y\}$, $\varphi(c) = \{y\}$. Clearly φ is lower perfectly continuous, $\varphi_-^{-1}(\{x\}) = \{a\}$ is not clopen, and so the multifunction φ is not strongly continuous.

2.8. **Example.** Let $X = Y = R$, the set of real numbers and let $\Im$ be the upper limit topology [20] on X and let U the usual topology on Y. Define a multifunction $\varphi : X \multimap Y$ by $\varphi(x) = \{x\}$ for each $x \in X$. Then φ is upper (lower) cl-supercontinuous but φ is not upper (lower) perfectly continuous. Moreover, φ is not upper (lower) almost perfectly continuous.

2.9. **Example.** Let $X = \{a, b, c\}$ be endowed with the topology $\Im_X = \{\varnothing, X, \{a\}\}$ and let $Y = \{x, y, z\}$ be given the topology $\Im_Y = \{\varnothing, Y, \{x\}\}$. Define $\varphi : X \multimap Y$ by $\varphi(a) = \{x, y\}$, $\varphi(b) = \{x\}$, $\varphi(c) = \{y, z\}$. Clearly φ is upper almost cl-supercontinuous but φ is not upper cl-supercontinuous, since $\{x\}$ is an open set in Y but $\varphi_-^{-1}(\{x\}) = \{b\}$ is not cl-open in X. Moreover, φ is not lower cl-supercontinuous. Further observe that φ is upper (lower) almost perfectly continuous but not upper (lower) perfectly continuous.

2.10. **Example.** Let X be a completely regular space which is not zero dimensional. Then every upper (lower) semicontinuous multifunction $\varphi : X \multimap Y$ is upper (lower) z-supercontinuous but not necessarily upper (lower) cl-supercontinuous.

2.11. **Example.** Let $X = \{a, b, c, d\}$ be equipped with the topology $\Im_X = \{\varnothing, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and let $Y = \{p, q, r, s\}$ be endowed with topology $\Im_Y = \{\varnothing, Y, \{p\}, \{r\}, \{p, q\}, \{p, r\}, \{p, q, r\}, \{p, r, s\}\}$. Define $\varphi : X \multimap Y$ by $\varphi(a) = \{q\}$, $\varphi(b) = \{p, q\}$, $\varphi(c) = \{p\}$, $\varphi(d) = \{p\}$. The sets $\{r\}$ and $\{p, q\}$ are regular open sets. Clearly φ is upper (lower) almost z-supercontinuous but not upper (lower) z-supercontinuous.

3. Strongly continuous multifunctions

3.1. **Theorem.** If $\varphi : X \multimap Y$ is a strongly continuous multifunction and $\varphi : Y \multimap Z$ is any multifunction, then $\psi o \varphi$ is strongly continuous.

3.2. **Theorem.** Let $\varphi : X \multimap Y$ be strongly continuous and let $A \subset X$. Then $\varphi_{|A} : A \multimap Y$ is strongly continuous.
Proof. Let B be a subset of Y. Since φ is strongly continuous, $\varphi_-^{-1}(B)$ is clopen in X. Again since $(\varphi_{|A})_-^{-1}(B) = A \cap \varphi_-^{-1}(B)$ is clopen in A, the multifunction $\varphi_{|A} : A \multimap Y$ is strongly continuous.

3.3. **Theorem.** Let $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ be two strongly continuous multifunctions. Then $\varphi \cup \psi : X \multimap Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is strongly continuous.
Proof. Let B be a subset of Y. Since φ and ψ are strongly continuous, $\varphi_-^{-1}(B)$ and $\psi_-^{-1}(B)$ are clopen sets in X. Since $(\varphi \cup \psi)_-^{-1}(B) = \varphi_-^{-1}(B) \cap \psi_-^{-1}(B)$ and since finite intersection of clopen sets is clopen, $(\varphi \cup \psi)_-^{-1}(B)$ is clopen in X. Thus $\varphi \cup \psi$ is strongly continuous.

4. Upper(almost) perfectly continuous multifunctions

4.1. **Theorem.** A multifunction $\varphi : X \multimap Y$ is upper (almost) perfectly continuous if and only if for every (regular) closed set $A \subset Y$ the set $\varphi_+^{-1}(A)$ is a clopen subset of X.
Proof. Suppose $\varphi : X \multimap Y$ is upper (almost) perfectly continuous. Let A be a (regular) closed subset of Y. Then $Y - A$ is a (regular) open set in Y. Since φ is upper (almost) perfectly continuous,$\varphi_-^{-1}(Y - A) = X - \varphi_+^{-1}(A)$ is clopen in X and so $\varphi_+^{-1}(A)$ is clopen in X.
Conversely suppose that $\varphi_+^{-1}(A)$ is clopen in X for every (regular) closed $A \subset Y$. Let U be a (regular) open subset of Y. Then $Y - U$ is a (regular) closed subset of Y. By hypothesis, $\varphi_+^{-1}(Y - U) = X - \varphi_-^{-1}(U)$

is clopen in X and so $\varphi_-^{-1}(U)$ is clopen in X. Thus φ is upper (almost) perfectly continuous.

4.2. **Theorem.** Let $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ be multifunctions. The following statements are true.
(a) If φ is upper perfectly continuous and ψ is upper semicontinuous, then the multifunction $\psi \circ \varphi$ is upper perfectly continuous.
(b) If φ is upper almost perfectly continuous and ψ is upper completely continuous, then $\psi \circ \varphi$ is upper perfectly continuous.
(c) If φ is upper almost perfectly continuous and ψ is upper almost completely continuous, then their composition $\psi \circ \varphi$ is upper almost perfectly continuous.
(d) If φ is upper perfectly continuous and ψ is upper almost continuous, then $\psi \circ \varphi$ is upper almost perfectly continuous.
Proof. The assertions (b) and (c) are proved in [7]. To prove (a) let B be an open subset of Z. Since ψ is upper semicontinuous, $\psi_-^{-1}(B)$ is open in Y. Again since φ is upper perfectly continuous, $\varphi_-^{-1}(\psi_-^{-1}(B)) = (\psi \circ \varphi)_-^{-1}(B)$ is clopen in X and so $\psi \circ \varphi$ is upper perfectly continuous.
(d) Let B be a regular open subset of Z. Since ψ is upper almost continuous, $\psi_-^{-1}(B)$ is open in Y. Again since φ is upper perfectly continuous, $\varphi_-^{-1}(\psi_-^{-1}(B)) = (\psi \circ \varphi)_-^{-1}(B)$ is clopen in X and so $\psi \circ \varphi$ is upper almost perfectly continuous.

4.3. **Corollary.** If $\varphi : X \multimap Y$ is upper perfectly continuous and if Z is a superspace of Y, then $\psi : X \multimap Z$ defined by $\psi(x) = \varphi(x)$ for each $x \in X$ is upper perfectly continuous.
Proof. This is immediate in view of Theorem 4.1 and the fact that $\psi = \imath \circ \varphi$, where $\imath$ denotes the inclusion mapping which is upper semicontinuous.

The following theorem embodies a sufficient condition for the preservation of upper almost perfectly continuity under the expansion of range.

4.4. **Theorem.** Let $\varphi : X \multimap Y$ be an upper almost perfectly continuous multifunction and let Z be the superspace of Y such that the intersection of every regular open in Z with Y is a regular open set in Y. Then the multifunction $\psi : X \multimap Z$ defined by $\psi(x) = \varphi(x)$ for each $x \in X$ is upper almost perfectly continuous.

Proof. Let W be a regular open set in Z. By hypothesis, $W \cap Y$ is regular open set in Y. Again since $\varphi : X \multimap Y$ is upper almost perfectly continuous, $\varphi_-^{-1}(W \cap Y)$ is a clopen set in X. Now it is clear that $\psi_-^{-1}(W) = \varphi_-^{-1}(W \cap Y)$ and so $\psi : X \multimap Z$ is upper almost perfectly continuous.

4.5. **Theorem.** If $\varphi : X \multimap Y$ is upper perfectly continuous and $\varphi(X)$ is endowed with the subspace topology, then the multifunction $\varphi : X \multimap \varphi(X)$ is upper perfectly continuous.
Proof. This is immediate in view of the fact that for every open set $V \subset Y$, $\varphi_-^{-1}(V) = \varphi_-^{-1}(V \cap \varphi(X))$.

4.6. **Theorem.** If a multifunction $\varphi : X \multimap Y$ is upper almost perfectly continuous and $\varphi(X)$ is δ-embedded in Y, then the multifunction $\varphi : X \multimap \varphi(X)$ is also upper almost perfectly continuous.
Proof. Let V be a regular open set in $\varphi(X)$. Since $\varphi(X)$ is δ-embedded in Y, there exist a regular open set W in Y such that $V = W \cap \varphi(X)$. Since φ is upper almost perfectly continuous, $\varphi_-^{-1}(W)$ is clopen in X. Now $\varphi_-^{-1}(V) = \varphi_-^{-1}(W \cap \varphi(X)) = \varphi_-^{-1}(W)$ and hence φ is upper almost perfectly continuous.

4.7. **Theorem.** Let $\varphi : X \multimap Y$ be an upper (almost) perfectly continuous multifunction and let $A \subset X$. Then the multifunction $\varphi_A = \varphi_{|A} : A \multimap Y$ is upper (almost) perfectly continuous.
Proof. Let U be a (regular) open set in Y. Since φ is upper (almost) perfectly continuous, $\varphi_-^{-1}(U)$ is clopen in X. Since $(\varphi_A)_-^{-1}(U) = A \cap \varphi_-^{-1}(U)$, which is clopen in A and so φ_A is upper (almost) perfectly continuous.

4.8. **Theorem.** Let $\Omega = \{X_\alpha : \alpha \in \Lambda\}$ be a locally finite clopen cover of X and let $\varphi : X \multimap Y$ be a multifunction. For each $\alpha \in \Lambda$, let $\varphi_\alpha = \varphi_{|X_\alpha} : X_\alpha \multimap Y$ be the restriction of φ to X_α. Then φ is upper (almost) perfectly continuous if and only if each φ_α is upper (almost) perfectly continuous.
Proof. Necessity is easy to see. To prove sufficiency, let V be a (regular) open set in Y. Then $\varphi_-^{-1}(V) = \bigcup_{\alpha\in\Lambda}(\varphi_\alpha)_-^{-1}(V) = \bigcup_{\alpha\in\Lambda}(\varphi_-^{-1}(V) \cap X_\alpha)$. Since each $\varphi_-^{-1}(V) \cap X_\alpha$ is clopen in X_α and hence in X. Thus $\varphi_-^{-1}(V)$ is open being the union of clopen sets. In view of local finiteness of Ω, the collection $\{\varphi_-^{-1}(V) \cap X_\alpha : \alpha \in \Lambda\}$ is a locally finite collection of clopen sets. Therefore $\varphi_-^{-1}(V)$ is also closed being the

union of a locally finite collection of clopen sets and hence clopen. Consequently, φ is upper (almost) perfectly continuous.

4.9. **Theorem.** Let $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ be two upper (almost) perfectly continuous multifunctions. Then $\varphi \cup \psi : X \multimap Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is upper (almost) perfectly continuous.

Proof. Let U be a (regular) open set in Y. Since φ and ψ are upper (almost) perfectly continuous, $\varphi_-^{-1}(U)$ and $\psi_-^{-1}(U)$ are clopen sets in X. Again since $(\varphi \cup \psi)_-^{-1}(U) = \varphi_-^{-1}(U) \cap \psi_-^{-1}(U)$ and since the finite intersection of clopen sets is clopen, $(\varphi \cup \psi)_-^{-1}(U)$ is clopen in X. Thus $\varphi \cup \psi$ is upper (almost) perfectly continuous.

4.10. **Theorem.** Let $\varphi : X \multimap Y$ be upper perfectly continuous. Then $[\varphi_+^{-1}(B)]_{cl} \subset \varphi_+^{-1}(\overline{B})$ for every subset B of Y.

Proof. Let B be any subset of Y. Then $\overline{B}$ is a closed subset of Y. Since φ is upper perfectly continuous, $\varphi_+^{-1}(\overline{B})$ is clopen in X. Again since $\varphi_+^{-1}(B) \subset \varphi_+^{-1}(\overline{B})$, $[\varphi_+^{-1}(B)]_{cl} \subset [\varphi_+^{-1}(\overline{B})]_{cl} = \varphi_+^{-1}(\overline{B})$.

4.11. **Remark.** Converse of Theorem 4.10 is not true. For example, let $X = Y = R$, the set of real numbers and let $\Im$ be the upper limit topology on X and let U be the usual topology on Y. Define $\varphi : X \multimap Y$ by $\varphi(x) = \{x\}$ for each $x \in X$. Then it is easily verified that for every subset B of Y, $[\varphi_+^{-1}(B)]_{cl} \subset \varphi_+^{-1}(\overline{B})$ but φ is not upper perfectly continuous.

4.12. **Theorem.** Let $\varphi : X \multimap Y$ be a multifunction and let $g : X \multimap X \times Y$ defined by $g(x) = \{x\} \times \varphi(x)$ for each $x \in X$ be the graph multifunction. If g is upper (almost) perfectly continuous, then φ is upper (almost) perfectly continuous and the space X is equipped with an (almost) partition topology.

Proof. Suppose that g is upper (almost) perfectly continuous. By Theorem4.1 the multifunction $\varphi = p_y \circ g$ is upper (almost) perfectly continuous, where $p_y : X \times Y \longrightarrow Y$ denotes the projection mapping which is upper semicontinuous as well as upper almost completely continuous. Now to show that X is endowed with an (almost) partition topology, let U be a (regular) open set in X. Then $U \times Y$ is a (regular) open set in $X \times Y$. Since g is upper (almost) perfectly continuous, $g_-^{-1}(U \times Y)$ is clopen in X. It is easily verified that $g_-^{-1}(U \times Y) = U$, and so U is clopen in X. Thus X is endowed with an (almost) partition topology.

4.13. **Theorem.** If $\varphi : X \multimap Y$ is an upper (almost) perfectly continuous multifunction where Y is a regular space and $\varphi(x)$ is closed for each $x \in X$, then the graph Γ_φ of φ is a cl-closed subset of $X \times Y$ with respect to X.
Proof. Let $(x, y) \notin \Gamma_\varphi$. Then $y \notin \varphi(x)$. Since Y is a regular space, there exist disjoint open sets V_y and $V_{\varphi(x)}$ containing y and $\varphi(x)$, respectively. It is easily verified that the sets V_y and $V_{\varphi(x)}$ may be chosen to be regular open. Since φ is upper (almost) perfectly continuous, $U_x = \varphi_-^{-1}(V_{\varphi(x)})$ is a clopen set containing x. We assert that $(U_x \times V_y) \cap \Gamma_\varphi = \varnothing$. For, if $(h, k) \in (U_x \times V_y) \cap \Gamma_\varphi$, then $h \in \varphi_-^{-1}(V_x)$, $k \in V_y$ and $k \in \varphi(h)$. Hence $\varphi(h) \subset V_{\varphi(x)}$ and $k \in \varphi(h) \cap V_y$ which contradicts the fact that V_y and $V_{\varphi(x)}$ are disjoint. Thus the graph Γ_φ is cl-closed in $X \times Y$ with respect to X.

4.14. **Theorem.** Let $\varphi : X \multimap Y$ be an upper perfectly continuous multifunction such that $\varphi(x)$ is compact for each $x \in X$. If A is a mildly compact set in X, then $\varphi(A)$ is compact.
Proof. Let Ω be an open cover of $\varphi(A)$ .Then Ω is also an open cover of $\varphi(a)$ for each $a \in A$. Since each $\varphi(a)$ is compact, there exists a finite subset $\beta_a \subset \Omega$ such that $\varphi(a) \subset \bigcup_{B\in\beta_a} B = V_a$(say). Again since φ is upper perfectly continuous, $U_a = \varphi_-^{-1}(V_a)$ a clopen set containing a. Let $Q = \{U_a \mid a \in A\}$. Then Q is a clopen covering of A. Since A is mildly compact, there exists a finite subset $\{a_1, ..., a_n\}$ of A such that $A \subset \bigcup_{i=1}^n U_{a_i} \subset \bigcup_{i=1}^n \varphi_-^{-1}(V_{a_i})$. Therefore $\varphi(A) \subset \varphi(\bigcup_{i=1}^n \varphi_-^{-1}(V_{a_i})) = \bigcup_{i=1}^n \varphi(\varphi_-^{-1}(V_{a_i})) \subset \bigcup_{i=1}^n V_{a_i}$, where $V_{a_i} = \bigcup_{B\in\beta_{a_i}} B$, $i = 1, ..., n$ and each β_{a_i} is finite. Thus $\varphi(A)$ is compact.

We may recall that a multifunction $\varphi : X \multimap Y$ is called ***nonmingled*** [21] if for $x, y \in X$, $x \neq y$ the image sets $\varphi(x)$ and $\varphi(y)$ are either disjoint or identical.

4.15. **Theorem.** Let $\varphi : X \multimap Y$ be a closed, open and upper perfectly continuous nonmingled multifunction such that $\varphi(x)$ is paracompact for each $x \in X$. If A is cl-paracompact, then $\varphi(A)$ is paracompact.

Proof. Let Ψ be an open cover of $\varphi(A)$. Then Ψ is also an open cover of $\varphi(x)$ for each $x \in A$. Since $\varphi(x)$ is paracompact, Ψ has a locally finite open refinement Ψ_x such that $\varphi(x) \subset \bigcup \Psi_x = V_x$(say). Since φ is upper perfectly continuous, $U_x = \varphi_-^{-1}(V_x)$ is a clopen set containing x. Now $\{U_x \mid x \in A\}$ is a clopen cover of A. Since A is cl- paracompact, it has a locally finite open refinement $\Omega = \{W_\alpha \mid \alpha \in \Lambda\}$ such that $A \subset \bigcup_{\alpha\in\Lambda} W_\alpha$. So for each $\alpha \in \Lambda$ there exists $x_\alpha \in A$ such that $W_\alpha \subset U_{x_\alpha}$ and hence $\varphi(W_\alpha) \subset \varphi(U_{x_\alpha}) \subset \bigcup \Psi_{x_\alpha}$. Let $\Re_\alpha = \{\varphi(W_\alpha) \cap V \mid V \in \Psi_{x_\alpha}\}$, and let $\Re = \{R \mid R \in \Re_\alpha, \alpha \in \Lambda\}$. We shall show that $\Re$ is a locally finite open refinement of Ψ which covers $\varphi(A)$. Since φ is open, each $\varphi(W_\alpha)$ is open and so each $R \in \Re$ is open. Let $R \in \Re$. Then $R \in \Re_\alpha$ for some $\alpha \in \Lambda$, i.e. $R = \varphi(W_\alpha) \cap V \subset V \subset U$ for some $U \in \Psi$. This shows that $\Re$ is an open refinement of Ψ. Now to show that $\Re$ is locally finite, let $y \in \varphi(A)$. Then $y \in \varphi(x)$ for some $x \in A$. Since Ω is locally finite, for each $x \in A$ we can choose an open neighborhood G_x of x which intersects only finitely many members $W_{\alpha_1}, W_{\alpha_2} ..., W_{\alpha_n}$ of Ω. It follows that $H_0 = \varphi(G_x)$ is an open neighborhood of y which intersects only finitely many members $\varphi(W_{\alpha_1})$, $\varphi(W_{\alpha_2}) ..., \varphi(W_{\alpha_n})$ of the family $\{\varphi(W_\alpha) \mid \alpha \in \Lambda\}$. Furthermore, each $\Re_{\alpha_k}$ $(k = 1, ..., n)$ is locally finite, hence there exists an open neighborhood H_k $(k = 1, ..., n)$ of y which intersects only finitely many members of $\Re_{\alpha_k}$ $(k = 1, ..., n)$. Finally let $H = \bigcap_{k=1}^n H_k$. Then H is an open neighborhood of y which intersects at most finitely many members of $\Re$. Hence $\Re$ is locally finite. Thus $\varphi(A) \subset \varphi(\bigcup_{\alpha\in\Lambda} W_\alpha) \subset \bigcup_{\alpha\in\Lambda} \varphi(W_\alpha) \subset \bigcup_{\alpha\in\Lambda}(\bigcup \Re_\alpha) = \bigcup\{R : R \in \Re\}$. So $\Re$ is a locally finite open refinement of Ψ which covers $\varphi(A)$ and thus $\varphi(A)$ is paracompact.

4.16. **Corollary.** Let $\varphi : X \multimap Y$ be a closed, open, upper perfectly continuous, nonmingled multifunction from a space X onto Y such that $\varphi(x)$ is a paracompact set in Y for each $x \in X$. If X is a cl-paracompact space, then Y is paracompact.

4.17. **Theorem.** $\varphi : X \multimap Y$ be a closed, open, and upper perfectly continuous, nonmingled multifunction from a space X into a P-space Y such that $\varphi(x)$ is paraLindelöf. If A is a $cl-$ paraLindelöf, then so is $\varphi(A)$.

Proof. of Theorem 4.17 is similar to that of Theorem 4.15 and hence omitted.

4.18. **Theorem.** Let $\varphi : X \multimap Y$ be an upper (almost) perfectly continuous multifunction such that $\varphi(x) \cap \varphi(y) = \varnothing$ for each $x \neq y$ in X and $\varphi(x)$ is closed for each $x \in X$. If Y is a normal space, then X is an ultra Hausdorff space.

Proof. Let $x, y \in X$, $x \neq y$. Then $\varphi(x) \cap \varphi(y) = \varnothing$. Since Y is normal, there exist disjoint open sets U_1 and V_1 containing $\varphi(x)$ and $\varphi(y)$ respectively. Then $U = \overline{U_1}^{\circ}$ and $V = \overline{V_1}^{\circ}$ are disjoint regular open sets containing $\varphi(x)$ and $\varphi(y)$, respectively. Since φ is upper (almost) perfectly continuous, $\varphi_-^{-1}(U)$ and $\varphi_-^{-1}(V)$ are disjoint clopen sets containing x and y respectively and so X is ultra Hausdorff.

4.19. **Theorem.** Let $\varphi, \psi : X \multimap Y$ be upper (almost) perfectly continuous multifunctions from a topological space X into a normal space Y such that $\varphi(x)$ and $\psi(x)$ are closed for each $x \in X$. Then the set $E = \{x \in X : \varphi(x) \cap \psi(x) \neq \varnothing\}$ is a cl-closed subset of X.

Proof. To prove that E is a cl-closed, we shall show that $X \setminus E$ is cl-open. To this end, let $x \in X \setminus E$. Then $\varphi(x) \cap \psi(x) = \varnothing$. Since Y is normal, there exist disjoint open sets U_1 and V_1 containing $\varphi(x)$ and $\psi(x)$ respectively. Then $U = \overline{U_1}^{\circ}$ and $V = \overline{V_1}^{\circ}$ are disjoint regular open sets containing $\varphi(x)$ and $\psi(x)$ respectively. Since φ and ψ are upper (almost)perfectly continuous, $\varphi_-^{-1}(U)$ and $\psi_-^{-1}(V)$ are clopen sets containing x. Let $G_1 = \varphi_-^{-1}(U)$ and $G_2 = \psi_-^{-1}(V)$. Then $G = G_1 \cap G_2$ is a clopen set containing x. Since U and V are disjoint, $G \subset X \setminus E$ and hence $X \setminus E$ is cl-open.

4.20. **Theorem.** Let $\varphi : X \multimap Y$ be an upper (almost) perfectly continuous multifunction from a topological space X into a normal space Y such that $\varphi(x)$ is closed for each $x \in X$. Then the set $A = \{(x, y) \in X \times X : \varphi(x) \cap \varphi(y) \neq \varnothing\}$ is a cl-closed subset of $X \times X$.

Proof. Let $(x, y) \notin A$. Then $\varphi(x) \cap \varphi(y) = \varnothing$. Since Y is normal, there exist disjoint open sets U and V containing $\varphi(x)$ and $\varphi(y)$ respectively. As in the proof of Theorem 4.19 the sets U and V may be taken to be regular open. Since φ is upper (almost) perfectly continuous, $\varphi_-^{-1}(U)$ and $\varphi_-^{-1}(V)$ are disjoint clopen sets containing x and y respectively. Let $G_1 = \varphi_-^{-1}(U)$ and $G_2 = \varphi_-^{-1}(V)$. Then $G_1 \times G_2$ is a clopen set in $X \times X$ containing (x, y). We claim that $(G_1 \times G_2) \cap A = \varnothing$. For if $(G_1 \times G_2) \cap A \neq \varnothing$. Then $(x, y) \in G_1 \times G_2$ and $(x, y) \in A$. This in turn implies that U and V are not disjoint which is a contradiction. Thus $(G_1 \times G_2) \cap A = \varnothing$ and so $G_1 \times G_2 \subset X \times X \setminus A$. Hence $X \times X \setminus A$

being the union of clopen sets is cl-open and so A is a cl-closed subset of $X \times X$.

5. Lower (almost) perfectly continuous multifunctions

5.1. **Theorem.** A multifunction $\varphi : X \multimap Y$ is lower (almost) perfectly continuous if and only if for every (regular) closed set $A \subset Y$ the set $\varphi_{-}^{-1}(A)$ is a clopen subset of X.
Proof is similar to that of theorem 4.1 except for obvious modifications and hence omitted.

5.2. **Theorem.** Let $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ be multifunctions. The following statements are true.
(a) If φ lower perfectly continuous and ψ is lower semicontinuous multifunction, then the multifunction $\psi \circ \varphi$ is lower perfectly continuous.
(b) If φ is lower almost perfectly continuous and ψ is lower completely continuous, then $\psi \circ \varphi$ is lower perfectly continuous.
(c) If φ is lower almost perfectly continuous and ψ is lower almost completely continuous, then their composition $\psi \circ \varphi$ is lower almost perfectly continuous.
(d) If φ is lower perfectly continuous and ψ is lower almost continuous, then $\psi \circ \varphi$ is lower almost perfectly continuous.
Proof. The assertion (b) and (c) are dealt with in [7]. To prove (a) let B be an open subset of Z. Since ψ is lower semicontinuous, $\psi_{+}^{-1}(B)$ is open in Y. Again since φ is lower perfectly continuous, $\varphi_{+}^{-1}(\psi_{+}^{-1}(B)) = (\psi \circ \varphi)_{+}^{-1}(B)$ is clopen in X and so $\psi \circ \varphi$ is lower perfectly continuous.
(d) Let B be a regular open subset of Z. Since ψ is lower almost continuous, $\psi_{+}^{-1}(B)$ is open in Y. Again since φ is lower perfectly continuous, $\varphi_{+}^{-1}(\psi_{+}^{-1}(B)) = (\psi \circ \varphi)_{+}^{-1}(B)$ is clopen in X and so $\psi \circ \varphi$ is lower almost perfectly continuous.

5.3. **Theorem.** Let $\varphi : X \multimap Y$ be a (almost) lower perfectly continuous multifunction and let $A \subset X$. Then the multifunction $\varphi_{|A} : A \multimap Y$ is lower (almost) perfectly continuous.
Proof is similar to that of Theorem 4.7 except for obvious changes.

5.4. **Theorem.** If $\varphi : X \multimap Y$ and $\psi : X \multimap Y$ are lower (almost) perfectly continuous multifunctions, then the multifunction $\varphi \cup \psi : X \multimap Y$ defined by $(\varphi \cup \psi)(x) = \varphi(x) \cup \psi(x)$ for each $x \in X$, is lower (almost) perfectly continuous.

Proof is similar to that of Theorem 4.9 except for obvious modifications.

5.5. **Corollary.** If a multifunction $\varphi : X \multimap Y$ is lower almost perfectly continuous and $\varphi(X)$ is δ-embedded in Y, then the multifunction $\varphi : X \multimap \varphi(X)$ is lower almost perfectly continuous.

Proof. Let V be a regular open set in $\varphi(X)$. Since $\varphi(X)$ is δ-embedded, there exist a regular open set W in Y such that $V = W \cap \varphi(X)$. Since φ is lower almost perfectly continuous, $\varphi_+^{-1}(W)$ is clopen in X. Now $\varphi_+^{-1}(V) = \varphi_+^{-1}(W \cap \varphi(X)) = \varphi_+^{-1}(W)$ and hence φ is lower almost perfectly continuous.

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