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RELATED FIXED POINT THEOREMS FOR MAPPINGS SATISFYING CONTRACTIVE CONDITIONS OF INTEGRAL TYPE

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Abstract. We prove related fixed point theorems for mappings satisfying contractive conditions of integral type in two complete metric spaces which generalize Theorem 1 of [4] and Theorem 2 of [5].

1. Introduction

The following Theorems were proved by [4] and [5] respectively.

Theorem 1. *Let (X, d) and (Y, ρ) be complete metric spaces, let T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$\begin{aligned} \rho(Tx, TSy) &\leq c \max\{d(x, Sy), \rho(y, Tx), \rho(y, TSy)\}, \\ d(Sy, STx) &\leq c \max\{\rho(y, Tx), d(x, Sy), d(x, STx)\} \end{aligned}$$

for all x in X and y in Y, where $0 \leq c < 1$. Then, ST has a unique fixed point u in X and TS has a unique fixed point v in Y. Further, $Tu = v$ and $Sv = u$.

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Theorem 2. *Let (X,d) and (Y,ρ) be complete metric spaces, let T be a continuous mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities*

$$\begin{aligned} d(STx,STx') &\leq c\max\{d(x,x'),d(x,STx),d(x',STx'),\rho(Tx,Tx')\}, \\ d(TSy,TSy') &\leq c\max\{\rho(y,y'),\rho(y,TSy),\rho(y',TSx'),d(Sy,Sy')\} \end{aligned}$$

for all x,x' in X and y,y' in Y, where $0 \leq c < 1$. Then, ST has a unique fixed point z in X and TS has a unique fixed point ω in Y. Further, $Tz=\omega$ and $S\omega=z$.

Several authors proved fixed point and common fixed point theorems in metric spaces, see [1], [2], [3], [6] and [7].

It is our purpose in this paper to prove related fixed point theorems for mappings satisfying contractive conditions of integral type in two complete metric spaces. Our theorems generalize Theorem 1 of [4] and Theorem 2 of [5].

2. Main Results

Now, we prove the following a related fixed point Theorem in two complete metric spaces.

Theorem 3. *Let (X,d) and (Y,ρ) be complete metric spaces, T a mapping of X into Y and S a mapping of Y into X satisfying the inequalities*

$$\int_0^{d(Sy,STx)} \varphi(t)dt \leq c\int_0^{\max\{\rho(y,Tx),d(x,Sy),d(x,STx)\}} \varphi(t)dt, \tag{2.1}$$

$$\int_0^{\rho(Tx,TSy)} \varphi(t)dt \leq c\int_0^{\max\{d(x,Sy),\rho(y,Tx),\rho(y,TSy)\}} \varphi(t)dt \tag{2.2}$$

for all x in X and y in Y, where $0 \leq c < 1$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is non-increasing, a Lebesgue integrable mapping which is summable in each compact subset of $\mathbb{R}_+$ and such that

$$\int_0^{\epsilon} \varphi(t)dt > 0 \quad \textit{for each} \quad \epsilon > 0. \tag{2.3}$$

Then, ST has a unique fixed point u in X and TS has a unique fixed point v in Y. Further, $Tu=v$ and $Sv=u$.

Proof. Let x_0 be an arbitrary point in X. We define the sequences $\{x_n\}$and $\{y_n\}$ in X and Y inductively by

$$x_n = Sy_n \text{ and } y_n = Tx_{n-1}, \text{ for } n = 1, 2, ... \tag{2.4}$$

If $x_n = x_{n+1}$ for some n, we can put $u = x_n$ and then putting $v = Tu$, we obtain $STu = u$ and $TSv = v$. Similarly, if $y_n = y_{n+1}$ for some n, we can put $v = y_n$ and then putting $u = Sv$, we get $STu = u$ and $TSv = v$. We will now suppose that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for $n = 0, 1, 2, ...$

Using (2.1) and (2.4) we have

$$\begin{aligned}
\int_0^{d(x_n,x_{n+1})} \varphi(t)dt &= \int_0^{d(Sy_n,STx_n)} \varphi(t)dt \\
&\leq c\int_0^{\max\{\rho(y_n,Tx_n),d(x_n,Sy_n),d(x_n,STx_n)\}} \varphi(t)dt \\
&= c\int_0^{\max\{\rho(y_n,y_{n+1}),0,d(x_n,x_{n+1})\}} \varphi(t)dt.
\end{aligned}$$

Then

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq c\int_0^{\rho(y_n,y_{n+1})} \varphi(t)dt. \tag{2.5}$$

Using inequality (2.2) and (2.4) we get

$$\begin{aligned}
\int_0^{\rho(y_n,y_{n+1})} \varphi(t)dt &= \int_0^{\rho(Tx_{n-1},TSy_n)} \varphi(t)dt \\
&\leq c\int_0^{\max\{d(x_{n-1},Sy_n),\rho(y_n,Tx_{n-1}),\rho(y_n,TSy_n)\}} \varphi(t)dt \\
&= c\int_0^{\max\{d(x_{n-1},x_n),0,\rho(y_n,y_{n+1})\}} \varphi(t)dt
\end{aligned}$$

and so

$$\int_0^{\rho(y_n,y_{n+1})} \varphi(t)dt \leq c\int_0^{d(x_{n-1},x_n)} \varphi(t)dt. \tag{2.6}$$

Using (2.5) and (2.6) we obtain

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq c^2\int_0^{d(x_{n-1},x_n)} \varphi(t)dt$$

and

$$\int_0^{\rho(y_n,y_{n+1})} \varphi(t)dt \le c^2 \int_0^{d(y_{n-1},y_n)} \varphi(t)dt.$$

By induction, it follows that

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le c^{2n} \int_0^{d(x_0,x_1)} \varphi(t)dt \tag{2.7}$$

and

$$\int_0^{\rho(y_n,y_{n+1})} \varphi(t)dt \le c^{2n} \int_0^{\rho(y_0,y_1)} \varphi(t)dt. \tag{2.8}$$

Let $m, n \in \mathbb{N}$ such that $m > n$. Using the triangular inequality, we have

$$d(x_n, x_m) \le \sum_{i=1}^{m-1} d(x_i, x_{i+1}).$$

Since φ is non-increasing, it follows that

$$\int_0^{d(x_n,x_m)} \psi(t)dt \le \sum_{i=n}^{m-1} \int_0^{d(x_i,x_{i+1})} \psi(t)dt. \tag{2.9}$$

Using (2.7) and (2.9) we get

$$\int_0^{d(x_n,x_m)} \psi(t)dt \le \sum_{i=n}^{\infty} c^{2i} \int_0^{d(x_0,x_1)} \psi(t)dt = \frac{c^{2n}}{1-c^2} \int_0^{d(x_0,x_1)} \psi(t)dt. \tag{2.10}$$

Taking the limit as $m, n \to \infty$ in (2.10), it follows that $\{x_n\}$ is a Cauchy sequence in X. Similarly, we can prove that $\{y_n\}$ is a Cauchy sequence in Y. Since (X, d) and (Y, ρ) are complete, $\{x_n\}$ and $\{y_n\}$ converge to limits u and v respectively.

If $u \neq Sv$, using (2.1) we get

$$\begin{aligned}
\int_0^{d(Sv,u)} \varphi(t)dt &= \lim_{n\to\infty} \int_0^{d(Sv,STx_{n-1})} \varphi(t)dt \\
&\le c \lim_{n\to\infty} \int_0^{\max\{\rho(v,y_n),d(x_{n-1},Sv),d(x_{n-1}x_n)\}} \varphi(t)dt \\
&= c \int_0^{d(u,Sv)} \varphi(t)dt \\
&< \int_0^{d(u,Sv)} \varphi(t)dt,
\end{aligned}$$

a contradiction. Hence

$$\int_0^{d(u,Sv)} \varphi(t)dt = 0$$

and (2.3) implies that $Sv = u$. If $Tu \neq v$, using (2.2) we have

$$\begin{aligned}
\int_0^{\rho(Tu,v)} \varphi(t)dt &= \lim_{n\to\infty} \int_0^{\rho(Tu,TSy_{n-1})} \varphi(t)dt \\
&\leq c \lim_{n\to\infty} \int_0^{\max\{d(u,x_{n-1}),\rho(y_{n-1},Tu),\rho(y_{n-1},y_n)\}} \varphi(t)dt \\
&= c \int_0^{\rho(v,Tu)} \varphi(t)dt \\
&< \int_0^{\rho(v,Tu)} \varphi(t)dt,
\end{aligned}$$

a contradiction and so $Tu = v$. Then, $STu = Sv = u$ and $TSv = Tu = v$.

To prove the uniqueness, suppose that ST has a second distinct fixed point u'.

Using inequality (2.1) we obtain

$$\begin{aligned}
\int_0^{d(u,u')} \varphi(t)dt &= \int_0^{d(Sv,STu')} \varphi(t)dt \qquad (2.11)\\
&\leq c \int_0^{\max\{\rho(v,Tu'),d(u',Sv),d(u',STu')\}} \varphi(t)dt \\
&= c \int_0^{\rho(Tu,Tu')} \varphi(t)dt.
\end{aligned}$$

Using inequality (2.2) we have

$$\begin{aligned}
\int_0^{\rho(Tu,Tu')} \varphi(t)dt &= \int_0^{\rho(Tu,TSTu')} \varphi(t)dt \qquad (2.12)\\
&\leq c \int_0^{\max\{d(u,STu'),\rho(Tu',Tu),\rho(Tu',TSTu')\}} \varphi(t)dt \\
&= c \int_0^{d(u,u')} \varphi(t)dt.
\end{aligned}$$

Using (2.11) and (2.12) we get

$$\begin{aligned}\int_0^{d(u,u')}\varphi(t)dt &\le c^2\int_0^{d(u,u')}\varphi(t)dt\\ &< \int_0^{d(u,u')}\varphi(t)dt,\end{aligned}$$

a contradiction and so $u = u'$. We can prove similarly that v is the unique fixed point of TS.

This completes the proof of the Theorem. ∎

Remark 1. *If $\varphi(t) = 1$ in Theorem 3, we obtain Theorem 1 of [4].*

If $(X, d) = (Y, \rho)$ in Theorem 3, we get the following Corollary.

Corollary 1. *Let (X, d) be a complete metric space, and S, T be mappings of X into itself satisfying the inequalities*

$$\begin{aligned}\int_0^{d(Sy,STx)}\varphi(t)dt &\le c\int_0^{\max\{d(y,Tx),d(x,Sy),d(x,STx)\}}\varphi(t)dt,\\ \int_0^{d(Tx,TSy)}\varphi(t)dt &\le c\int_0^{\max\{d(x,Sy),d(y,Tx),d(y,TSy)\}}\varphi(t)dt,\end{aligned}$$

for all x, y in X, where $0 \le c < 1$ and φ satisfies (2.3). Then, ST has a unique fixed point u in X and TS has a unique fixed point v in X. Further, $Tu = v$ and $Sv = u$.

If $S = T$ in Corollary 1 we obtain the following Corollary.

Corollary 2. *Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying the inequality*

$$\int_0^{d(Tx,T^2y)}\varphi(t)dt \le c\int_0^{\max\{d(x,Ty),d(y,Tx),d(y,T^2y)\}}\varphi(t)dt,$$

for all x, y in X, where $0 \le c < 1$ and φ satisfies (2.3). Then, T has a unique fixed point u in X

3. An Other related fixed point Theorem

Now, we prove an other related fixed point Theorem in two complete metric spaces.

Theorem 4. *Let (X, d) and (Y, ρ) be complete metric spaces, T a mapping of X into Y and S a mapping of Y into X satisfying the inequalities*

(3.1)
$$\int_0^{d(STx,STx')} \varphi(t)dt \leq c\int_0^{\max\{d(x,x'),d(x,STx),d(x',STx'),\rho(Tx,Tx')\}} \varphi(t)dt,$$

(3.2)
$$\int_0^{\rho(TSy,TSy')} \varphi(t)dt \leq c\int_0^{\max\{\rho(y,y'),\rho(y,TSy),\rho(y',TSy'),d(Sy,Sy')\}} \varphi(t)dt$$

for all x, x' in X and y, y' in Y, where $0 \leq c < 1$ and φ satisfies (2.3). *If T or S is continuous, then ST has a unique fixed point u in X and TS has a unique fixed point v in Y. Further, $Tu = v$ and $Sv = u$.*

Proof. Let x_0 be an arbitrary point in X. As in Theorem 5, we define the sequences $\{x_n\}$and $\{y_n\}$ in X and Y inductively by (2.4). If $x_n = x_{n+1}$ and $y_n = y_{n+1}$ for some $n = 0, 1, 2, ...$, we can put $u = x_n$ and $v = y_n$. Now, we suppose that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all $n = 0, 1, 2, ...$

Using (3.1) and (2.4) we have

$$\begin{aligned}\int_0^{d(x_n,x_{n+1})} \varphi(t)dt &= \int_0^{d(STx_{n-1},STx_n)} \varphi(t)dt \\ &\leq c\int_0^{\max\{d(x_{n-1},x_n),d(x_{n-1},STx_{n-1}),d(x_n,STx_n),\rho(Tx_{n-1},Tx_n)\}} \varphi(t)dt \\ &= c\int_0^{\max\{d(x_{n-1},x_n),d(x_n,x_{n+1}),\rho(y_n,y_{n+1})\}} \varphi(t)dt.\end{aligned}$$

Then

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq c\int_0^{\max\{d(x_{n-1},x_n),\rho(y_n,y_{n+1})} \varphi(t)dt. \tag{3.3}$$

Using inequality (3.2) and (2.4) we get

$$\begin{aligned}\int_0^{\rho(y_n,y_{n+1})}\varphi(t)dt &= \int_0^{\rho(TSy_{n-1},TSy_n)}\varphi(t)dt\\ &\leq c\int_0^{\max\{\rho(y_{n-1},y_n),\rho(y_{n-1},TSy_{n-1}),\rho(y_n,TSy_n),d(Sy_{n-1},Sy_n)\}}\varphi(t)dt\\ &= c\int_0^{\max\{\rho(y_{n-1},y_n),\rho(y_n,y_{n+1}),d(x_{n-1},x_n)\}}\varphi(t)dt.\end{aligned}$$

Therefore

$$\int_0^{\rho(y_n,y_{n+1})}\varphi(t)dt \leq c\int_0^{\max\{d(x_{n-1},x_n),\rho(y_{n-1},y_n)\}}\varphi(t)dt. \tag{3.4}$$

Using (3.3) and (3.4) we obtain

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \leq c\int_0^{\max\{d(x_{n-1},x_n),\rho(y_{n-1},y_n)\}}\varphi(t)dt.$$

By induction, it follows that

$$\int_0^{d(x_n,x_{n+1})}\varphi(t)dt \leq c^n\int_0^{\max\{d(x_0,x_1),\rho(y_0,y_1)\}}\varphi(t)dt. \tag{3.5}$$

Similarly, we get

$$\int_0^{\rho(y_n,y_{n+1})}\varphi(t)dt \leq c^n\int_0^{\max\{d(x_0,x_1),\rho(y_0,y_1)\}}\varphi(t)dt. \tag{3.6}$$

As in the proof of Theorem 3, it follows that $\{x_n\}$ is a Cauchy sequence in X. Similarly, we can prove that $\{y_n\}$ is a Cauchy sequence in Y. Since (X,d) and (Y,ρ) are complete, $\{x_n\}$ and $\{y_n\}$ converge to u and v respectively. Suppose that T is continuous. It follows that

$$\lim_{n\to\infty} Tx_{n-1} = Tu = v = \lim_{n\to\infty} y_n.$$

If $u \neq Sv$, using (3.1) we get

$$\begin{aligned}
\int_0^{d(Sv,u)} \varphi(t)dt &= \lim_{n\to\infty} \int_0^{d(STu,STx_{n-1})} \varphi(t)dt \\
&\leq c \lim_{n\to\infty} \int_0^{\max\{d(u,x_{n-1}),d(u,STu),d(x_{n-1},STx_{n-1}),\rho(Tu,Tx_{n-1})\}} \varphi(t)dt \\
&= c\int_0^{d(u,Sv)} \varphi(t)dt \\
&< \int_0^{d(u,Sv)} \varphi(t)dt,
\end{aligned}$$

a contradiction. Hence, $Sv = u = STu$.

If $v \neq Tu$, using (3.2) we have

$$\begin{aligned}
\int_0^{\rho(Tu,v)} \varphi(t)dt &= \lim_{n\to\infty} \int_0^{\rho(TSv,TSy_{n-1})} \varphi(t)dt \\
&\leq c \lim_{n\to\infty} \int_0^{\max\{\rho(v,y_{n-1}),\rho(v,TSv),\rho(y_{n-1},TSy_{n-1}),d(Sv,Sy_{n-1})\}} \varphi(t)dt \\
&= c\int_0^{\rho(v,Tu)} \varphi(t)dt \\
&< \int_0^{\rho(v,Tu)} \varphi(t)dt,
\end{aligned}$$

a contradiction. Therefore, $Tu = v = TSv$. Similarly, The same results hold if S is continuous instead of T. To prove the uniqueness, suppose that ST has a second distinct fixed point u'.

Using inequality (3.1) we obtain

$$\begin{aligned}
\int_0^{d(u,u')} \varphi(t)dt &= \int_0^{d(STu,STu')} \varphi(t)dt \qquad (3.7)\\
&\leq c\int_0^{\max\{d(u,u'),d(u,STu),d(u',STu'),\rho(Tu,Tu')\}} \varphi(t)dt \\
&= c\int_0^{\rho(Tu,Tu')} \varphi(t)dt.
\end{aligned}$$

Using inequality (3.2) we have

$$\begin{aligned}
\int_0^{\rho(Tu,Tu')} \varphi(t)dt &= \int_0^{\rho(TSv,TSTu')} \varphi(t)dt \qquad (3.8)\\
&\leq c\int_0^{\max\{\rho(v,Tu'),\rho(v,TSv),\rho(Tu',TSTu'),d(Sv,STu')\}} \varphi(t)dt \\
&= c\int_0^{d(u,u')} \varphi(t)dt.
\end{aligned}$$

Using (3.7) and (3.8) we get

$$\begin{aligned}
\int_0^{d(u,u')} \varphi(t)dt &\leq c^2\int_0^{d(u,u')} \varphi(t)dt \\
&< \int_0^{d(u,u')} \varphi(t)dt,
\end{aligned}$$

a contradiction. Therefore, $u = u'$. We can prove similarly that v is a fixed point of TS.

This completes the proof of the Theorem. ∎

Remark 2. *If $\varphi(t) = 1$ in Theorem 4, we obtain Theorem 2 of [5].*

If $(X, d) = (Y, \rho)$ in Theorem 4, we get the following Corollary.

Corollary 3. *Let (X, d) be a complete metric space, S, T be mappings of X into itself satisfying the inequalities*

$$\begin{aligned}
\int_0^{d(STx,STy)} \varphi(t)dt &\leq c\int_0^{\max\{d(x,y),d(x,STx),d(y,STy),d(Tx,Ty)\}} \varphi(t)dt, \\
\int_0^{d(TSx,TSy)} \varphi(t)dt &\leq c\int_0^{\max\{d(x,y),d(x,TSx),d(y,TSy),d(Sx,Sy)\}} \varphi(t)dt
\end{aligned}$$

for all x, y in X, where $0 \leq c < 1$ and φ satisfies (2.3). If T or S is continuous, then ST has a unique fixed point u in X and TS has a unique fixed point v in X. Further, $Tu = v$ and $Sv = u$.

If $S = T$ in Corollary 3 we have the following Corollary.

Corollary 4. *Let (X, d) be a complete metric space and T be a continuous mapping of X into itself satisfying the inequality*

$$\int_0^{d(T^2x,T^2y)} \varphi(t)dt \leq c\int_0^{\max\{d(x,y),d(x,T^2x),d(y,T^2y),d(Tx,Ty)\}} \varphi(t)dt$$

for all x, y in X, where $0 \leq c < 1$ and φ satisfies (2.3). *Then, T has a unique fixed point u in X.*

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