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β-R_0 AND β-R_1 TOPOLOGICAL SPACES

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Abstract. We discuss here some properties of β-R_0 and β-R_1 topological spaces. A new separation axiom β-R_T is introduced.lt is proved that β-R_T is strictly weaker then β-R_0. It is also seen that digital line and digital plane both are β-R_0.

1. Introduction

The notion of R_0 topological spaces is introduced by Shanin [22] in 1943. By definition, topological space is R_0 if every open set contains the closure of each of its single tons. Later, Davis [5] rediscovered it and studied some properties of this weak separation axiom. Many researchers further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts([6, 8, 9, 16, 17]).In 1983, Mashhour et al. [1] introduced the notion of β-open sets which are also known as semi-preopen sets [4] in the literature. Since then, many interesting results and separation axioms using β-open sets received wide usage in general topology. Recently, Noiri et al. [19] introduced and investigated the fundamental properties of separation axiom β-R_0.

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Further Tahiliani [23] introduced the notion of β-R_1 spaces and investigated its properties. In this paper,we study some more properties of β-R_1 spaces. Every β-R_1 space is β-R_0 but not conversely. We also introduce a new separation axiom called β-R_T. It turns out that β-R_T is weaker then β-R_0 and we investigate relationship of β-R_T spaces with other weaker forms. It is also observed that that digital line and digital plane both are β-R_0. We may mention that the present work may be relevant to computer graphics ([12], [13]), biochemistry, quantam information system and dynamics [3].

Throughout this paper we consider spaces on which no separation axiom are assumed unless explicitly stated.The topology of a space (by space we always mean a topological space) is denoted by τ and (X,τ) will be replaced by X if there is no chance of confusion.For $A \subseteq X$, the closure and interior of A in X are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively. By $\beta O(X,\tau)$ or $\beta O(X)$, we denote the collection of all β-open sets of (X,τ).

Definition 1. Let A be a subset of the space (X,τ). Then

(i) A is said to be semi-open [14] (resp. β-open [1]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A)$ (resp. $A \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))))$.
(ii) A is said to be β-closed [1] if $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A))) \subseteq A$.
(iii) The intersection of all β-closed sets containing A is called β-closure [2] of A and it is denoted by $\mathrm{Cl}(A)$.
(iv) Let x be a point of a topological space X. The β-kernel of x [19], denoted by $\beta\mathrm{Ker}(\{x\})$ is defined to be the set $\cap\{O \in \beta O(X,x) \,|\, x \in O\}$.
(v) Let A be a subset of a topological space X. The β-kernel of A, denoted by $\beta\mathrm{Ker}(A)$ [23] or $\Lambda_{sp}(A)$ [19] is defined to be the set $\cap\{O \in \beta O(X,x) \,|\, O \supseteq A\}$.

Definition 2. (i) A topological space (X,τ) is called β-T_0 [15] if for any distinct pair of points x and y of X, there exists a β-open set of X containing x but not y or a β-open set containing y but not x.
(ii) A topological space (X,τ) is called semi-T_2 [14] (resp. β-T_2 [15]) if for any distinct pair of points x and y of X, there exist disjoint semi-open (resp. β-open sets) U and V containing x and y respectively.

(iii) A topological space (X,τ) is said to be a β-R_0 [19] if any β-open set contains the β-closure of each of its singletons. Equivalently $x \in \beta\mathrm{Cl}(\{y\})$ if and only if $y \in \beta\mathrm{Cl}(\{x\})$ for any x and y in X.

Lemma 3 ([2, Theorems 2.1, 2.2, 2.3])**.** *Let (X,τ) be a topological space and let A and B be subsets of X. Then the following hold:*

(i) *$x \in \beta\mathrm{Cl}(A)$ if and only if $A \cap V \neq \varnothing$ for every $V \in \beta O(X,\tau)$ such that $x \in V$.*
(ii) *A is β-closed in (X,τ) if and only if $A = \beta\mathrm{Cl}(A)$.*
(iii) *$\beta\mathrm{Cl}(A) \subseteq \beta\mathrm{Cl}(B)$ if $A \subseteq B$.*
(iv) *$\beta\mathrm{Cl}(\beta\mathrm{Cl}(A)) = \beta\mathrm{Cl}(A)$.*

2. β-R_0 Spaces

Lemma 4 ([23, Lemmas 4.3, 4.7 and 4.8])**.** *Let (X,τ) be a arbitrary topological space, A a subset of X. For any x, y in X, the following hold:*

(i) *$y \in \beta\mathrm{Ker}(\{x\})$ if and only if $x \in \beta\mathrm{Cl}(\{y\})$.*
(ii) *$\beta\mathrm{Ker}(A) = \{x \in X \mid \beta\mathrm{Cl}(\{x\}) \cap A \neq \varnothing\}$.*
(iii) *$\beta\mathrm{Ker}(\{x\}) \neq \beta\mathrm{Ker}(\{y\})$ if and only if $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$.*

Lemma 5 ([23, Theorem 4.11])**.** *A space (X,τ) is β-R_0 if and only if for any x in X, $\beta\mathrm{Cl}(\{x\}) \subseteq \beta\mathrm{Ker}(\{x\})$.*

Theorem 6 ([19, Theorem 5.5])**.** *A space (X,τ) is β-R_0 if and only if for every x in X, $\beta\mathrm{Ker}(\{x\}) \subseteq \beta\mathrm{Cl}(\{x\})$.*

The following lemma is special case of above theorem

Lemma 7. *A space (X,τ) is β-R_0 if and only if for any x in X, $\beta\mathrm{Cl}(\{x\}) = \beta\mathrm{Ker}(\{x\})$.*

Since $\beta\mathrm{Cl}(\{x\})$ is the intersection of all β-open sets containing x, Lemma 7 suggests a natural definition of β-R_0.

Definition 8. A topological space (X,τ) is β-R_0 if the intersection of all β-open sets containing x coincides with the intersection of all β-closed sets containing x.

Observe that Lemma 5 and Theorem 6 show the symmetry of β-R_0 spaces in another sense.

3. β-R_1 SPACES

Theorem 9. *Every topological space is β-T_0.*

Proof. Let x and y be any two points in X. If $\mathrm{Int}(\{x\})$ is non empty,then $\{x\}$ is open, thus β-open and we are done. If $\mathrm{Int}(\{x\})$ is empty,then $\{x\}$ is β-closed, i.e., $X \setminus \{x\}$ is β-open set containing y, and we are again done. □

Definition 10. A topological space (X, τ) is said to be β-R_1 [23] if for $x, y \in X$ and $\beta\mathrm{Cl}(\{x\}0 \neq \beta\mathrm{Cl}(\{y\})$, there exist disjoint β-open sets U and V such that $\beta\mathrm{Cl}(\{x\}) \subseteq U$ and $\beta\mathrm{Cl}(\{y\}) \subseteq V$.

Remark 11 ([23, Remark 4.17])**.** Every β-R_1 space is β-R_0 but not conversely.

Theorem 12. *A topological space X is β-T_2 if and only if it is β-R_1.*

Proof. Let X be a β-T_2 space. Then X is β-T_1. Also, by ([19, Proposition 3.1, Theorem 5.6] and Lemma 3(ii)) $\{x\} = \beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\}) = \{y\}$ for $x, y \in X$. As X is β-T_2, there exist disjoint β-open sets U and V such that $\beta\mathrm{Cl}(\{x\}) \subseteq U$ and $\beta\mathrm{Cl}(\{y\}) \subset V$. Thus X is β-R_1 space.

Conversely let X be β-R_1. Let x and y be any two distinct points of X. By Theorem 9, there exists a β-open set U such that $x \in U$ and $y \notin U$. Then $y \notin \beta\mathrm{Cl}(\{x\})$ and then $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$ by Lemma 4(i). Since X is β-R_1, there exist disjoint β-open sets U and V such that $x \in \beta\mathrm{Cl}(\{x\}) \subseteq U$ and $y \in \beta\mathrm{Cl}(\{y\}) \subset V$. Thus X is β-T_2. □

Definition 13. A point x in X is called a (β-θ-cluster point [18] of A if $\beta\mathrm{Cl}(U) \cap A \neq \varnothing$ for every β-open set U containing x. The set of all (β-θ-cluster point of A is said to be (β-θ-closure of A and is denoted by $\beta\mathrm{Cl}_\theta(A)$.

Lemma 14. *Let x and y be the points in a space (X, τ). Then $y \in \beta\mathrm{Cl}_\theta(\{x\})$ if and only if $x \in \beta\mathrm{Cl}_\theta(\{y\})$.*

Theorem 15. *A space (X, τ) is β-R_1 if and only if for each x in X, $\beta \in \mathrm{Cl}(\{x\}) = \beta\mathrm{Cl}_\theta(\{x\})$.*

Proof. Necessity. Assume that X is β-R_1 and $y \in \beta\mathrm{Cl}_\theta(\{x\}) \setminus \beta\mathrm{Cl}(\{x\})$. Then, there exists a β-open set U containing y such that $\beta\mathrm{Cl}(U) \cap \{x\} \neq \varnothing$ but $U \cap \{x\} \neq 0$. Thus $\beta\mathrm{Cl}(\{y\}) \subseteq U$,

$\beta\mathrm{Cl}(\{x\}) \cap U = \varnothing$. Hence $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$. Since X is β-R_1, there exist disjoint β-open sets U_1 and U_2 such that $\beta\mathrm{Cl}(\{x\}) \subseteq U_1$ and $\beta\mathrm{Cl}(\{y\}) \subseteq U_2$. Therefore, $X \backslash U_1$ is a β-closed, β-neighbourhood at y which does not contain x. Thus $y \in \beta\mathrm{Cl}(\{x\})$. This is a contradiction.

Sufficiency. Suppose that $\beta\mathrm{Cl}(\{x\}) = \beta\mathrm{Cl}_\theta(\{x\})$ for each x in X. We first prove that X is β-R_0. Let x belong to β-open set U and $y \notin U$. Since $\beta\mathrm{Cl}_\theta(\{y\}) = \beta\mathrm{Cl}(\{y\})X \setminus U$, we have $x \notin \beta\mathrm{Cl}_\theta(\{y\})$ and by Lemma 14, $y \in \beta\mathrm{Cl}_\theta(\{x\}) = \beta\mathrm{Cl}(\{x\})$. It follows that $\beta\mathrm{Cl}(\{x\}) \subset U$. Therefore (X,τ) is β-R_0. Now let $ab \in X$ with $\beta\mathrm{Cl}(\{a\}) \neq \beta\mathrm{Cl}(\{b\})$. By ([19, Theorem 5.6]), (X,τ) is β-T_1 and $b \notin \beta\mathrm{Cl}(\{a\})$ and hence there exists a β-open set U containing b such that $a \notin \beta\mathrm{Cl}_\theta(U)$. Therefore,we obtain that bell, $a \in X \setminus \beta\mathrm{Cl}(U)$ and $U \cap (X \setminus \beta\mathrm{Cl}(U) = 0$. This shows that (X,τ) is β-T_2. It follows from Theorem 12 that (X,τ) is β-R_1.

Theorem 16. *For a space (X,τ),the following statements are equivalent:*

(i) (X,τ) *is* β-R_1.
(ii) *If $x, y \in X$ such that $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$, then there exist β-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.*

Proof. (1)$\Rightarrow$(2). Let x, y in X be such that $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$, and hence $x \neq y$. Therefore,there exist disjoint β-open sets U_1 and U_2 such that $x \in \beta\mathrm{Cl}(\{x\}) \subseteq U_1$ and $y \in \beta\mathrm{Cl}(\{y\}) \subseteq U_2$. Then $F_1 = X \setminus U_2$ and $F_2 = X \setminus U_1$ are β-closed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$.
(2)$\Rightarrow$(1). Suppose that x and y are distinct points of X, such that $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$. Therefore, there exist β-closed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$ and $X = F_1 \cup F_2$. Now set $U_1 = X \setminus F_2$ and $U_2 = X \setminus F_1$. Then we obtain that $x \in U_1$, $y \in U_2$, $U_1 \cap U_2 = \varnothing$ and U_1, U_2 are β-open. This shows that (X,τ) is β-T_2. Hence by Theorem 12, (X,τ) is β-R_1.

4. A weaker form of β-R_0 spaces

We begin with following definitions

Definition 17. A topological space (X,τ) is said to be

(i) β-R_H if for any points x and y in X, $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$ implies $\beta\mathrm{Cl}(\{x\}) \cap \beta\mathrm{Cl}(\{y\})$ is either empty or equals to $\{x\}$ or equals to $\{y\}$.
(ii) β-R_D if for x in X, $\beta\mathrm{Cl}(\{x\}) \cap \beta\mathrm{Ker}(\{x\}) = \{x\}$ implies that $(\beta\mathrm{Cl}(\{x\}) \setminus \{x\})$ is β-closed.
(iii) weakly β-R_0 if and only if $\cap\{\beta\mathrm{Cl}(\{x\}) : x \in X\} = \varnothing$.

It follows from Lemma 5 and Theorem 6 that if X is not β-R_0, then there is some x such that $\beta\mathrm{Ker}(\{x\}) \setminus \beta\mathrm{Cl}(\{x\}) \neq \varnothing$, and there is some x such that $\beta\mathrm{Cl}(\{x\}) \setminus \beta\mathrm{Ker}(\{x\}) \neq \varnothing$. This suggests a new separation axiom. In the following definition, a set that contains at most one point is said to be degenerate.

Definition 18. A topological space (X,τ) is a β-R_T space if for any x, both $\beta\mathrm{Ker}(\{x\}) \setminus \beta\mathrm{Cl}(\{x\})$ and $\beta\mathrm{Cl}(\{x\}) \setminus \beta\mathrm{Ker}(\{x\})$ are degenerate.

Every β-R_0 space is weakly β-R_0 but not conversely and obviously, every β-R_0 space is β-R_T. In general the converse may not be true.

Example 19. Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a,b\}, \{b,c\}\}$. Then (X,τ) is β-R_T but not β-R_0.

Example 20. Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, X, \{a\}, \{a,b\}, \{a,c\}\}$. Then (X,τ) is R_H and β-R_D but not β-R_0 since $\beta\mathrm{Cl}(\{a\}) = X$ and $\beta\mathrm{Ker}(\{a\}) = \{a\}$.

Example 21. Let $X = \{a,b,c\}$ and $\tau = \{\varnothing, X, \{a\}, \{a,b\}\}$. Then (X,τ) is weakly Ro but not β-R_T since $\beta\mathrm{Cl}(\{a\}) = X$ and $\beta\mathrm{Ker}(\{a\}) = \{a\}$. Also it is not β-R_0.

Theorem 22. *If (X,τ) is β-R_T, then (X,τ) is β-R_D.*

Proof. Suppose that X is β-R_T and denote $(x)_p = \beta\mathrm{Cl}(\{x\}) \cap \beta\mathrm{Ker}(\{x\})$. Then $\beta\mathrm{Cl}(\{x\}) = (x)_p \cup D$ and $\beta\mathrm{Ker}(\{x\}) = (x)_p \cup E$, where D and E are degenerate sets such that $D \not\subset \beta\mathrm{Ker}(\{x\})$ and $E \not\subset \beta\mathrm{Cl}(\{x\})$. If $(x)_p = \{x\}$, then $\beta\mathrm{Cl}(\{x\}) = (x) \cup D$ and $\beta\mathrm{Ker}(\{x\}) = (x) \cup E$. We prove that $(\beta\mathrm{Cl}(\{x\}) \setminus \{x\})$ is β-closed. Let U be a β-open set containing $\beta\mathrm{Ker}(\{x\})$. Then $X \setminus U$ is a β-closed set, and $(X \setminus U) \cap \beta\mathrm{Cl}(\{x\}) = D$ or $\varnothing$. If $(X \cap U) \cap \beta\mathrm{Cl}(\{x\}) = D$, then D is the intersection of β-closed sets and hence D is also β-closed. If $(X \setminus U) \cap \beta\mathrm{Cl}(\{x\}) = \varnothing$, then $\beta\mathrm{Cl}(\{x\}) \subseteq U$ and $D \subseteq U$.

Since $D \not\subset \beta\mathrm{Ker}(\{x\})$, there is a β-open set V such that $x \in V$ and $D \nsubseteq V$. Then $\beta\mathrm{Cl}(\{x\}) \cap X \setminus V) = D$ is a β-closed set. Therefore $(\beta\mathrm{Cl}(\{x\}) \setminus \{x\})$ is β-closed whenever $(x)_p = \{x\}$. Hence X is β-R_D. $\square$

Theorem 23. *If (X,τ) is β-R_T, then (X,τ) is β-R_H.*

Proof. Let X be β-R_T and $x, y \in X$. Suppose that $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$ and there is an $a \in X$ such that $a \neq x$, $a \neq y$ but $a \in \beta\mathrm{Cl}(\{x\}) \cap \beta\mathrm{Cl}(\{y\})$. Then $a \in \beta\mathrm{Cl}(\{x\})$ and $a \in \beta\mathrm{Cl}(\{y\})$. Hence $x \in \beta\mathrm{Ker}(\{a\})$ and $y \in \beta\mathrm{Ker}(\{a\})$. Since $\beta\mathrm{Ker}(\{a\}) = (a)_p \cup E$, where E is a degenerate set and $E \nsubseteq \beta\mathrm{Cl}(\{a\})$, there are following four cases for x, y in $\beta\mathrm{Ker}(\{a\})$:

(i) $x \in (a)_p$ and $y \in (a)_p$. We have $x \in \beta\mathrm{Cl}(\{a\})$ and $y \in \beta\mathrm{Cl}(\{a\})$ and also $a \in \beta\mathrm{Cl}(\{x\})$ and $a \in \beta\mathrm{Cl}(\{y\})$. Hence $\beta\mathrm{Cl}(\{y\}) = \beta\mathrm{Cl}(\{a\}) = \beta\mathrm{Cl}(\{x\})$ which is a contradiction.

(ii) $\{x\} = E$ and $y \in (a)_p$. We have $x \notin \beta\mathrm{Cl}(\{a\})$ and $y \in \beta\mathrm{Cl}(\{a\})$. Since $a \in \beta\mathrm{Cl}(\{y\})$, we have $\beta\mathrm{Cl}(\{y\}) = \beta\mathrm{Cl}(\{a\})$. We have following two cases to consider between y and $\beta\mathrm{Cl}(\{x\})$.

Case 1. $y \in \beta\mathrm{Cl}(\{x\})$. Then $\beta\mathrm{Cl}(\{y\}) = \beta\mathrm{Cl}(\{a\})$. Since $x \in \beta\mathrm{Cl}(\{a\})$, $x \in X \setminus \beta\mathrm{Cl}(\{a\})$. As $X \setminus \beta\mathrm{Cl}(\{a\})$ is β-open, $\beta\mathrm{Ker}(\{x\}) \subseteq X \setminus \beta\mathrm{Cl}(\{a\})$ and hence $(\beta\mathrm{Cl}(\{x\}) \setminus \beta\mathrm{Ker}(\{x\})) \supseteq \beta\mathrm{Cl}(\{a\}) = \{y, a\}$. Hence $(\beta\mathrm{Cl}(\{x\}) \setminus \beta\mathrm{Ker}(\{x\}))$ is not a degenerate set, a contradiction to the fact that X is β-R_T.

Case 2. $y \notin \beta\mathrm{Cl}(\{x\})$. Since $y \in \beta\mathrm{Cl}(\{a\})$ and $a \in \beta\mathrm{Cl}(\{x\})$, we have $y \in \beta\mathrm{Cl}(x)$, a contradiction.

(iii) $x \in (a)_p$ and $\{y\} = E$. Similar to case (2).

(iv) $\{x\} = \{y\} = E$. We have $\beta\mathrm{Cl}(\{x\}) = \beta\mathrm{Cl}(\{y\})$. But this is impossible. Therefore if $\beta\mathrm{Cl}(\{x\}) \neq \beta\mathrm{Cl}(\{y\})$, we have $\beta\mathrm{Cl}(\{x\}) \cap \beta\mathrm{Cl}(\{y\})$ is either $\varnothing$, $\{x\}$ or $\{y\}$. It follows that X is β-R_H.

From the definitions mentioned above,we have the following diagram

$$\begin{array}{ccccc} \beta\text{-}R_1 & \rightarrow & \beta\text{-}R_0 & \rightarrow & \text{weakly } \beta\text{-}R_0 \\ & & \downarrow & & \\ \beta\text{-}R_D & \leftarrow & \beta\text{-}R_T & \rightarrow & \beta\text{-}R_H \end{array}$$

None of the above implication is reversible as can be seen from the examples above.

5. Applications

Definition 24. (i) The digital line [10, 12, 13] or so called the Khalimsky line (Z, k) is the set of all integers Z, equipped with the topology k generated by $\{\{2n-1, 2n, 2n+1\} | n \in Z\}$. The digital line or an interval as its subspace is a topological model of a 1-dimensional computer screen and the points are the pixels.

(ii) The digital n-space [7] is a topological product of the n-copies of the digital line (Z, k) and is denoted by (Z^n, k^n). For $n = 2$, (Z^2, k^2) is called as the digital plane. The digital plane or a rectangular portion of the plane as its subspace is a topological model of a 2-dimensional computer screen and the points are the pixels. Many application of digital plane are important in the theory of computer graphics ([11, 12]).

A subset V is open in (Z, k) if and only if whenever $x \in V$ and x is an even integer, then $x-1, x+1 \in V$ (cf. [11, page 175]). It is clear that a singleton $\{2s+1\}$ is open, a singleton $\{2m\}$ is closed and a subset $\{2k-1, 2k, 2k+1\}$ is the smallest open set containing $2k$, where s, m and k are any integers

Definition 25. (i) A subfamily m of the power set $P(X)$ of a non empty set X is called the minimal structure on X (briefly m-structure) ([21, Definition 3.1]) if $\varnothing \in m$ and $X \in m$.

By (X, m), we denote the non empty set X with a minimal structure m on X and we call the pair (X, m) an m-space. Each member of m is said to be an m-open set and the complement of an m-open set is said to be m-closed. For a subset A of X, the m-closure of A ([21, Definition 3.2]), denoted by $m\mathrm{Cl}(A)$, is defined as follows:

$$m\mathrm{Cl}(A) = \cap\{F \mid A \subseteq F, X \setminus F \in m\}.$$

(ii) An m-space (X, m) is said to be m-R_0 [20] if for each open set U and each $x \in U$, $m\mathrm{Cl}(\{x\}) \subseteq U$.

(iii) An m-space (X, m) is said to be m-R_1 [20] if for each points x, y in X with $m\mathrm{Cl}(\{x\}) \neq m\mathrm{Cl}(\{y\})$, there exist disjoint m-open sets U and V such that $m\mathrm{Cl}(\{x\}) \subseteq U$ and $m\mathrm{Cl}(\{y\}) \subseteq V$.

6. Conclusion

(i) The digital line and digital plane both are β-R_1 and β-R_0.

Proof. The digital line (resp. digital plane) is semi-T_2 ([7, Theorem 2.3], resp. [7, Theorem 4.8(i)]) and hence β-T_2. By Theorem 12,they are β-R_1 and hence β-R_0. □

We now show an alternative proof of the fact that digital line is β-R_1:

We now give an alternative proof of the fact that digital line is β-R_1 by considering the following:

Lemma 26. *For any point x of (Z, k), we have $\beta\mathrm{Cl}\{x\} = \{x\}$.*

Proof. Case 1. Let $x = 2s$(even), $s \in Z$. Now, we prove that $\{2s\}$ is closed in (Z, k) and it is not open. Then $\mathrm{Cl}(\{x\}) = \{x\}$, $x = 2s$ and $\mathrm{Int}(\{x\}) = \varnothing$. Now as k is generated by $\{(2m-1, 2m, 2m+1)|m \in Z\}$, $Z \setminus \{x\} = \cup\{(2t-1, 2t, 2t+1)|t \neq s, t \in Z\} \in k$. Thus, $\{x\} = \{2s\}$ is closed, $\mathrm{Cl}(\{x\}) = \{x\}$, $x = 2s4$, $s \in Z$ and $\{x\} = 2s \notin \{2m-1, 2m, 2m+1\}$. So $\mathrm{Int}(\{2s\}) = 0$. Therefore, we have that $\beta\mathrm{Cl}(\{x\}) = \beta\mathrm{Cl}(\{2s\}) = \{2s\} \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(2s))) = \{2s\} \cup \varnothing = \{2s\}$ and so $\beta\mathrm{Cl}(\{2s\}) = \{2s\}$.

Case 2. Let $x = 2s+1$(odd), $s \in Z$. Then we claim that $\{x\}$ is open in (Z, k). Now $\{x\} = \{2s+1\} = \{2s-1, 2s, 2s+1\} \cap \{2s+1, 2s+2, 2s+3\}$. So $\{x\} \in k$ and hence $\mathrm{Int}\{2s+1\} = \{2s+1\}$. For this point $\{x\} = \{2s+1\}$, we have that $\mathrm{Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\}$. Now $2s \in \mathrm{Cl}(\{2s+1\})$ holds. Indeed, for any open set V_{2s} containing $2s$, $V_{2s} \supseteq \{2s-1, 2s, 2s+1\}$ and $2s \in \{2s-1, 2s, 2s+1\}$. Thus, $V_{2s} \cap \{2s+1\} = \{2s+1\} \neq \varnothing$. Hence $2s \in \mathrm{Cl}(\{2s+1\})$. As a similar proof, we have that $2s+2 \in \mathrm{Cl}(\{2s+1\})$. However for point $y \notin \{2s, 2s+1, 2s+2\}$, $y \notin \mathrm{Cl}(\{2s+1\})$ as $\{2s-1\} \cap \{2s+1\} = \varnothing$. So $2s-1 \notin \mathrm{Cl}(\{2s+1\})$. Hence $\mathrm{Cl}(\{2s+1\}) = \{2s, 2s+1, 2s+2\}$. Now, $\beta\mathrm{Cl}(\{x\}) = \beta\mathrm{Cl}(\{2s+1\}) = \{2s+1\} \cup \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(\{2s+1\}) = \{2s+1\} \cup \mathrm{Int}(\mathrm{Cl}(\{2s+1\})) = \{2s+l\} \cup \mathrm{Int}\{2s, 2s+1, 2s+2\} = \{2s+1\} \cup \{2s+1\} = \{2s+1\}$.

Theorem 27. *The digital line is β-R_1.*

Proof. Let p and q be the two points of (Z, k) such that $\beta\mathrm{Cl}(\{p\}) \neq \beta\mathrm{Cl}(\{q\})$. Then $p \neq q$.

Case 1. $p = 2k$ and $q = 2s$, where $k \neq s$ and $k < s$: For this case we take $U = \{2k-1, p\}$ and $V = \{q, 2s+1\}$. Then U

and V are disjoint β-open sets containing $\beta\mathrm{Cl}(\{p\}) = p$ and $\beta\mathrm{Cl}(\{q\}) = q$.

Case 2. $p = 2k$ and $q = 2s + 1$, where $p < q$: For this case also,we put $U = \{2k - 1, p\}$ and $V = \{q, 2s + 2\}$. Then U and V are disjoint β-open sets containing p and q respectively.

Case 3. $p = 2k + 1$ and $q = 2s + 1$, where $p < q$: For this case, $U = \{2k, p\}$ and $V = \{q, 2s + 2\}$, then U and V are disjoint β-open sets containing $\beta\mathrm{Cl}(\{p\}) = p$ and $\beta\mathrm{Cl}(\{q\}) = q$ respectively. Hence (Z, k) is β-R_1.

(ii) By using β-open sets, several modifications of β-R_0 and β-R_1 spaces are introduced and investigated. Since $\beta O(X)$ is an example of m-structure, we put $m = \beta O(X)$, then several characterizations of the family are obtained (Theorems 12, 15,16) from those of m-R_0 and m-R_1 spaces established in Sections 4 and 5 of [20].

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