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SOME PROPERTIES OF A R-RANDERS QUARTIC SPACE

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Abstract. It is well known that a Randers metric is a deformation of a Riemannian metric $\alpha(x,y) = \sqrt{a_{ij}(x)\, y^i y^j}$ using a 1-form $\beta(x,y) = b_i(x)\, y^i$. In this paper we consider a deformation of a 4-th root metric (or a quartic metric) $F(x,y) = \sqrt[4]{a_{hijk}(x)\, y^h y^i y^j y^k}$ using the Riemannian metric $\alpha(x,y)$. We call it a R-Randers quartic metric and we are going to study some of its properties.

1. Introduction

Let M be a real n-dimensional smooth manifold and (TM, τ, M) the tangent bundle of M. In a local chart (U, x^i) on M, a tangent vector $y \in T_xM$, $x \in M$ has the form $y = y^i \frac{\partial}{\partial x^i}|_x$.

A Finsler metric on M is a positive function $F : TM \to R_+$ satisfying the following properties:

i) F is of C^∞-class on $\widetilde{TM} = TM - \{0\}$ and only continuous on $\{0\}$

ii) F is positive homogenous of order one with respect to y:

$$F(x, \lambda y) = \lambda F(x,y), \lambda > 0; \tag{1}$$

iii)$\forall (x,y) \in \widetilde{TM}$ the symmetric bilinear form $g_{ij}(x,y)$ is positive and nondegenarate, where

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(2) $$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}.$$

The pair $F^n = (M, F)$ is called a Finsler space and the bilinear form $g_{ij}(x,y)$ is called the fundamental tensor of the Finsler space.

The metric $F(x,y) = \alpha(x,y) + \beta(x,y)$, where $\alpha(x,y) = \sqrt{a_{ij}(x)\, y^i y^j}$ is a Riemannian metric and $\beta(x,y) = b_i(x)\, y^i$ is a 1-form, is a Finsler metric. The space $F^n = (M, F = \alpha + \beta)$ is called a Randers space.

2. Quartic Finsler space

Let $F^n = (M, F)$ be a Finsler space with the metric F given by

(3) $$F(x,y) = \sqrt[4]{a_{hijk}(x)\, y^h y^i y^j y^k},$$

where $a_{hijk}(x)$ are the components of a symetric tensor field of (0,4)-type.

We call this space a *quartic Finsler space* and we denote it by QF^n.

For an easier calculation we also denote:

(4) $$\begin{aligned} a_{hijk}(x)\, y^h &= F a_{ijk}(x,y); \\ a_{hijk}(x)\, y^h y^i &= F^2 a_{jk}(x,y); \\ a_{hijk}(x)\, y^h y^i y^j &= F^3 a_k(x,y). \end{aligned}$$

We have

(5) $$l_i = a_i$$

and

(6) $$h_{ij} = 3\left(a_{ij} - a_i a_j\right).$$

The fundamental tensor of QF^n space is

(7) $$g_{ij} = 3a_{ij} - 2a_i a_j.$$

If $g^{ij} = \left(g_{ij}\right)^{-1}$, then

$$g^{ij} = \frac{1}{3}\left(a^{ij} + 2a^i a^j\right), \tag{8}$$

where $a^{ij} = (a_{ij})^{-1}$ i $a^i = a^{ir} a_r$.
The generalised Cristoffel symbols of 4-th order are given by

$$\Gamma^s_{hijk} = \frac{1}{6} a^{sp} \left(\frac{\partial a_{ijkp}}{\partial x^h} + \frac{\partial a_{jkph}}{\partial x^i} + \frac{\partial a_{kphi}}{\partial x^j} + \frac{\partial a_{iphj}}{\partial x^k} - \frac{\partial a_{hijk}}{\partial x^p} \right), \tag{9}$$

and the components of the Cartan tensor field are

$$C_{ijk} = \frac{3}{F}\left(a_{ijk} - a_{ij}a_k - a_{jk}a_i - a_{ki}a_j + 2a_i a_j a_k\right). \tag{10}$$

From (7) we have

$$F^2 = g_{ij} y^i y^j = 3a_{ij} y^i y^j - 2a_i y^i a_j y^j. \tag{11}$$

And if we take into account

$$a_i y^i = \frac{1}{F^3} a_{ijkh} y^j y^k y^h y^i = \frac{1}{F^3} F^4 = F,$$

we obtain

$$F^2 = 3a_{ij} y^i y^j - 2F^2,$$

or

$$F^2 = a_{ij} y^i y^j. \tag{12}$$

So, we can construct a Finsler connection based on a_{ij} .

3. R-Randers Quartic Space

Let $QF^n = (M, F)$ be a quartic Finsler space with F given by (3) and a Riemannian metric $\alpha(x, y) = \sqrt{b_{ij}(x)\, y^i y^j}$ on TM. We construct the function

$$L(x, y) = F(x, y) + \alpha(x, y), \forall\,(x, y) \in TM \tag{13}$$

and we call it a *R-Randers quartic metric.* The space $RRQ^n = (M, L)$ will be called a *R-Randers quartic space.*

We denote
${}^{F}l_i = \frac{\partial F}{\partial y^i} = a_i, {}^{\alpha}l_i = \frac{\partial \alpha}{\partial y^i} = \frac{b_{ij}y^j}{\alpha}$ i ${}^{L}l_i = \frac{\partial L}{\partial y^i} = a_i + {}^{\alpha}l_i$
and we calculate the angular metric of the $RRQ^n = (M, L)$:

$$ {}^{L}h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = \frac{L}{F}{}^{F}h_{ij} + \frac{L}{\alpha}{}^{\alpha}h_{ij} \tag{14}$$

or

$$ {}^{L}h_{ij} = \frac{L}{F}3\left(a_{ij} - a_i a_j\right) + \frac{L}{\alpha}\left(b_{ij} - {}^{\alpha}l_i^{\alpha}l_j\right). \tag{15}$$

Proposition 3.1 The tensor field g_{ij} of a R-Randers Quartic space is given by

$$ {}^{L}g_{ij} = \frac{L}{F}3\left(a_{ij} - a_i a_j\right) + \frac{L}{\alpha}\left(b_{ij} - {}^{\alpha}l_i^{\alpha}l_j\right) + \left(a_i + {}^{\alpha}l_i\right)\left(a_j + {}^{\alpha}l_j\right), \tag{16}$$

or, equivalently

$$ {}^{L}g_{ij} = \frac{L}{F}g_{ij} + \frac{L}{\alpha}b_{ij} - \frac{\alpha}{F}a_i a_j - \frac{F}{\alpha}{}^{\alpha}l_i{}^{\alpha}l_j + a_i{}^{\alpha}l_j + a_j{}^{\alpha}l_i. \tag{17}$$

By a direct calculation we obtain

Proposition 3.2 The components ${}^{L}C_{ijk}$ of the Cartan tensor field in a R-Randers quartic space are given by

(18)

$$ \begin{aligned} {}^{L}C_{ijk} &= \tfrac{L}{F}{}^{F}C_{ijk} + \tfrac{1}{2}\left[\tfrac{{}^{\alpha}l_k F - \alpha a_k}{F^2}\left({}^{F}g_{ij} - a_i a_j\right) - \tfrac{{}^{\alpha}l_k F - \alpha a_k}{\alpha^2}\left(b_{ij} - {}^{\alpha}l_i{}^{\alpha}l_j\right)\right. \\ &-\tfrac{\alpha}{F^2}\left({}^{F}h_{ik}a_j + {}^{F}h_{jk}a_i\right) - \tfrac{F}{\alpha^2}\left({}^{\alpha}h_{ik}{}^{\alpha}l_j + {}^{\alpha}h_{jk}{}^{\alpha}l_i\right) \\ &+\tfrac{1}{F}\left({}^{F}h_{ik}{}^{\alpha}l_j + {}^{F}h_{jk}{}^{\alpha}l_i\right) + \tfrac{1}{\alpha}\left({}^{\alpha}h_{ik}a_j + {}^{\alpha}h_{jk}a_i\right). \end{aligned} $$

Now our goal is to write down the geodesic spray coefficients of a R-Randers quartic metric. We begin with the fact the Riemannian metric $b_{ij}$is constant with respect to its Levi-Civita connection:

$$ b_{ij,x^k} = \gamma_{ijk} + \gamma_{jik}, \tag{19}$$

where

$$ \gamma_{ijk} = \frac{1}{2}\left(b_{ij,x^k} + b_{ik,x^j} - b_{jk,x^i}\right). \tag{20}$$

We have the following computations:

$$\begin{aligned}(21)\qquad &\frac{\partial\alpha}{\partial x^k}=\frac{1}{2\alpha}b_{pq,x^k}y^py^q=\frac{\alpha^2}{2\alpha}b_{pq,x^k}l^pl^q=\\ &=\frac{\alpha}{2}\left(\gamma_{pqk}+\gamma_{qpk}\right)l^pl^q=\frac{\alpha}{2}\left(\gamma_{nnk}+\gamma_{nnk}\right)=\alpha\gamma_{nnk}.\end{aligned}$$

$$(22)\qquad \frac{\partial F}{\partial x^k}=\frac{\alpha^4}{4F^3}a_{mnpq}l^ml^nl^pl^q.$$

$$(23)\qquad \frac{\partial}{\partial x^k}\left(\frac{L}{F}\right)=\frac{4F^4\alpha\gamma_{nnk}-\alpha^5a_{mnpq,x^k}l^ml^nl^pl^q}{4F^5}\overset{not}{=}\frac{B_{nnk}}{4F^5}.$$

$$(24)\qquad \frac{\partial}{\partial x^k}\left(\frac{L}{\alpha}\right)=-\frac{4F^4\alpha\gamma_{nnk}-\alpha^5a_{mnpq,x^k}l^ml^nl^pl^q}{4F^3\alpha^2}\overset{not}{=}-\frac{B_{nnk}}{4F^3\alpha^2}.$$

$$\begin{aligned}(25)\qquad &\frac{\partial^\alpha l_i}{\partial x^k}=\frac{\partial}{\partial x^k}\left(\frac{b_{ip}y^p}{\alpha}\right)=\frac{b_{ip,x^k}y^p\alpha-b_{ip}y^p\alpha_{,x^k}}{\alpha^2}\\ &=\frac{\alpha\left(\gamma_{ipk}+\gamma_{pik}\right)y^p-b_{ip}y^p\alpha\gamma_{nnk}}{\alpha^2}\\ &=\gamma_{ink}+\gamma_{nik}-{}^\alpha l_i\gamma_{nnk}\overset{not}{=}A_{ink}\end{aligned}$$

$$(26)\quad \frac{\partial^F l_i}{\partial x^k}=\frac{\partial}{\partial x^k}\left(a_i+{}^\alpha l_i\right)=\gamma_{ink}+\gamma_{nik}-{}^\alpha l_i\gamma_{nnk}+a_{i,x^k}=A_{ink}+a_{i,x^k}.$$

We obtain

$$\begin{aligned}(27)\qquad &\frac{\partial^L g_{ij}}{\partial x^k}=\frac{3}{4F^5}B_{nnk}\left(a_{ij}-a_ia_j\right)+\frac{L}{F}3\left(a_{ij,x^k}-a_{i,x^k}-a_ia_{j,x^k}\right)\\ &-\frac{1}{4F^3\alpha^2}B_{nnk}\left(b_{ij}-{}^\alpha l_i{}^\alpha l_j\right)+\frac{L}{\alpha}\left(b_{ij,x^k}-A_{ink}l_j-A_{jnk}l_i\right)\\ &+\left(A_{ink}+a_{i,x^k}\right)^Ll_j+\left(A_{jnk}+a_{j,x^k}\right)^Ll_i\end{aligned}$$

and

(28)

$$\begin{aligned}&2\gamma_{ijk}=\frac{3}{4F^5}\left[B_{nnk}\left(a_{ij}-a_ia_j\right)+B_{nnj}\left(a_{ik}-a_ia_k\right)-B_{nni}\left(a_{jk}-a_ja_k\right)\right]\\ &+\frac{3L}{F}\left(2f_{ijk}-a_{i,x^k}a_j-a_ia_{j,x^k}-a_{i,x^j}a_k-a_ia_{k,x^j}+a_{j,x^i}a_k+a_ja_{k,x^i}\right)\\ &-\frac{1}{4F^3\alpha^2}\left[B_{nnk}\left(b_{ij}-{}^\alpha l_i{}^\alpha l_j\right)+B_{nnj}\left(b_{ik}-{}^\alpha l_i{}^\alpha l_k\right)-B_{nni}\left(b_{jk}-{}^\alpha l_j{}^\alpha l_k\right)\right]\\ &+\frac{\alpha}{2}\left[2\gamma_{ijk}-A_{ink}{}^\alpha l_j-A_{jnk}{}^\alpha l_i-A_{inj}{}^\alpha l_k-A_{knj}{}^\alpha l_i+A_{jni}{}^\alpha l_k+A_{kni}{}^\alpha l_j\right]\\ &+\left(A_{ink}+a_{i,x^k}\right){}^Ll_j+\left(A_{jnk}+a_{j,x^k}\right){}^Ll_i+\left(A_{inj}+a_{i,x^j}\right){}^Ll_k+\left(A_{knj}+a_{k,x^j}\right){}^Ll_i\\ &-\left(A_{jni}+a_{j,x^i}\right){}^Ll_k-\left(A_{kni}+a_{k,x^i}\right){}^Ll_j.\end{aligned}$$

Replacing (28) in $G^s = {}^L g^{is}\gamma_{ijk}y^j y^k$ it results the spray coefficients of RRQ-space.

Theorem 3.1 Let $QF^n = (M, F)$ be a quartic Finsler space of scalar curvature $K(x, y)$. A R-Randers quartic space $RRQ^n = (M, L)$ is of the same scalar curvature $K(x, y)$ if and only if

$$\frac{1}{L^3}{}^L R_{ij} - \frac{1}{F^3}{}^F R_{ij} = K(x, y)\frac{1}{\alpha}{}^\alpha h_{ij}. \tag{29}$$

Proof: The R-Randers quartic metric L is of scalar curvature $K(x, y)$ if and only if ${}^L R_{ij} = L^2 K(x, y){}^L h_{ij}$, or equivalently

$${}^L R_{ij} = L^3 K(x, y)\frac{1}{F}{}^F h_{ij} + L^3 K(x, y)\frac{1}{\alpha}{}^\alpha h_{ij}.$$

The quartic metric F is of scalar curvature $K(x, y)$ if and only if ${}^F R_{ij} = F^2 K(x, y){}^F h_{ij}$. Replacing this relation in the expression of we get immediately the conclusion.

Theorem 3.2 Let $RRQ^n = (M, L)$ be an n-dimensional R-Randers quartic space. If the three metrics L,F i α are of the same constant curvature K and $\frac{1}{\alpha}l_i l_j + \frac{3}{F}a_i a_j = \frac{3}{F}a_{ij}$, then

$${}^L R_{ij} = \frac{L^3}{(n-1)\alpha}{}^\alpha R_{ij}. \tag{30}$$

Proof: The Riemann space$(M, \alpha)$is of constant curvature if and only if it is an Einstein space. So, ${}^\alpha R_{ij} = (n-1)K b_{ij}$. We replace this in (3.18) and we take into account that $\frac{1}{\alpha}l_i l_j + \frac{3}{F}a_i a_j = \frac{3}{F}a_{ij}$.

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