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ORLICZ-POINCARÉ INEQUALITIES AND EMBEDDINGS OF ORLICZ-SOBOLEV SPACES ON METRIC SPACES

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Abstract. The main result of this paper shows that an Orlicz-Sobolev space with zero boundary values on a doubling metric measure space with homogeneous dimension s, corresponding to an Orlicz function generalizing t^q with $q < s$, is continuously embedded in an Orlicz space generalizing L^{q^*}, where $q^* = \frac{sq}{s-q}$. In order to prove this embedding result, we use an optimal result of Heikkinen [18] describing sharp self-improving properties of Orlicz-Poincaré inequalities in connected metric spaces. We also prove an Orlicz-Poincaré inequality for functions vanishing on large subsets of balls and some counterparts of the results mentioned above for Orlicz-Sobolev spaces of Hajłasz type.

1. Introduction

During the last fifteen years, the theory of Sobolev spaces has been extended to the setting of metric measure spaces, including the study of Hajłasz spaces [12], Newtonian spaces [27], Cheeger spaces [9].

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This extension was motivated by the study of several aspects of geometric analysis and nonlinear potential theory, such as quasiconformal theory on Carnot groups, nonlinear potential theory with weighted Sobolev spaces, analysis on fractals, potential theory on infinite graphs. Recently, the theory of Orlicz-Sobolev spaces has been also extended to metric measure spaces [1], [29].

Sobolev inequalities and Sobolev-Poincaré inequalities play a very important role in proving regularity properties, such as Harnack's inequality and Hölder continuity, for solutions to nonlinear degenerate elliptic equations, as well for minimizers of variational integrals of Dirichlet type. It is known that every Euclidean space $\mathbb{R}^n$ supports a $(1,1)-$Poincaré inequality, therefore, by Hölder's inequality, supports a $(1,p)-$Poincaré inequality for every p with $1 \leq p < \infty$. The classical Sobolev embedding theorem for exponent below the dimension says that $W^{1,p}(\mathbb{R}^n)$ continuously embeds in $L^{np/(n-p)}(\mathbb{R}^n)$ if $1 \leq p < n$. The corresponding Sobolev-Poincaré inequality shows that

$$\left(\frac{1}{\mu(B)}\int_B |u-u_B|^{\frac{np}{n-p}}\,d\mu\right)^{(n-p)/np} \leq c(n,p)diam\,(B)\left(\frac{1}{\mu(B)}\int_B |\nabla u|^p\,d\mu\right)^{1/p},$$

for every $u \in W^{1,p}(B)$, where $1 \leq p < n$ and $B \subset \mathbb{R}^n$ is a ball.

Throughout this paper we deal with a metric measure space (X,d,μ), which is a metric space (X,d) equipped with a Borel regular outer measure μ, that is finite and positive on balls.

Poincaré inequalities in doubling metric measure spaces have several self-improving features.

In a doubling measure space a weak $(1,q)-$Poincaré inequality implies a Sobolev-Poincaré inequality called weak $(t,q)-$Poincaré inequality, for some $t > 1$, as it was proved by Hajłasz and Koskela [13]. Let (X,d,μ) be a doubling metric measure space with a homogeneous dimension s. Assume that the pair (u,g) satisfies a weak $(1,q)-$Poincaré inequality in X, with constants $c_P > 0$ and $\tau \geq 1$. Then for every $1 \leq t \leq \frac{sq}{s-q}$ if $q < s$ and for every $t \geq 1$ if $q \geq s$ the pair (u,g) satisfies the weak $(t,q)-$Poincaré inequality

(1.1)
$$\left(\frac{1}{\mu(B)}\int_B |u-u_B|^t\,d\mu\right)^{1/t} \leq cr\left(\frac{1}{\mu((1+\delta)\tau B)}\int_{(1+\delta)\tau B} g^q d\mu\right)^{1/q},$$

for every ball $B = B(x,r)$ in X. Here $\delta > 0$ is a constant and $c = c(s, C_s, C_P, \tau, \delta)$ does not depend neither on B, nor on the pair (u, g).

Moreover, if $q > s$ then u has a locally Hölder continuous representative satisfying

$$|u(x) - u(y)| \le C r^{s/q} d(x,y)^{1-s/q} \left(\frac{1}{\mu(B(a,5\tau r))} \int_{5\tau B} g^q d\mu \right)^{1/q}$$

for all $x, y \in B$, where B is an arbitrary ball of radius r and $C > 0$ is a constant.

Tuominen introduced in [29] the Orlicz-Sobolev counterparts of the weak $(1,p)-$Poincaré inequality and of the weak $(p,p)-$Poincaré inequality, namely the weak $(1,\Phi)-$Poincaré inequality and the weak $(\Phi,\Phi)-$Poincaré inequality, respectively. By Jensen's inequality, it follows that a $(1,\Phi)-$Poincaré inequality follows

Heikinnen has proved in [18] that Orlicz-Sobolev inequalities have sharp self-improving properties in connected metric spaces, extending the above result of Hajłasz and Koskela, but also some sharp inequalities proved by Cianchi for Orlicz-Sobolev spaces on $\mathbb{R}^n$ in [6],[7],[8].

The aim of this paper is to prove a result similar to a generalization of the weak $(q,q)-$Poincaré inequality for Newtonian spaces for boundary values, in the case when the exponent q is smaller that the homogeneous dimension s of the doubling metric space. This result shows that an Orlicz-Sobolev space with zero boundary values, corresponding to an Orlicz function generalizing t^q with $q < s$, is continuously embedded in an Orlicz space generalizing L^{q^*}, where $q^* = \frac{sq}{s-q}$ is the Sobolev conjugate of q. We compare this embedding result to a $(\Phi,\Phi)-$Poincaré inequality for Orlicz-Sobolev functions with zero boundary values, proved in [23]. Our main tool here is a part of the result of Heikinnen [18, Theorem 1.7]. The main result, Theorem 2 is based on an auxiliary result, Theorem 1 which provides an Orlicz-Poincaré inequality for Orlicz-Sobolev functions vanishing on large subsets of balls. Theorem 1 is a partial extension to the Orlicz-Sobolev spaces of Lemma 2.1 proved in [21] for Newtonian spaces. Finally, we prove some counterparts of Theorem 1 and Theorem 2 for Orlicz-Sobolev spaces of Hajłasz type, using a $(\Phi,\Phi)-$Orlicz-Sobolev inequality proved by Aïssaoui in [3].

2. Preliminaries

Let us recall some important notions from the theory of Orlicz spaces [25]. A function $\Psi : [0,\infty] \to [0,\infty]$ is called a Young function if it has the form

$$\Psi(t) = \int_0^t \psi(s)ds, t \in [0,\infty),$$

where $\psi : [0,\infty] \to [0,\infty]$ is increasing, left-continuous, neither identically zero nor identically infinite on $(0,\infty)$. An $N-$function is a continuous Young function $\Psi : [0,\infty] \to [0,\infty]$ satisfying $\Psi(t) = 0$ only if $t = 0$, $\lim\limits_{t\to\infty} \frac{\Psi(t)}{t} = \infty$ and $\lim\limits_{t\to 0} \frac{\Psi(t)}{t} = 0$.

For each Young function Ψ, the complementary function $\widehat{\Psi}$ is defined by

$$\widehat{\Psi}(s) = \sup\{st - \Psi(t) : t \geq 0\}.$$

The complementary function $\widehat{\Psi}$ of a Young function Ψ is also a Young function, and Ψ is the complementary function of $\widehat{\Psi}$.

A Young function $\Psi : [0,\infty) \to [0,\infty)$ said to to satisfy a Δ_2-condition if there is a constant $C_\Phi > 0$ such that $\Phi(2t) \leq C_\Phi \Phi(t)$ for every $t \in [0,\infty)$. A Young function satisfying a Δ_2-condition is called *doubling* (globally). Every doubling Young function is strictly increasing and continuous. The Δ_2-condition for an increasing Young function Φ implies the power growth estimate: $\Phi(\lambda t) \leq C_\Phi \lambda^{\log_2 C_\Phi} \Phi(t)$, for all $\lambda \geq 1, t \geq 0$.

Let $\Psi : [0,\infty] \to [0,\infty]$ be a Young function. Let $(X, \mathcal{A}, \mu)$ be a measure space with μ a complete and $\sigma-$finite measure and let $\Omega \subset X$ be an open set.

The Orlicz space $L^\Psi(\Omega)$ is defined by

$$L^\Psi(\Omega) = \left\{ u : \Omega \to [-\infty,\infty] : u \text{ measurable}, \int_\Omega \Psi(\lambda |u|) d\mu < \infty \text{ for some } \lambda > 0 \right\}.$$

The Orlicz space $L^\Psi(\Omega)$ is a Banach space with the Luxemburg norm defined by

$$\|u\|_{L^\Psi(\Omega)} = \inf\left\{ k > 0 : \int_\Omega \Psi\left(\frac{|u|}{k}\right) d\mu \leq 1 \right\}.$$

An equivalent norm on $L^\Psi(\Omega)$, called the Orlicz norm, is defined by

$$|u|_{L^\Psi(\Omega)} = \sup\left\{ \int_\Omega |uv|\, d\mu : v : \Omega \to [-\infty,\infty], \int_\Omega \Psi(|v|) d\mu \le 1 \right\}.$$

Recall the following formulas for the Luxemburg norm, respectively for the Orlicz norm of a characteristic function . Let Φ be a Young function and let $\widehat{\Phi}$ be its complementary Young function. Suppose that $A \subset X$ with $0 < \mu(A) < \infty$. Then the Luxemburg norm of χ_A is

$$\|\chi_A\|_{L^\Phi(X)} = 1/\Phi^{-1}\left(\frac{1}{\mu(A)}\right) \tag{2.1}$$

and the Orlicz norm of χ_A is

$$|\chi_A|_{L^{\hat{\Phi}}(X)} = \mu(A)\Phi^{-1}\left(\frac{1}{\mu(A)}\right), \tag{2.2}$$

We use the strongest form of the generalized Hölder's inequality for a pair of complementary Young functions Φ and $\widehat{\Phi}$ [26, Theorem 8, page 17]:

$$\int_X |u(x)v(x)|\, d\mu \le \|u\|_{L^\Phi(X)} \, |v|_{L^{\hat{\Phi}}(X)} \tag{2.3}$$

Remark 1. *Since μ is finite on balls, for every doubling $N-$function $\Phi : [0,\infty) \to [0,\infty)$ we have $L^\Phi(X) \subset L^1_{loc}(X)$, by* [25, Proposition 3.1.7].

Definition 1. *The measure μ on the metric space (X, d, μ) is said to be doubling if there is a constant $C_d \ge 1$ such that*

$$\mu(B(x,2r)) \le C_d \mu(B(x,r)) \tag{2.4}$$

for every ball $B(x,r) \subset X$.

For every doubling measure μ there are some positive constants C_s and s so that

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \ge C_s \left(\frac{r}{r_0}\right)^s, \tag{2.5}$$

for all $0 < r \le r_0$ and $x \in B(x_0, r_0)$. Here s is called a *homogeneous dimension* of the metric measure space X.

The doubling property of the measure allows several extensions the setting of metric measure spaces of some classical results, such as Vitali covering theorem, Lebesgue's differentiation theorem and the maximal function theorem [16]. In harmonic analysis, doubling metric measure spaces are extensively used and are called homogeneous spaces [10].

We cannot speak of weak partial derivatives for a real function defined on a general metric space. A substitute for the norm of the gradient in analysis on metric measure spaces is the concept of upper gradient, introduced by Heinonen and Koskela in [17]. Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0, +\infty]$ is said to be an *upper gradient* of u in X if

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_{\gamma} g \, ds, \tag{2.6}$$

for every compact rectifiable path $\gamma : [0, 1] \to X$.

Upper gradients are unstable under changes $\mu-$a.e. and under limits, therefore a more general notion of weak upper gradient, which is more flexible, is more apropriate for the purposes of the analysis on metric measure spaces [15]. The notion of weak upper gradient is defined with respect to a Banach function space and is essential in introducing and studying some Sobolev-type spaces on metric measure spaces.

Let (X, Σ, μ) be a complete $\sigma-$finite measure space and $\mathcal{M}^+(X)$ be the collection of all measurable functions $f : X \to [0, +\infty]$. Let $\mathbf{N} : \mathcal{M}^+(X) \to [0, \infty]$ be a *Banach function norm* [4]. The collection B of the $\mu-$measurable functions $f : X \to [-\infty, +\infty]$ for which $\mathbf{N}(|f|) < \infty$ is called a *Banach function space* on X. For $f \in B$ define $\|f\|_B = \mathbf{N}(|f|)$. Then $(B, \|\cdot\|_B)$ is a complete normed space. Some important examples of Banach function spaces are Lebesgue spaces, Orlicz spaces, Lorentz spaces and Marcinkiewicz spaces.

Let $(B, \|\cdot\|_B)$ be a Banach function space corresponding to a metric measure space (X, d, μ).

A function $g : X \to [0, \infty]$ is said to be a Hajłasz gradient of u if there exists a set $E \subset X$ with $\mu(E) = 0$ such that $|u(x) - u(y)| \leq d(x, y)[g(x) + g(y)]$ for every $x, y \in X \setminus E$. The Sobolev-type space $M^{1,B}(X)$ is defined to be the space of functions $u \in B$ having a Hajłasz gradient in B. The space $M^{1,B}(X)$ is a Banach space with the norm

$\|u\|_{M^{1,B}(X)} = \|u\|_B + \inf \|g\|_B$, where the infimum is taken over all the Hajłasz gradients g of u satisfying $g \in B$.

The spaces $M^{1,B}(X)$ were introduced for the first time for $B = L^p$ by Hajłasz [12], starting from a Lipschitz-like pointwise estimate for Sobolev functions on Euclidean domains, and the extension to the case $B = L^\Psi$ has been made by Aïssaoui [3].

The notion of modulus of a path family is indispensable in order to define the extensions to metric measure spaces, based on upper gradients,of Sobolev spaces and Orlicz-Sobolev spaces.

Definition 2. *Let Γ be a family of paths in X. The $B-$modulus of the family Γ, denoted by $M_B(\Gamma)$ is definde to be the number* $\inf \|\rho\|_B$ *,where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ such that $\int_\gamma \rho ds \geq 1$ for all locally rectifiable paths $\gamma \in \Gamma$.*

$B-$modulus is an outer measure on the family of all paths in X [24]. The notion of modulus is fundamental for quasiconformal theory, both in the Euclidean setting and in the metric measure space setting.

Definition 3. *Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0, +\infty]$ is a $B-$weak upper gradient of u if (2.6) holds for all compact rectifiable paths $\gamma : [0, 1] \to X$ except for a path family Γ_0 in X with $M_B(\Gamma_0) = 0$.*

The collection $\widetilde{N}^{1,B}(X)$ of all functions $u \in B$ having a $B-$weak upper gradient $g \in B$ is a vector space. For $u \in \widetilde{N}^{1,B}(X)$ define $\|u\|_{1,B} = \|u\|_B + \inf_g \|g\|_B$, where the infimum is taken over all $B-$weak upper gradients $g \in B$ of u. The seminormed space $(\widetilde{N}^{1,B}(X), \|\cdot\|_{1,B})$ is turned into a normed space via the equivalence relation: $u \sim v \Leftrightarrow \|u - v\|_{1,B} = 0$. It turns out that $N^{1,B}(X) = \widetilde{N}^{1,B}(X)/\sim$ is a Banach space with the norm $\|u\|_{N^{1,B}} := \|u\|_{1,B}$ [24].

The space $N^{1,B}(X)$ was introduced and studied for the first time for $B = L^p(X)$ by Shanmugalingam [28], as a Sobolev-type space called Newtonian space, which is denoted by $N^{1,p}(X)$. The extension to the case $B = L^\Psi(X)$ was made by Tuominen [29] and by Aïssaoui [1], by introducing and studying the Orlicz-Sobolev space $N^{1,\Psi}(X)$.

Note that in the case where $X = \Omega \subset \mathbb{R}^n$ is a domain and $B = L^\Psi(X)$, with Ψ a doubling Young function, it turns out that

$N^{1,B}(X) = W^{1,\Psi}(\Omega)$ as Banach spaces and the norms are equivalent [29, Theorem 6.19].

A *capacity* with respect to the space $N^{1,B}(X)$, called $B-$capacity, is defined by

$$Cap_B(E) = \inf\{\|u\|_{N^{1,B}} : u \in N^{1,B}(X), u \geq 1 \text{ on } E\}.$$

The $B-$capacity Cap_B is an outer measure on X and represents the correct gauge for distinguishing between two functions in $N^{1,B}(X)$ [24]. In the case $B = L^\Psi(X)$ the $B-$capacity is denoted by Cap_Ψ and is called $\Psi-$capacity [29].

We will use Orlicz-Sobolev spaces with zero boundary values $N_0^{1,\Psi}(E)$ with $E \subset X$ [23]. We define the Banach-Sobolev spaces with zero boundary values $N_0^{1,B}(E)$ with $E \subset X$, where B is a Banach function space. In the case $B = L^\Psi(X)$ we denote $N_0^{1,B}(E)$ by $N_0^{1,\Psi}(E)$.

In the setting of metric measure spaces various Sobolev-type spaces with zero boundary values have been introduced, as follows: Hajłasz-Sobolev spaces $M_0^{1,p}$ by Kilpeläinen, Kinnunen and Martio in [19], Newtonian space $N_0^{1,p}$ by Shanmugalingam in [28], [27], Orlicz-Sobolev spaces $M_0^{1,\Phi}$, $N_0^{1,\Phi}$ by Aïssaoui [2] and Banach-Sobolev spaces $M_0^{1,B}$, $N_0^{1,B}$ in [24].

Denote by $\widetilde{N}_0^{1,B}(E)$ be the collection of functions $u : E \to R$ for which there exists $\overline{u} \in \widetilde{N}^{1,B}(X)$ such that $\overline{u} = u$ $\mu-$a.e. on E and $Cap_B(\{x \in X \backslash E : \overline{u}(x) \neq 0\}) = 0$. If $u, v \in \widetilde{N}_0^{1,B}(E)$ define $u \simeq v$ if $u = v$ $\mu-$a.e. on E. Then $\simeq$ is an equivalence relation. We consider the quotient space $N_0^{1,B}(E) = \widetilde{N}_0^{1,B}(E)/\simeq$. A norm on $N_0^{1,B}(E)$ is unambiguously defined by $\|u\|_{N_0^{1,B}(E)} := \|\overline{u}\|_{N^{1,B}(X)}$.

For $u : E \to R$ we denote by $\widetilde{u}$ the extension by zero to X, defined by $\widetilde{u}(x) = u(x)$ if $x \in E$ and $\widetilde{u}(x) = 0$ if $x \in X \setminus E$.

If $\overline{u} \in \widetilde{N}^{1,B}(X)$ corresponds to $u \in \widetilde{N}_0^{1,B}(E)$ as in the above definition, define $\widetilde{\widetilde{u}}(x) = \overline{u}(x)$ if $x \in E$ and $\widetilde{\widetilde{u}}(x) = 0$ if $x \in X \setminus E$. Since $\widetilde{\widetilde{u}} = \overline{u}$ in the complement of a set of $B-$capacity zero, it follows that $\widetilde{\widetilde{u}} \in \widetilde{N}^{1,B}(X)$ and $\widetilde{\widetilde{u}}$ defines the same equivalence class in $N^{1,B}(X)$ as $\overline{u}$.

Roughly speaking, for an open set E we have $u \in N_0^{1,B}(E)$ if u has an extension belonging to $N^{1,B}(X)$ and vanishing on the boundary of

E. Every Lipschitz function with compact support in an open set $\Omega \subset X$ belongs to $N_0^{1,B}(\Omega)$. It turns out that $N_0^{1,B}(E)$ is a closed subspace of the Banach space $N^{1,B}(X)$.

Similarly, we can generalize the definition given by Aïssaoui in [2] to the Orlicz-Sobolev space with zero boundary values $M_0^{1,\Phi}(E)$, based on the notion of Hajłasz gradient. A *capacity* with respect to the space $M^{1,B}(X)$ is defined by

$$C_B(E) = \inf\{\|u\|_{N^{1,B}} : u \in M^{1,B}(X), u \geq 1 \text{ on a neighborhood of } E\}.$$

A function $u : X \to [-\infty, \infty]$ is said to be C_B-quasicontinuous in X if for every $\varepsilon > 0$ there is a set $E \subset X$ such that $C_B(E) < \varepsilon$ and the restricion of u to $X \setminus E$ is continuous.

We say that $u \in M_0^{1,B}(E)$ if there is a C_B-quasicontinuous function $\overline{u} \in M^{1,B}(E)$ such that $\overline{u} = u$ $\mu-$a.e. on E and $C_B(\{x \in X\backslash E : \overline{u}(x) \neq 0\}) = 0$. In the case $B = L^\Phi(X)$ we have $M_0^{1,B}(E) = M_0^{1,\Phi}(E)$.

In the classical theory of Sobolev spaces on $\mathbb{R}^n$, a $(1,p)-$Poincaré inequality provides a control on the average oscillation of a function on a ball in terms of the average value of the p-th power of the gradient. The $(1,p)-$Poincaré inequality was extended to metric measure spaces, being an important tool in dealing with quasiconformal theory and nonlinear potential theory in this general setting.

Denote the mean value of a function $u \in L^1(A)$ over A by $u_A := \frac{1}{\mu(A)} \int_B u d\mu$, where $0 < \mu(A) < \infty$. For a ball $B = B(x,r)$ we denote $\tau B = B(x, \tau r)$.

Definition 4. [14] *Let Ω be an open subset of the metric measure space X. A pair formed by $u \in L^1_{loc}(\Omega)$ and a Borel measurable function $g : \Omega \to [0,\infty]$ is said to satisfy a weak $(1,p)-$Poincaré inequality, $1 \leq p < \infty$, in Ω if there exist some constants $C_P > 0$ and $\tau \geq 1$ such that for every ball $B = B(x,r)$ satisfying $\tau B \subset \Omega$,*

$$\frac{1}{\mu(B)} \int_B |u - u_B| \; d\mu \leq C_P r \left(\frac{1}{\mu(\tau B)} \int_{\tau B} g^p \; d\mu \right)^{1/p}.$$

It is said that Ω supports a weak $(1,p)-$Poincaré inequality if the above inequality holds for every $u \in L^1_{loc}(\Omega)$ and every upper gradient g of u, with fixed constants C_P and τ. We may replace in the above definition upper gradients by $p-$weak upper gradients, since every $p-$weak upper gradient can be approximated in L^p-norm by a sequence of upper gradients [15, Lemma 2.4].

The weak $(1,p)-$ Poincaré inequality has been generalized for Orlicz-Sobolev spaces by Tuominen, as follows:

Definition 5. [29, Definition 5.2] *Let $\Phi : [0,\infty) \to [0,\infty)$ be a strictly increasing Young function and $\Omega \subset X$ an open set. We say that a function $u \in L^1_{loc}(\Omega)$ and a Borel measurable non-negative function g on Ω satisfy a $(1,\Phi)-$weak Poincaré inequality in Ω if there exist some constants $C_{P,\Phi} > 0$ and $\tau \geq 1$ such that*

$$\frac{1}{\mu(B)} \int_B |u - u_B|\, d\mu \leq C_{P,\Phi} r \Phi^{-1} \left(\frac{1}{\mu(\tau B)} \int_{\tau B} \Phi(g) d\mu \right). \tag{2.7}$$

for each ball $B = B(x,r)$ satisfying $\tau B \subset \Omega$. It is said that Ω supports a weak $(1,\Psi)-$Poincaré inequality if the above inequality holds for each function $u \in L^1_{loc}(\Omega)$ and every upper gradient g of u, with fixed constants.

Remark 2. *Every $\Phi-$weak upper gradient can be approximated in $L^\Phi-$norm by a sequence of upper gradients* [29, Lemma 4.3]. *If Φ is doubling, we may replace in the above definition upper gradients by $\Phi-$weak upper gradients.*

Suppose that X supports a weak $(1,\Phi)-$Poincaré inequality. Then small spheres in X are non-empty: for each $x \in X$ and every $r > 0$ such that $B(x,r) \neq X$, there is a point on the sphere $S(x,r) = \{y \in X : d(y,x) = r\}$. If $S(x,r)$ would be empty, then the ball $B(x,r)$ would be pathwise disconnected from $X \setminus B(x,r)$. The characteristic function $u = \chi_{B(x,r)}$ would have zero as a $\Phi-$weak upper gradient, since for every path $\gamma : [0,1] \to X$ we have either $\{\gamma(0), \gamma(1))\} \subset B(x,r)$ or $\{\gamma(0), \gamma(1))\} \subset X \setminus B(x,r)$, therefore $|u(\gamma(1)) - u(\gamma(0))| = 0$. Let $B = B(x,R)$, where $R > r$ is such that $B(x,R) \setminus B(x,r) \neq \varnothing$. Then $u_B = \frac{\mu(B(x,r))}{\mu(B)} < 1$. By the weak $(1,\Phi)-$Poincaré inequality for the pair (u,g) on B, where $u = \chi_{B(x,r)}$ and $g = 0$, we have $\int_B |u - u_B|\, d\mu \leq 0$. Then $u = u_B$ $\mu - a.e.$ on B, which is false, since $u = 1$ on the set of positive measure $B(x,r)$. It follows that $S(x,r)$ is non-empty.

3. Poincaré inequalities for Orlicz-Sobolev functions with zero boundary values and an embedding result

Let $s > 1$. Denote $s' = \frac{s}{s-1}$. In the following, we assume that Φ is a Young function satisfying

$$\int_0^1 \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt < \infty \text{ and } \int_1^\infty \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt = \infty, \tag{3.1}$$

As in [5] and [18], consider $\Psi_s(r) = \left(\int_0^r \left(\frac{t}{\Phi(t)}\right)^{s'-1} dt\right)^{1/s'}$. Then, define $\Phi_s = \Phi \circ \Psi_s^{-1}$.

The following result is a part of Theorem 1.7 in [18, Theorem 1.7], which extends to Orlicz-spaces the result of Hajłasz and Koskela [13, Theorem 5.1] saying that " in a doubling measure space a weak $(1,p)$ −Poincaré inequality implies a weak (t,p) −Poincaré inequality", where t is smaller than a homogeneous dimension of the doubling metric space.

Lemma 1. [18, Theorem 1.7] *Assume that (X, d, μ) is doubling and supports a weak $(1, \Phi)-$Poincaré inequality 2.7). Let s be a homogeneous dimension of (X, d, μ). Let B be a ball of radius r,let $\delta > 0$ and let τ be the constant from the weak $(1, \Phi)-$Poincaré inequality and $\widehat{B} = (1+\delta)\tau B$.*

If the Young function Φ satisfies (3.1), then $N^{1,\Phi}(\widehat{B}) \subset L^{\Phi_s}(B)$. Moreover, for every $u \in N^{1,\Phi}(\widehat{B})$ and every $\Phi-$weak upper gradient g of u, we have

$$\|u - u_B\|_{L^{\Phi_s}(B)} \le cr\mu(B)^{-1/s} \|g\|_{L^{\Phi}(\widehat{B})} . \tag{3.2}$$

Here $c = c\,(s, C_s, C_{P,\Phi}, \tau, \delta) > 0$ is a constant.

Remark 3. *In the classical case of Lebesgue spaces, $\Phi(t) = t^p$, with $p \ge 1$, satisfies conditions (3.1) if and only if $p < s$. In this case we have $\Psi_s(r) = k_1 r^{(s-p)/s}$, where $k_1 = \left(\frac{s-1}{s-p}\right)^{(s-1)/s}$. Hence we obtain $\Phi_s(t) = kt^{sp/(s-p)}$, where $k = k_1^{sp/(p-s)}$.*

Theorem 1. *Let X be a doubling metric measure space, with a homogeneous dimension s, supporting a weak $(1,\Phi)-$Poincaré inequality, where the Young function Φ satisfies (3.1). Assume that there exists a decreasing function $\varphi:(0,1]\to[1,\infty)$ with $\varphi(t)>1$ for every $0<t<1$, such that $\Phi_s^{-1}(\lambda t)\geq\lambda\varphi(\lambda)\Phi_s^{-1}(t)$ for all $\lambda\in(0,1]$ and $t\in[0,\infty)$. Let B be a ball of radius r, $\delta>0$ and let τ be the constant from the weak $(1,\Phi)-$Poincaré inequality and $\widehat{B}=(1+\delta)\tau B$. Suppose that $u\in N^{1,\Phi}(X)$ and g is a $\Phi-$weak upper gradient of u in X. Let $A=\{x\in B:u(x)\neq 0\}$. If $\mu(A)\leq\gamma\mu(B)$ for some γ with $0<\gamma<1$, then there is a constant $C>0$ so that*

$$\|u\|_{L^{\Phi_s}(B)}\leq Cr\mu(B)^{-1/s}\|g\|_{L^{\Phi}(\widehat{B})} \tag{3.3}$$

The constant C depends only on $\varphi(\gamma)$ and on the constant c of (3.2).

Proof. By the subadditivity of Luxemburg's norm,

$$\|u\|_{L^{\Phi_s}(B)}\leq\|u-u_B\|_{L^{\Phi_s}(B)}+\|u_B\|_{L^{\Phi_s}(B)}. \tag{3.4}$$

By (2.1),

$$\|u_B\|_{L^{\Phi_s}(B)}=\frac{|u_B|}{\Phi_s^{-1}\left(\frac{1}{\mu(B)}\right)}\leq\frac{1}{\mu(B)\Phi_s^{-1}\left(\frac{1}{\mu(B)}\right)}\int_B|u|\,d\mu= \tag{3.5}$$

$$=\frac{1}{\mu(B)\Phi_s^{-1}\left(\frac{1}{\mu(B)}\right)}\int_A|u|\,d\mu.$$

Using Hölder's inequality (2.3) and the monotonicity of Luxemburg's norm, we get

$$\int_A|u|\,d\mu\leq\|u\|_{L^{\Phi_s}(B)}|\chi_A|_{L^{\widehat{\Phi_s}}(X)}.$$

Using (2.1), this implies

$$\int_A|u|\,d\mu\leq\mu(A)\Phi_s^{-1}\left(\frac{1}{\mu(A)}\right)\|u\|_{L^{\Phi_s}(B)}. \tag{3.6}$$

Using (3.4), (3.5) and (3.6), we obtain

$$\left(1-\frac{\Theta_s\left(\frac{1}{\mu(A)}\right)}{\Theta_s\left(\frac{1}{\mu(B)}\right)}\right)\|u\|_{L^{\Phi_s}(B)}\leq\|u-u_B\|_{L^{\Phi_s}(B)}. \tag{3.7}$$

Here we denoted $\Theta_s(t) = \frac{1}{t}\Phi_s^{-1}(t)$. Since the function Φ_s is convex, the function Θ_s is decreasing. By our assumption, $\Phi_s^{-1}(\lambda t) \geq \lambda\varphi(\lambda)\Phi_s^{-1}(t)$ for all $\lambda \in (0,1]$ and $t \in [0,\infty)$. It follows that $\Theta_s(\lambda t) \geq \lambda\Theta_s(t)$ whenever $\lambda \in (0,1]$ and $t \in [0,\infty)$. In particular, $\Theta_s\left(\frac{1}{\mu(B)}\right) \geq \Theta_s\left(\frac{\gamma}{\mu(A)}\right) \geq \varphi(\gamma)\,\Theta_s\left(\frac{1}{\mu(A)}\right)$. Then (3.7) implies

$$\|u\|_{L^{\Phi_s}(B)} \leq \frac{\varphi(\gamma)}{\varphi(\gamma)-1}\|u-u_B\|_{L^{\Phi_s}(B)}.$$

By Heikkinen's result (3.2), the above inequality implies

$$\|u\|_{L^{\Phi_s}(B)} \leq Cr\mu(B)^{-1/s}\|g\|_{L^{\Phi}(\widehat{B})},$$

where $C = c\frac{\varphi(\gamma)}{\varphi(\gamma)-1}$.

Corollary 1. *Let X be a doubling metric measure space, with a homogeneous dimension s and supporting a weak $(1,q)-$Poincaré inequality, where $1 \leq q < s$. Let B be a ball of radius r. Suppose that $u \in N^{1,q}(X)$ and g is a $q-$weak upper gradient of u in X. Let $A = \{x \in B : u(x) \neq 0\}$. If $\mu(A) \leq \gamma\mu(B)$ for some γ with $0 < \gamma < 1$, then there is a constant $C' > 0$ so that*

$$\left(\frac{1}{\mu(B)}\int_B |u|^{q^*}\,d\mu\right)^{\frac{1}{q^*}} \leq C'r\left(\frac{1}{\mu(\widehat{B})}\int_{\widehat{B}} |g|^q\,d\mu\right)^{1/q}. \tag{3.8}$$

Here $q^* = \frac{sq}{s-q}$ and $\widehat{B} = (1+\delta)\tau B$, where $\delta > 0$ and τ are as in $(1,q)-$Poincaré inequality (1.1). The constant C' depends only on s, q, γ and on c_P *of the $(1,q)-$Poincaré inequality* .

Proof. Take $\Phi(t) = t^q$ in the above theorem. Since $1 \leq q < s$, conditions (3.1) are satisfied. We have $\Phi_s(t) = k(s,q)\,t^{q^*}$, where $k(s,q) = \left(\frac{s-q}{s-1}\right)^{\frac{q(s-1)}{s-q}}$. By (3.3), $(k(s,q))^{1/q*}\|u\|_{L^{q^*}(B)} \leq Cr\mu(B)^{-1/s}\|g\|_{L^q(\widehat{B})}$, hence

$$\left(\frac{1}{\mu(B)}\int_B |u|^{q^*}\,d\mu\right)^{\frac{1}{q^*}} \leq (k(s,q))^{-1/q*}\,Cr\left(\frac{1}{\mu(B)}\int_{\widehat{B}} |g|^q\,d\mu\right)^{1/q}.$$

By the doubling property of the measure (2.4), $\mu(\widehat{B}) \leq C_d\left((1+\delta)\tau\right)^{\log_2 C_d}$. Then the above inequality implies (3.8), where $C' = \left(k\left(s,q\right)\right)^{-1/q*} C \left(C_d\left((1+\delta)\tau\right)^{\log_2 C_d}\right)^{1/q}$.

Note that, by Hölder's inequality, (3.8) implies

$$\left(\frac{1}{\mu(B)}\int_B |u|^t \, d\mu\right)^{\frac{1}{t}} \leq C'r\left(\frac{1}{\mu(\widehat{B})}\int_{\widehat{B}} |g|^q \, d\mu\right)^{1/q}$$

for all $1 \leq t \leq q^*$. This shows that Theorem 1 partially extends Lemma 2.1 of [21] from Newtonian spaces to Orlicz-Sobolev spaces. The term "partially" refers to the fact that in Lemma 2.1 of [21] the case $t > q^*$ was also covered. On the other hand, in Lemma 2.1 of [21] it was assumed that $u \in N^{1,p}(X)$ for some $p > q$, but it suffices to assume that $u \in N^{1,q}(X)$.

Remark 4. *We assumed in Theorem 1 that there exists a decreasing function* $\varphi : (0,1] \to [1,\infty)$ *with* $\varphi(t) > 1$ *for every* $0 < t < 1$*, such that*

$$\Phi_s^{-1}(\lambda t) \geq \lambda\varphi(\lambda)\,\Phi_s^{-1}(t) \tag{3.9}$$

for all $\lambda \in (0,1]$ *and* $t \in [0,\infty)$*. This condition is satisfied in the classical case, where* $\Phi(t) = t^q$*, since then* $\frac{\Phi_s^{-1}(\lambda t)}{\lambda\Phi_s^{-1}(t)} = \lambda^{(1-q^*)}/q^*$ *is decreasing in* $\lambda > 0$ *and is greater than* 1 *for all* $\lambda \in (0,1]$.

Lemma 2.1. of [21] implies the following Poincaré inequality for Newton-Sobolev functions with zero boundary values [21, p. 407]:

there exists $c > 0$ such that for every ball $B = B(z,r)$ with $0 < r < diam(X)/3$ and every $u \in N_0^{1,q}(B)$ we have

$$\left(\frac{1}{\mu(B)}\int_B |u|^t \, d\mu\right)^{1/t} \leq cr\left(\frac{1}{\mu(B)}\int_B |g_u|^q \, d\mu\right)^{1/q}$$

Here $1 \leq t \leq \frac{sq}{s-q}$ if $q < s$ and $t \geq 1$ if $q \geq s$.

We extend the above inequality to the setting of Orlicz-Sobolev spaces. We need a result that guarantees that, extending with zero a weak upper gradient of a function in an Orlicz-Sobolev space with zero boundary values, we get a weak upper gradient of a representative of the extension with zero of the function.

Recall that a metric space is called *proper* if every closed ball is compact.

Lemma 2. [23] *If $\Phi : [0,\infty) \to [0,\infty)$ is a doubling $N-$function and X is proper, doubling and supporting a weak $(1,\Phi)-$Poincaré inequality, then $Lip_C(\Omega)$ is a dense subspace of $N_0^{1,\Phi}(\Omega)$, for every open set $\Omega \subset X$.*

A consequence of the above density result is

Corollary 2. [23]**.** *If $u \in N_0^{1,\Phi}(\Omega)$ has an upper gradient $g \in L^\Phi(\Omega)$ in the open set Ω, then the function $\widetilde{g}$ is a $\Phi-$weak upper gradient of $\widetilde{\widetilde{u}}$ in X.*

Theorem 2. *Let X be a doubling metric measure space, with a homogeneous dimension s, proper and supporting a weak $(1,\Phi)-$Poincaré inequality, where Φ is a doubling $N-$function satisfying (3.1). Assume that there exists a decreasing function $\varphi : (0,1] \to [1,\infty)$ with $\varphi(t) > 1$ for every $0 < t < 1$, such that $\Phi_s^{-1}(\lambda t) \geq \lambda\varphi(\lambda)\Phi_s^{-1}(t)$ for all $\lambda \in (0,1]$ and $t \in [0,\infty)$. Let B be a ball of radius r,with $0 < r < diam(X)/3$. Suppose that $u \in N_0^{1,\Phi}(B)$ and g is a $\Phi-$weak upper gradient of u in B.Then*

$$\|u\|_{L^{\Phi_s}(B)} \leq Cr\mu(B)^{-1/s}\|g\|_{L^\Phi(B)}. \tag{3.10}$$

Here $C > 0$ is the constant of (3.3).

Proof. We may assume without loss of generality that g is an upper gradient of u in B. Indeed, assume that (3.10) holds under this assumption. For every $\Phi-$weak upper gradient g of u in B there is a decreasing sequence of upper gradients $(g_n)_{n\geq 1}$ of u in B, such that $\lim_{n\to\infty}\|g_n - g\|_{L^\Phi(X)} = 0$. By our assumption, $\|u\|_{L^{\Phi_s}(B)} \leq Cr\mu(B)^{-1/s}\|g_n\|_{L^\Phi(B)}$ for every $n \geq 1$ and letting n tend to infinity we get (3.10).

Let $\widetilde{g}$ be the extension with zero of g to X. Let $\overline{u} \in N^{1,\Phi}(X)$ such that $\overline{u} = u$ $\mu-$a.e. on B and $Cap_\Phi(\{x \in X \setminus B : \overline{u}(x) \neq 0\}) = 0$.

Define $\widetilde{\widetilde{u}}(x) = \overline{u}(x)$ if $x \in B$ and $\widetilde{\widetilde{u}}(x) = 0$ if $x \in X \setminus B$. Then $\widetilde{\widetilde{u}} = \overline{u}$ in $N^{1,\Phi}(X)$. According to Corollary 2, $\widetilde{g}$ is a $\Phi-$weak upper gradient of $\widetilde{\widetilde{u}}$ in X.

Denote $B = B(z, r)$. Since X supports a weak $(1, \Phi)-$Poincaré inequality and $B(z, 2r) \neq X$, it follows that the sphere $S(z, 2r)$ is non-empty. Let $y \in S(z, 2r)$. Since $B(y, r) \subset B(z, 3r) \cap (X \setminus B(z, r))$, clearly $\mu(B(z, 3r)) \geq \mu(B(z, r)) + \mu(B(y, r))$. By (2.5), $\mu(B(y, r)) \geq C_s 3^{-s} \mu(B(z, 3r))$. The latter two inequalities imply $\mu(B(z, r)) \leq (1 - C_s 3^{-s}) \mu(B(z, 3r))$. Denoting $A = \{x \in B(z, 3r) : u(x) \neq 0\}$, we have $A \subset B(z, r)$. Then

$$\mu(A) \leq \gamma \mu(B(z, 3r)),$$

where $\gamma = (1 - C_s 3^{-s})$, $0 < \gamma < 1$.

By Theorem 1,

$$\left\| \widetilde{\widetilde{u}} \right\|_{L^{\Phi_s}(B(z,3r))} \leq C r \mu(B(z, 3r))^{-1/s} \left\| \widetilde{g} \right\|_{L^{\Phi}(B(z,3(1+\delta)\tau r))}.$$

Taking account that $\left\| \widetilde{\widetilde{u}} \right\|_{L^{\Phi_s}(B(z,3r))} = \|u\|_{L^{\Phi_s}(B(z,r))}$, that $\|\widetilde{g}\|_{L^{\Phi}(B(z,3(1+\delta)\tau r))} = \|g\|_{L^{\Phi}(B(z,r))}$ and $\mu(B(z, 3r))^{-1/s} < \mu(B(z, 3r))^{-1/s}$, the above inequality yields (3.10).

Remark 5. *Theorem 2 says that $N_0^{1,\Phi}(B) \subset L^{\Phi_s}(B)$ as a continuous embedding, under the given assumptions.*

Next we compare inequality (3.10) with the following $(\Phi, \Phi)-$Poincaré inequality proved in [23, Theorem 2]:

$$\int\limits_{\Omega} \Phi\left(\frac{|u|}{r}\right) d\mu \leq C \int\limits_{\Omega} \Phi(g) d\mu. \tag{3.11}$$

Inequality (3.11) holds under the following assumptions. X is a proper metric space, equipped with a doubling measure, supporting a weak $(1, \Psi)-$Poincaré inequality for some strictly increasing Young function Ψ. Here Φ is a doubling $N-$ function, such that $\Phi \circ \Psi^{-1}$ is an $N-$function satisfying a ∇_2-condition. $\Omega = B(x, r)$ with $r < \frac{diam(X)}{3}$, $u \in N_0^{1,\Psi}(\Omega)$ and $g \in L^{\Psi}(\Omega)$ is $\Psi-$weak upper gradient of u in Ω. The constant $C > 0$ does not depend on u or on Ω.

Theorem 2 of [23] implies the following inequality:

$$\|u\|_{L^{\Phi}(\Omega)} \leq C_0 \; diam(\Omega) \, \|g\|_{L^{\Phi}(\Omega)} . \tag{3.12}$$

Here the Young functions Φ, Ψ and the metric measure space X satisfy the conditions stated above. Ω is a bounded open non-empty set with $diam(\Omega) < diam(X)/3$, $u \in N_0^{1,\Phi}(\Omega)$ and $g \in L^{\Phi}(\Omega)$ is a $\Psi-$weak upper gradient of u in Ω.The constant $C_0 > 0$ does not depend on u or on Ω.

Definition 6. *Let Φ_1, $\Phi_2 : [0, \infty)$ be Young functions. It is said that Φ_2 dominates Φ_1 near infinity if there exist two positive constants c, T such that*

$$\Phi_1 (t) \leq \Phi_2 (ct) \;\text{ for all } t \geq T.$$

Lemma 3. [25] *Let Φ_1, $\Phi_2 : [0, \infty)$ be Young functions. If $\mu(X) < \infty$ and Φ_2 dominates Φ_1 near infinity, then $L^{\Phi_2}(X)$ is continuously embedded in $L^{\Phi_1}(X)$. Moreover, there exists a constant $M > 0$ depending only on Φ_1 and Φ_2 such that $\|u\|_{L^{\Phi_1}(X)} \leq M \|u\|_{L^{\Phi_2}(X)}$ for all $u \in L^{\Phi_2}(X)$.*

Lemma 4. *Let Φ be a Young function satisfying (3.1) for some $s > 1$. Then*

a) Φ_s dominates Φ near infinity;

b) There exists a constant $M = M(\Phi_1, \Phi_2) > 0$ such that for every bounded open set Ω in a metric measure space X,

$$\|u\|_{L^{\Phi}(\Omega)} \leq M \|u\|_{L^{\Phi_s}(\Omega)}$$

for all $u \in L^{\Phi_s}(\Omega)$.

Proof.

a) The function Ψ_s is concave, therefore $\frac{\Psi_s(t)}{t}$ is decreasing on $(0, \infty)$. Taking $T = 1$ and $c = \Psi_s(1)$, we have $\Psi_s(t) \leq ct$ for all $t \geq T$, hence $\Phi(t) = \Phi_s(\Psi_s(t)) \leq \Phi_s(ct)$ for all $t \geq T$.

b) Let $\Omega \subset X$ be a bounded open set. Since $\mu(\Omega) < \infty$ and Φ_s dominates Φ near infinity, $L^{\Phi_s}(\Omega) \subset L^{\Phi}(\Omega)$. Moreover, the above lemma shows that there exists $M = M(\Phi, s) > 0$, not depending on Ω, such that $\|u\|_{L^{\Phi}(\Omega)} \leq M \|u\|_{L^{\Phi_s}(\Omega)}$ for all $u \in L^{\Phi_s}(\Omega)$.

Corollary 3. *Under the assumptions of Theorem 2, there exists a constant $M > 0$, depending only on Φ and s, such that*

$$\|u\|_{L^{\Phi}(B)} \leq MCr\mu(B)^{-1/s} \|g\|_{L^{\Phi}(B)} . \tag{3.13}$$

for every ball B of radius r,with $0 < r <$diam$(X)/3$, whenever $u \in N_0^{1,\Phi}(B)$ and g is a $\Phi-$weak upper gradient of u in B.

Note that inequalities (3.13) and (3.12) for $\Omega = B$ a ball of radius r display some similarities, although different sets of assumptions have been used in their proofs. Inequality (3.12) is more convenient for applications to variational problems, since the coefficient of $\|g\|_{L^{\Phi}(\Omega)}$ in right hand side does not depend on Ω. For balls with the measure big enough, inequality (3.13) implies inequality (3.12) for $\Omega = B$ a ball of radius r. For balls with the measure small enough, inequality (3.12) for $\Omega = B$ a ball of radius r implies inequality (3.13).

We will prove some results similar to Theorem 1 and Theorem 2 for Hajłasz type Orlicz-Sobolev spaces.

The starting point is the following $(\Phi, \Phi)-$Orlicz-Sobolev inequality proved by Aïssaoui in [3], that plays here the role of (3.1).

Lemma 5. [3, Proposition 3.9] *If Φ is an $N-$function, then for every function $u \in M^{1,\Phi}(X)$ with a Hajłasz gradient $g \in L^{\Phi}(X)$ and for every measurable set $E \subset X$ with $0 < \mu(E) < \infty$,*

$$\|u - u_E\|_{L^{\Phi}(E)} \leq 2 \; diam\,(E) \, \|g\|_{L^{\Phi}(E)} . \tag{3.14}$$

Compared to (3.14) inequality (3.1) has been obtained at the expense of many additional assumptions and has a much more complicated proof.

Proposition 1. *Let X be a metric measure space. Assume that Φ is a strictly increasing Young function for which there exists a decreasing function $\varphi : (0, 1] \to [1, \infty)$ with $\varphi\,(t) > 1$ for every $0 < t < 1$, such that $\Phi_s^{-1}(\lambda t) \geq \lambda\varphi\,(\lambda)\,\Phi_s^{-1}(t)$ for all $\lambda \in (0, 1]$ and $t \in [0, \infty)$.*

Let Ω be an open bounded non-empty set. Suppose that $u \in N^{1,\Phi}(X)$ and g is a $\Phi-$weak upper gradient of u in X. Let $A = \{x \in \Omega : u(x) \neq 0\}$. If $\mu(A) \leq \gamma\mu(B)$ for some γ with $0 < \gamma < 1$, then

$$\|u\|_{L^{\Phi}(\Omega)} \leq 2\frac{\varphi\,(\gamma)}{\varphi\,(\gamma) - 1} \; diam\,(\Omega) \, \|g\|_{L^{\Phi}(\Omega)} . \tag{3.15}$$

Proof. As in the proof of Theorem 1, we obtain

$$\left(1 - \frac{\Theta\left(\frac{1}{\mu(A)}\right)}{\Theta\left(\frac{1}{\mu(\Omega)}\right)}\right) \|u\|_{L^\Phi(\Omega)} \leq \|u - u_\Omega\|_{L^\Phi(\Omega)} . \tag{3.16}$$

Here we denoted $\Theta(t) = \frac{1}{t}\Phi^{-1}(t)$.

By our assumptions on Φ^{-1} and A, $\Theta\left(\frac{1}{\mu(\Omega)}\right) \geq \Theta\left(\frac{\gamma}{\mu(A)}\right) \geq \varphi(\gamma)\,\Theta\left(\frac{1}{\mu(A)}\right)$, therefore (3.16) implies

$$\|u\|_{L^\Phi(\Omega)} \leq \frac{\varphi(\gamma)}{\varphi(\gamma) - 1} \|u - u_\Omega\|_{L^\Phi(\Omega)} .$$

Using (3.14) the above inequality implies (3.15).

Corollary 4. *Let X and Φ be as in the above Proposition. Assume that the sphere $S(x, R)$ is nonempty whenever $B(x, R) \neq X$. Suppose that $B = B(z, r) \subset X$ is a ball with $0 < r < diam(X)/3$. If $u \in M_0^{1,\Phi}(B)$ and $\overline{g} \in L^\Phi(X)$ is a Hajłasz gradient in X of the extension $\overline{u} \in M^{1,\Phi}(X)$ of u, then*

$$\|u\|_{L^\Phi(B(z,r))} \leq 12 \frac{\varphi(\gamma)}{\varphi(\gamma) - 1}\; r \;\|\overline{g}\|_{L^\Phi(B(z,3r))} \tag{3.17}$$

Proof. For $u \in M_0^{1,\Phi}(B)$ consider the extension $\overline{u} \in M^{1,\Phi}(X)$ of u such that $\overline{u} = u$ $\mu-$a.e. in B and $C_\Phi(\{x \in X \setminus B : \overline{u}(x) \neq 0\}) = 0$. Denote by $\overline{g}$ a Hajlasz gradient of $\overline{u}$ in X. Since $C_\Phi(E) = 0$ implies $\mu(E) = 0$, by [3, Lemma 5.1], we have $\overline{u} = 0$ $\mu-$a.e. in $X \setminus B$. Denote $A = \{x \in B(z, 3r) : u(x) \neq 0\}$. There exists a set $A_1 \subset B(z, 3r)$ with $\mu(A_1) = 0$ such that $A \subset B(z, r) \cup A_1$, hence $\mu(A) \leq \mu(B(z, r))$.

As in the proof of Theorem 2, taking a point $y \in S(z, 2r)$ we conclude that

$$\mu(B(z, r)) \leq \gamma\mu(B(z, 3r)),$$

where $\gamma = (1 - C_s 3^{-s})$, $0 < \gamma < 1$. Then $\mu(A) \leq \gamma\mu(B(z, 3r))$.

By the above Proposition, $\|\overline{u}\|_{L^\Phi(B(z,3r))} \leq 12r\frac{\varphi(\gamma)}{\varphi(\gamma)-1}\,\|\overline{g}\|_{L^\Phi(B(z,3r))}$.

But $\|\overline{u}\|_{L^\Phi(B(z,3r))} = \|u\|_{L^\Phi(B(z,R))}$, therefore (3.17) follows.

Remark 6. *If $u \in M_0^{1,\Phi}(B)$ has a Hajłasz gradient $g \in L^{\Phi}(B)$, it is not true in general that the extension to X with zero $\widetilde{g}$ of g is a Hajłasz gradient of $\overline{u}$. If under some additional assumptions on the balls in X the extension property from Corollary 2 would be true for $M_0^{1,\Phi}$, then we would obtain from (3.17)*

$$\|u\|_{L^{\Phi}(B(z,r))} \leq 12\frac{\varphi(\gamma)}{\varphi(\gamma)-1}\ r\ \ \|g\|_{L^{\Phi}(B(z,r))}.$$

References

[1] N. Aïssaoui, **Another extension of Orlicz-Sobolev spaces to metric spaces**, Abstr. Appl. Anal. 1 (2004), 1-26

[2] N. Aïssaoui, **Orlicz-Sobolev spaces with zero boundary values on metric spaces**, Southwest J. Pure Appl. Math., 1 (2004), 10-32

[3] N. Aïssaoui, **Strongly nonlinear potential theory on metric spaces**, Abstr. Appl. Anal. 7 (2002), no.7, 357-374

[4] C. Bennett and R. Sharpley, **Interpolation of Operators**, Pure and Applied Mathematics, vol. 129, Academic Press, London, 1988.

[5] A. Cianchi, **Optimal Orlicz-Sobolev embeddings**, Rev. Mat. Iberoamericana 20 (2004), 427-474.

[6] A. Cianchi, **Continuity properties of functions from Orlicz-Sobolev spaces and embedding theorems**, Ann. Sc. Norm. Super. Pisa, Cl. Sci. IV 23(1996), 576-608

[7] A. Cianchi, **A Sharp embedding for Orlicz-Sobolev spaces**, Indiana Univ. Math. J. 45 (1996), 39-65.

[8] A. Cianchi,**A fully anisotropic Sobolev inequality**, Pacific J. Math. 19(2000), 283-295.

[9] J. Cheeger, **Differentiability of Lipschitz functions on metric measure spaces**, Geom. Funct. Anal. 9 (1999), 428-517.

[10] R. R. Coifman and G. Weiss, **Analyse harmonique non-commutative sur certains espaces homogènes**. Lecture Notes in Mathematics, vol. 242. Springer, Berlin (1971)

[11] H. Federer and W. P. Ziemer, **The Lebesgue set of a function whose distribution derivatives are pth power summable**, Indiana Univ. Math. J. 22 (1972), 139-158

[12] P. Hajłasz,**Sobolev spaces on an arbitrary metric space**, Potential Anal. 5 (1996), 403-415.

[13] P. Hajłasz and P. Koskela, **Sobolev met Poincaré**, Mem. Amer. Math. Soc. 145 (2000), no. 688, 101 pp.

[14] P. Hajłasz and P. Koskela, **Sobolev meets Poincaré**, C.R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 10, 1211-1215.

[15] P. Koskela and P. MacManus, **Quasiconformal mappings and Sobolev spaces**, Studia Math. 131 (1998), 1-17.

[16] J. Heinonen,**Lectures on Analysis on Metric Spaces**, Springer Verlag, New York, 2001.

[17] J. Heinonen and P. Koskela, **Quasiconformal maps on metric spaces with controlled geometry**, Acta Math. 181 (1998), 1-61.

[18] T. Heikkinen,**Sharp self-improving properties of generalized Orlicz-Poincaré inequalities in connected metric measure spaces**, Preprint 327, Department of Mathematics and Statistics, University of Jyväskylä, 2006.

[19] T. Kilpeläinen, J. Kinnunen and O. Martio, **Sobolev spaces with zero boundary values on metric spaces**, Potential Anal., 12 (2000), 233-247.

[20] J. Kinnunen and V. Latvala, **Lebesgue points for Sobolev functions on metric spaces**, Rev. Mat. Iberoamericana, 18 (3) (2002), 685-700.

[21] J. Kinnunen and N. Shanmugalingam,**Regularity of quasiminimizers on metric spaces**, Manuscripta Math., 105 (2001), 401-423.

[22] P. MacManus, **Poincaré inequalities and Sobolev spaces**, Publ. Mat. (2002), Spec. Vol. , 181-197.

[23] M. Mocanu,**A Poincaré inequality for Orlicz-Sobolev functions with zero boundary values on metric spaces**, Complex Anal. Oper. Theory, to appear (DOI 10.1007/s11785-010-0068-3)

[24] M. Mocanu,**A generalization of Orlicz-Sobolev spaces on metric measure spaces via Banach function spaces**, Complex Var. Elliptic Equ., 55 (1-3)(2010), 253-267

[25] M. M. Rao and Z. D. Ren,**Theory of Orlicz Spaces**, Monographs and Textbooks in Pure and Applied Mathematics 146, Marcel Dekker Inc., New York, 1991.

[26] M. M. Rao and Z. D. Ren, **Applications of Orlicz Spaces**, Monographs and Textbooks in Pure and Applied Mathematics, 250 Marcel Dekker, New York, 2002.

[27] N. Shanmugalingam, Newtonian spaces: an extension of Sobolev spaces to metric measure spaces, Rev. Mat. Iberoamericana 16 (2000), no.2, 243-279.

[28] N. Shanmugalingam, **Newtonian spaces: an extension of Sobolev spaces to metric measure spaces**, PhD thesis, University of Michigan, 1999.

[29] H. Tuominen, **Orlicz-Sobolev spaces on metric measure spaces**, Ann. Acad. Sci. Fenn., Diss.135 (2004) 86 pp.

[30] H. Tuominen, **Pointwise behaviour of Orlicz-Sobolev functions**, Ann. Mat. Pura Appl. 188 (2009), no.1, 35-59.

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