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IMPROVED ESTIMATORS FOR BIG FACTORIALS

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Abstract: The aim of this paper is to introduce some new estimators for big factorials connected to Stirling and Burnside formula. Some results stated by Mortici in [A method which generates sharp estimations for big factorials J. Adv. Math. Studies 1(2008) 71-74] are extended.

1. INTRODUCTION

There are many situations when for solving the practical problems, big factorials must be estimated. Large factorials have many applications in number theory, combinatorics, probability and statistics, while asymptotic estimates of big factorials are very useful for counting purposes. One of the most known and maybe the most used formula for estimation the large factorials is the Stirling's formula:

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n, \tag{1.1}$$

in the sense that the ratio between the left-hand side and the right-hand side tends to 1, as n tends to infinity. For details, see [4, 6].

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Many authors approached the problem of finding improvements of Stirling formula (1.1).

The starting idea of the extensions from this paper is the following approximation formula, due to Burnside:

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \beta_n \tag{1.2}$$

and rediscovered by J. L. Spouge in [5]. The sequence $(\beta_n/n!)_{n\geq 1}$ is decreasing and consequently, $\beta_n > n!$. For proofs and other details, see [1, 7].

Mortici [3] proposed the approximation formula

$$n! \sim \frac{\sqrt{2\pi}}{e^{n+3/8}}\left(n^2+\frac{3}{4}n+\frac{1}{12}\right)^{\frac{n}{2}+\frac{1}{4}}$$

as an intermediate between the Stirling formula and the Burnside formula. We introduce in this paper the improved estimate

$$n! \sim \frac{\sqrt{2\pi}}{e^{1/6}}\left(\frac{n^3+2n^2+\frac{5}{4}n+\frac{5}{24}}{e^3}\right)^{\frac{n}{3}+\frac{1}{6}}$$

and finally, some computations are made.

2. THE MAIN RESULTS

Recently, Mortici [2] introduced the following family of approximations

$$n! \approx \sqrt{2\pi e}\cdot e^{-x}\left(\frac{n+x}{e}\right)^{n+\frac{1}{2}} = \mu_n(x),$$

depending on the real parameter $x \in [0,1]$. For $x = 1$, the following interesting approximation formula follows:

$$n! \approx \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} = \alpha_n$$

where the sequence $(\alpha_n/n!)_{n\geq 1}$ converges to 1, increasingly, since

$$\frac{\alpha_n/n!}{\alpha_{n-1}/(n-1)!} = \frac{1}{e}\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} > 1.$$

The function $\mu_n(x)$ is strictly increasing on $[0,1/2]$ and strictly decreasing on $[1/2,1]$, because

$$\frac{d}{dx}\left(\ln \mu_n(x)\right) = \frac{\frac{1}{2}-x}{x+n},$$

so a direct consequence is that $\mu_n(0) < n! < \mu_n(1/2) > n! > \mu_n(1)$, or

$$\sigma_n < n! < \beta_n > n! > \alpha_n.$$

For the continuous function $f_n(x) = \alpha_n^x \beta_n^{1-x} - n!$, we have $f_n(0) = \beta_n - n! > 0$ and $f_n(1) = \alpha_n - n! < 0$, so by a simple continuity argument, it results that for every integer $n \geq 1$, there exists $\theta_n \in (0,1)$ such that

$$\alpha_n^{\theta_n} \beta_n^{1-\theta_n} = n!. \tag{2.1}$$

More precisely, we have

$$\theta_n = \frac{\ln \beta_n - \ln n!}{\ln \beta_n - \ln \alpha_n} = \frac{\ln \sqrt{2\pi} + \left(n+\frac{1}{2}\right)\left(\ln\left(n+\frac{1}{2}\right)-1\right) - \sum_{k=1}^{n} \ln k}{\frac{1}{2} + \left(n+\frac{1}{2}\right)\ln \frac{2n+1}{2n+2}},$$

while some of the values of the sequence $(\theta_n)_{n\geq 1}$ are given below:

n	θ_n
10	0.343 69
20	0.338 70
50	0.335 52
250	0.333 78
10 000	0.333 34
50 000	0.333 34
100 000	0.333 33
1 000 000	0.333 33

As we can easily see, the values of the sequence $(\theta_n)_{n\geq 1}$ become closer to the value 1/3. Then we expect to obtain very good approximations if we replace θ_n by 1/3 in relation (2.1):

$$n! \approx \alpha_n^{1/3}\beta_n^{2/3} = \frac{\sqrt{2\pi}}{e^{1/6}}\left(\frac{n^3+2n^2+\frac{5}{4}n+\frac{1}{4}}{e^3}\right)^{\frac{n}{3}+\frac{1}{6}} = \tau_n. \tag{2.2}$$

Being a mean between the under-approximation α_n and the upper-approximation β_n, it is clear that our approximation τ_n is better than both α_n and β_n.

Moreover, as we will see in the next section, τ_n is comparable with γ_n from the Gosper's formula

$$n! \approx \sqrt{\left(2n+\frac{1}{3}\right)\pi} \cdot \left(\frac{n}{e}\right)^n = \gamma_n, \tag{2.3}$$

which is considered one of the most accurate approximation for large factorials.

3. FURTHER IMPROVEMENTS AND CONCLUSIONS

As usually, we are concerned now to compare our formula with other known formulas. Before this, we present some ideas to improve the approximation formula (2.2). More precisely, we are looking for an estimator μ_n where we replace the polynomial from (2.2) by other polynomial Q, such that, at least

$$\lim_{n\to\infty} \left(\frac{n^3+2n^2+\frac{5}{4}n+\frac{1}{4}}{Q(n)}\right)^{\frac{n}{3}+\frac{1}{6}} = 1. \tag{3.1}$$

After some simple computations, we can deduce that (3.1) is fulfilled as soon as

$$\deg\left(Q(n)-\left(n^3+2n^2+\frac{5}{4}n+\frac{1}{4}\right)\right) \leq 1,$$

that is $Q(n) = n^3+2n^2+xn+y$, for some x and y :

$$n! \approx \frac{\sqrt{2\pi}}{e^{1/6}} \left(\frac{n^3+2n^2+xn+y}{e^3}\right)^{\frac{n}{3}+\frac{1}{6}}. \tag{3.2}$$

One way to estimate the coefficients x and y is to imagine that the approximation (3.2) holds with equality for some values of n. This is an ideal case, but it allows us to estimate x and y as the components of the solution of the system

$$(S(m,n)) \qquad \begin{cases} \dfrac{\sqrt{2\pi}}{e^{1/6}} \left(\dfrac{n^3+2n^2+xn+y}{e^3}\right)^{\frac{n}{3}+\frac{1}{6}} = n! \\ \dfrac{\sqrt{2\pi}}{e^{1/6}} \left(\dfrac{m^3+2m^2+xm+y}{e^3}\right)^{\frac{m}{3}+\frac{1}{6}} = m! \end{cases}$$

The nice part of this problem is that for large values of the parameters m and n, the systems $S(m,n)$ have almost the same solution. We have used the Maple software to obtain the following solutions:

		Solution	
m	n	x	y
50	70	1.2500	0.20841
60	90	1.2500	0.20839
100	130	1.2500	0.20837
150	250	1.2500	0.20836
400	405	1.2500	0.20834
400	470	1.2500	0.20834
400	650	1.2500	0.20834

so we will take $x = 1.25$ and $y = \frac{5}{24} = 0.20834...$ to define the new formula:

$$n! \approx \frac{\sqrt{2\pi}}{e^{1/6}} \left(\frac{n^3 + 2n^2 + \frac{5}{4}n + \frac{5}{24}}{e^3} \right)^{\frac{n}{3}+\frac{1}{6}} = \mu_n. \tag{3.3}$$

This change of the last two coefficients leads to a considerable improvement of the formula (2.2).

For minor values of n, we have used Maple software to calculate the approximations β_n from (1.1), τ_n from (2.2), γ_n from (2.3) and the improved approximation μ_n from (3.3).

n	$\beta_n - n!$	$\tau_n - n!$	$\gamma_n - n!$	$\mu_n - n!$
5	0.91079	4.9869×10^{-2}	-2.9970×10^{-2}	$\mathbf{-6.4410 \times 10^{-4}}$
7	29.0490	1.1566	-0.66252	$\mathbf{-1.0044 \times 10^{-2}}$
10	14421.0	433.84	-239.18	$\mathbf{-2.4841}$
15	3.5191×10^9	7.2964×10^7	$-3.898\,8 \times 10^7$	$\mathbf{-2.6166 \times 10^5}$
17	8.4772×10^{11}	1.5632×10^{10}	-8.2904×10^9	$\mathbf{-4.8557 \times 10^7}$
20	4.9493×10^{15}	7.8278×10^{13}	-4.116×10^{13}	$\mathbf{-2.0179 \times 10^{11}}$

Other way to compare the accurateness of some approximations for large values of n is to introduce the number of exact decimal digits function (*edd*), by

$$edd(n) = -\lg \left| 1 - \frac{approx(n)}{n!} \right|,$$

where $approx(n)$ is the respective approximation. The next table shows again the great superiority of our final formula μ_n over the other considered formulas:

n	$edd(n)$			
	β_n	τ_n	γ_n	μ_n
50	3.0833	5.2686	5.5608	**8.3243**
100	3.3823	5.8640	6.1606	**9.3199**
250	3.7790	6.6559	6.9551	**11.037**
1000	4.3804	7.858	8.1586	**11.817**

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Finally, it is to be noticed that some asymptotic analysis arguments show that when considering the cubic family

$$\nu_n(a,b,c) := \sqrt{2\pi}\left(n^3 + an^2 + bn + c\right)^{\frac{n}{3}+\frac{1}{6}} e^{-\left(n+\frac{a}{3}\right)},$$

the best choice is

$$a_\# = \frac{3}{2} + \frac{\sqrt{75+20\sqrt{5}}}{10}, \quad b_\# = \frac{a_\#^2}{2} - \frac{a_\#}{2} + \frac{1}{4}, \quad c_\# = \frac{a_\#^3}{6} - \frac{a_\#^2}{2} + \frac{a_\#}{2} - \frac{1}{8}.$$

We omit the proof and here are some edd values for comparison:

n	$edd(n)$	
	$\nu_n(a_\#, b_\#, c_\#)$	μ_n
50	**9.9356**	8.3243
100	**11.1271**	9.3199
250	**12.7113**	11.037
1000	**15.1157**	11.817

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