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COMMON FIXED POINT FOR COMPATIBLE PAIRS OF MAPPINGS IN NON-COMPLETE METRIC SPACES

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Abstract. We prove the existence and uniqueness of a common fixed point for two compatible pairs of self-maps of a metric space. Instead of the completeness of the metric space, we use a weaker assumption, namely, the convergence of alternate images of an associated sequence. Our main result generalizes several known results from [1], [3], [4] and [5].

1. Introduction

Jungck [2] introduced the notion of compatible maps as a generalization of commuting maps and has shown that some fixed point results still hold when compatibility is assumed instead of commutativity.

Definition 1.1. [2](*Compatible Mappings*) *If* (X, d) *is a metric space, two mappings* $A, B : X \to X$ *are said to be compatible if, and only if, whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$, *for some* $t \in X$, *then* $d(ABx_n, BAx_n) = 0$.

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Commutativity of two maps implies compatibility, but the converse does not hold in general. Several fixed point results from [1] and [3] have been generalized by replacing the assumption of commutativity by that of compatibility, in [5] and [4], respectively. In continuation of the same theme, here also the results would be obtained under general compatibility.

In addition, the concept of associated sequence would be required as one of the tools in our main results.

Definition 1.2. (*Associated Sequence*) *If* (X, d) *is a metric space and* $S, I, T, J : X \to X$ *such that* $S(X) \subseteq J(X)$ *and* $T(X) \subseteq I(X)$, *then for each* $x_0 \in X$, *a sequence* $\{x_n\}$ *with* $Sx_{2n} = Jx_{2n+1}$ *and* $Tx_{2n+1} = Ix_{2n+2}$, *for every* $n \geq 0$, *is called associated sequence relative to* S, I, T *and* J.

The notion of associated sequence was used in [4] and [5] in fixed point results that do not assume the completeness of the metric space.

2. Main Result

Using hypothesis somewhat similar to [1], [3], [4] and [5], we obtain our main result under generalized contraction condition.

Theorem 2.1. *Let* (X, d) *be a metric space and* $S, I, J, T : X \to X$ *such that*

(i) $S(X) \subseteq J(X)$ *and* $T(X) \subseteq I(X)$,

(ii) $F : (R_+ \cup \{0\})^4 \to R$ *is a continuous function satisfying*

(a) $F(u, u, 0, 0) \leq 0$ *implies* $u = 0$,

(b) $F(u, 0, u, 0) \leq 0$ *implies* $u = 0$,

(c) $F(u, 0, 0, u) \leq 0$ *implies* $u = 0$,

such that for all $x, y \in X$,

$$F(d(Sx, Ty), d(Sx, Ix), d(Ix, Jy), d(Jy, Ty)) \leq 0 \tag{2.1}$$

(iii) *The pairs* (S, I) *and* (J, T) *are compatible,*

(iv) *One of* S, I, J *and* T *is continuous.*

(v) *There is an* $x_0 \in X$ *and an associated sequence* $\{x_n\}$ *of* x_0 *relative to these four self maps such that the sequence*

$$Sx_0, Tx_1, Sx_2, Tx_3, \cdots, Sx_{2n}, Tx_{2n+1}, \cdots \tag{2.2}$$

converges to a point $z \in X$.

Then z *is unique common fixed point of* S, I, J *and* T.

Proof. First suppose that the mapping I is continuous.
$Ix_{2n} = Tx_{2n-1} \to z$, and by continuity of I, $I^2x_{2n} = IIx_{2n} \to Iz$, as $n \to \infty$. Since $Sx_{2n} \to z$, by using compatibility of S and I, $\lim\limits_{n\to\infty} SIx_{2n} = \lim\limits_{n\to\infty} ISx_{2n} = Iz$.
Taking $x = Ix_{2n}$ and $y = x_{2n+1}$, the contraction condition (2.1) gives

$$F\left(d\left(SIx_{2n}, Tx_{2n+1}\right), d\left(SIx_{2n}, I^2x_{2n}\right), d\left(I^2x_{2n}, Jx_{2n+1}\right), d\left(Jx_{2n+1}, Tx_{2n+1}\right)\right) \leq 0$$

Letting $n \to \infty$,

$$F\left(d\left(Iz, z\right), d\left(Iz, Iz\right), d\left(Iz, z\right), d\left(z, z\right)\right) \leq 0$$
$$F\left(d\left(Iz, z\right), 0, d\left(Iz, z\right), 0\right) \leq 0$$

By definition of F, $d\left(Iz, z\right) = 0$. So $Iz = z$ and z is fixed point of I.
Now taking $x = z$ and $y = x_{2n+1}$, the contraction condition (2.1) gives

$$F\left(\ d\left(Sz, Tx_{2n+1}\right), d\left(Sz, Iz\right), d\left(Iz, Jx_{2n+1}\right), d\left(Jx_{2n+1}, Tx_{2n+1}\right)\right) \leq 0$$

Letting $n \to \infty$, $Jx_{2n+1}, Tx_{2n+1} \to z$, and using $Iz = z$, we get,

$$F\left(\ d\left(Sz, z\right), d\left(Sz, z\right), d\left(z, z\right), d\left(z, z\right)\right) \leq 0$$
$$F\left(\ d\left(Sz, z\right), d\left(Sz, z\right), 0, 0\right) \leq 0$$

By definition of F, $d\left(Sz, z\right) = 0$. So $Sz = z$ and z is fixed point of S.
Thus, $Iz = Sz = z$.
Since $S\left(X\right) \subseteq J\left(X\right)$, there is a point $z_1 \in X$ such that $z = Sz = Jz_1$.
Taking $x = z$ and $y = z_1$, the contraction condition (2.1) gives

$$F\left(d\left(Sz, Tz_1\right), d\left(Sz, Iz\right), d\left(Iz, Jz_1\right), d\left(Jz_1, Tz_1\right)\right) \leq 0$$
$$F\left(d\left(z, Tz_1\right), d\left(z, z\right), d\left(z, z\right), d\left(z, Tz_1\right)\right) \leq 0$$
$$F\left(d\left(z, Tz_1\right), 0, 0, d\left(z, Tz_1\right)\right) \leq 0$$

By definition of F, $d\left(z, Tz_1\right) = 0$. So $Tz_1 = z = Jz_1$.
Now for the constant sequence $y_n = z_1$, for every $n \geq 1$, clearly as $n \to \infty$, $Jy_n \to Jz_1 = z$ and $Ty_n \to Tz_1 = z$. Using compatibility of the pair of mappings J and T, $TJz_1 = JTz_1$, i.e., $Tz = Jz$.
Taking $x = z$ and $y = z$, the contraction condition (2.1) gives

$$F\left(d\left(Sz, Tz\right), d\left(Sz, Iz\right), d\left(Iz, Jz\right), d\left(Jz, Tz\right)\right) \leq 0$$
$$F\left(d\left(z, Tz\right), d\left(z, z\right), d\left(z, Jz\right), d\left(Jz, Jz\right)\right) \leq 0$$
$$F\left(d\left(z, Tz\right), 0, d\left(z, Jz\right), 0\right) \leq 0$$

By definition of F, $d(z,Tz)=0$. So $Tz=z$ and z is fixed point of T. Also $Jz=Tz=z$ and z is thus the common fixed point of all four maps S,I,J and T.
If instead of I, the mapping J is continuous, the proof runs in a parallel way.
Now suppose that the mapping S is continuous.
$Sx_{2n}\to z$, and by continuity of S, $S^2x_{2n}=SSx_{2n}\to Sz$, as $n\to\infty$.
$Ix_{2n}=Tx_{2n-1}\to z$, and by using continuity of S and compatibility of mappings S and I, $\lim\limits_{n\to\infty} ISx_{2n}=\lim\limits_{n\to\infty} SIx_{2n}=Sz$.
Taking $x=Sx_{2n}$ and $y=x_{2n+1}$, the contraction condition (2.1) gives

$$F\left(d\left(SSx_{2n},Tx_{2n+1}\right),d\left(SSx_{2n},ISx_{2n}\right),d\left(ISx_{2n},Jx_{2n+1}\right),\right.$$
$$\left.d\left(Jx_{2n+1},Tx_{2n+1}\right)\right)\leq 0$$

Letting $n\to\infty$,

$$F\left(\ d\left(Sz,z\right),d\left(Sz,Sz\right),d\left(Sz,z\right),d\left(z,z\right)\right)\leq 0$$
$$F\left(\ d\left(Sz,z\right),0,d\left(Sz,z\right),0\right)\leq 0$$

By definition of F, $d(Sz,z)=0$. So $Sz=z$ and z is fixed point of S.
Since $S(X)\subseteq J(X)$, there is a point $z_2\in X$ such that $z=Sz=Jz_2$.
Taking $x=Sx_{2n}$ and $y=z_2$, the contraction condition (2.1) gives

$$F\left(d\left(SSx_{2n},Tz_2\right),d\left(SSx_{2n},ISx_{2n}\right),d\left(ISx_{2n},Jz_2\right)\right.$$
$$\left.d\left(Jz_2,Tz_2\right)\right)\leq 0$$

Letting $n\to\infty$,

$$F\left(d\left(Sz,Tz_2\right),d\left(Sz,Sz\right),d\left(Sz,z\right),d\left(z,Tz_2\right)\right)\leq 0$$
$$F\left(d\left(z,Tz_2\right),0,0,d\left(z,Tz_2\right)\right)\leq 0$$

By definition of F, $d(z,Tz_2)=0$. So $Tz_2=z=Jz_2$.
Now for the constant sequence $y_n=z_2$, for every $n\geq 1$, clearly as $n\to\infty$, $Ty_n\to Tz_2=z$ and $Jy_n\to Jz_2=z$. Using compatibility of the pair of mappings J and T, $TJz_2=JTz_2$, i.e., $Tz=Jz$.
Taking $x=x_{2n}$ and $y=z$, the contraction condition (2.1) gives

$$F\left(d\left(Sx_{2n},Tz\right),d\left(Sx_{2n},Ix_{2n}\right),d\left(Ix_{2n},Jz\right),d\left(Jz,Tz\right)\right)\leq 0$$

Letting $n\to\infty$,

$$F\left(\ d\left(z,Tz\right),d\left(z,z\right),d\left(z,Tz\right),d\left(Tz,Tz\right)\right)\leq 0$$
$$F\left(\ d\left(z,Tz\right),0,d\left(z,Tz\right),0\right)\leq 0$$

By definition of F, $d(z, Tz) = 0$. So $Tz = z = Jz = Sz$.
Since $T(X) \subseteq I(X)$, there is a point $z_3 \in X$ such that $z = Tz = Iz_3$.
Taking $x = z_3$ and $y = z$, the contraction condition (2.1) gives

$$F(\, d(Sz_3, Tz), d(Sz_3, Iz_3), d(Iz_3, Jz), d(Jz, Tz)) \leq 0$$
$$F(\, d(Sz_3, z), d(Sz_3, z), d(z, z), d(z, z)) \leq 0$$
$$F(\, d(Sz_3, z), d(Sz_3, z), 0, 0) \leq 0$$

By definition of F, $d(Sz_3, z) = 0$. So $Sz_3 = z = Iz_3$.
Now for the constant sequence $y_n = z_3$, for every $n \geq 1$, clearly as $n \to \infty$, $Sy_n \to Sz_3 = z$ and $Iy_n \to Iz_3 = z$. Using compatibility of the pair of mappings S and I, $SIz_3 = ISz_3$, i.e., $Sz = Iz = z$, and z is fixed point of I also. Thus, $Sz = Tz = Jz = Iz = z$, and z is the common fixed point of all four maps S, I, J and T.
If instead of S, the mapping T is continuous, the proof runs in a parallel way.
Now for proving uniqueness of the common fixed point z, suppose that there are two common fixed points of S, I, J and T, viz., z and z'. So $Sz' = Tz' = Jz' = Iz' = z'$.
Taking $x = z$ and $y = z'$, the contraction condition (2.1) gives

$$F(\, d(Sz, Tz'), d(Sz, Iz), d(Iz, Jz'), d(Jz', Tz')) \leq 0$$
$$F(\, d(z, z'), d(z, z), d(z, z'), d(z', z')) \leq 0$$
$$F(\, d(z, z'), 0, d(z, z'), 0) \leq 0$$

By definition of F, $d(z, z') = 0$. So $z = z'$ and the common fixed point z of S, I, J and T is unique.
It is also immediately clear that z is a fixed point of the all composition maps like $SI, IS, SJ, JS, ST, TS, IJ, JI, IT, TI, JT$, and TJ.
This completes the proof. ∎

The above result generalizes many known results, as we will show in the sequel.

3. Special Cases Giving Established Results

By choosing a particular example of the function F defined by condition (ii) in Theorem 2.1, we get the following.

Corollary 3.1. *Let (X, d) be a metric space and $S, I, J, T : X \to X$ such that*

(i) *$S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$,*

(ii) *There is $0 \leq c < 1$, such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq c \cdot (d(Sx, Ix) + d(Ix, Jy) + d(Jy, Ty)) \tag{3.1}$$

(iii) *The pairs (S, I) and (J, T) are compatible,*

(iv) *One of S, I, J and T is continuous.*

(v) *There is an $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to these four self maps such that the sequence*

$$Sx_0, Tx_1, Sx_2, Tx_3, \cdots, Sx_{2n}, Tx_{2n+1}, \cdots \tag{2.2}$$

converges to a point $z \in X$.

Then z is unique common fixed point of S, I, J and T.

Proof. $d(x, y)$, being always nonnegative, is valid to be used as an argument of F in Theorem 2.1. We take $F : (R_+ \cup \{0\})^4 \rightarrow R$ to be $F(t_1, t_2, t_3, t_4) = t_1 - c \cdot (t_2 + t_3 + t_4)$. With this the contraction condition of Theorem 2.1 reduces to

$$d(Sx, Ty) - c \cdot (d(Sx, Ix) + d(Ix, Jy) + d(Jy, Ty)) \leq 0$$

that is

$$d(Sx, Ty) \leq c \cdot (d(Sx, Ix) + d(Ix, Jy) + d(Jy, Ty))$$

Since in every application of this in the proof of the Theorem 2.1, only one of the three terms on the right hand side remains nonvanishing and has also been equal to the left hand side as required, the proof follows by Theorem 2.1. ∎

The contraction condition (3.1) is quite natural in the light of trivial triangle inequality. It generalizes a result of Fisher [1].

Corollary 3.2. [1] *Suppose S, I, T and J are four self maps of a metric space (X, d) into itself satisfying the conditions* :

(i) *$S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$,*

(ii) *There is $0 \leq c < 1$, such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq c \cdot \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\} \tag{3.2}$$

(iii) *The pairs (S, I) and (J, T) are compatible,*

(iv) *One of S, I, J and T is continuous.*
Further if

(v) *X is complete,*

then S, I, T and J have a unique common fixed point $z \in X$. Further z is the unique fixed point of S and I, and of T and J.

Proof. Since the commutativity of pairs of mappings implies their compatibility, completeness of the space implies convergence of sequence of alternate images of terms of the associated sequence and, with $0 \leq a = d(Sx, Ix)$, $0 \leq b = d(Ix, Jy)$, $0 \leq c = d(Jy, Ty)$, $\max\{a, b, c\} \leq a + b + c$, the conditions here guarantee hypothesis of Corollary 3.1 and consequently required fixed point exists. This completes the proof. ∎

Corollary 3.3. [3, *Theorem* 2.1] *Let (X, d) be a metric space, mappings $S, I, T, J : X \to X$ be such that $S(X) \subseteq J(X)$, $T(X) \subseteq I(X)$, one of S, I, J, T is continuous, the pairs S, I and J, T are commuting pairs and*

$$d(Sx, Ty) \leq c \cdot \max\{d(Sx, Ix), d(Ix, Jy), d(Jy, Ty), \tag{3.3}$$
$$\frac{1}{2}(d(Sx, Jy) + d(Ix, Ty))\}$$

for all $x, y \in X$, where $0 \leq c < 1$. If there is an $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to these four self maps such that the sequence

$$Sx_0, Tx_1, Sx_2, Tx_3, \cdots, Sx_{2n}, Tx_{2n+1}, \cdots \tag{2.2}$$

converges to a point $z \in X$, then z is unique common fixed point of S, I, J and T. Consequently, z is also a fixed point of the composition maps $SI, IS, JT, TJ, SJ, JS, IT$ and TI.

Proof. Commutativity of the pairs of mappings implies their compatibility and sequence of alternate images of terms of associated sequence is already taken to be convergent. Now with $0 \leq a = d(Sx, Ix)$, $0 \leq b = d(Ix, Jy)$, $0 \leq c = d(Jy, Ty)$, $0 \leq d = d(Sx, Jy)$, and $0 \leq e = d(Ix, Ty)$, triangle inequality property of the metric assures that $d \leq a + b$ and $e \leq b + c$. Then,

$$\frac{1}{2}(d + e) \leq \frac{1}{2}(a + 2b + c) \leq a + b + c.$$

Thus, in totality, $\max\left\{a, b, c, \frac{1}{2}(d + e)\right\} \leq a + b + c$ and contraction condition (3.1) is weaker than (3.3). So, by Corollary 3.1 required fixed point exists. This completes the proof. ∎

V.Srinivas has achieved significant generalization of Corollary 3.2. But his result is also seen to be just a particular case of Corollary 3.1.

Corollary 3.4. [5, *Theorem* 1] *Suppose S, I, T and J are four self maps of metric space (X, d) into itself satisfying the conditions*

(i) $S(X) \subseteq J(X)$ *and* $T(X) \subseteq I(X)$,

(ii) *There is* $0 \leq c < 1$, *such that for all* $x, y \in X$,

$$d(Sx, Ty) \leq c \cdot \max\{d(Ix, Jy), d(Ix, Sx), d(Jy, Ty)\} \tag{3.2}$$

(iii) *One of* S, I, J *and* T *is continuous*

and if the pairs (S, I) and (T, J) are compatible, and if there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the four self maps such that the sequence

$$Sx_0, Tx_1, Sx_2, Tx_3, \cdots, Sx_{2n}, Tx_{2n+1}, \cdots \tag{2.2}$$

converges to some $z \in X$, then S, I, J and T have a unique common fixed point $z \in X$. Further z is unique common fixed point of S and I and of T and J.

Proof. Most of the things are as per requirement in Corollary 3.1 except the contraction condition (3.2). Of course, (3.2) is stronger than (3.1), since the maximum of finitely many non-negative numbers does not exceed their sum and the proof follows as a particular case. ∎

Corollary 3.3 by authors happens to be generalization of Corollary 3.2 by Fisher in terms of sequence convergence instead of completeness of the space and weaker contraction condition. Corollary 3.4 by V.Srinivas is generalization of Corollary 3.2 by Fisher in terms sequence convergence instead of completeness of the space and compatibility in place of commutativity of pairs of maps. Authors [4] have successfully merged the advantages of both generalizations in following.

Corollary 3.5. [4, *Theorem* 3.1] *Let (X, d) be a metric space and $S, I, J, T : X \to X$ such that*

(i) $S(X) \subseteq J(X)$ *and* $T(X) \subseteq I(X)$,

(ii) *There is* $0 \leq c < 1$, *such that for all* $x, y \in X$,

$$\begin{aligned} d(Sx, Ty) \leq c \cdot \max\{&d(Sx, Ix), d(Ix, Jy), d(Jy, Ty), \\ &\frac{1}{2}(d(Sx, Jy) + d(Ix, Ty))\} \end{aligned} \tag{3.3}$$

(iii) *The pairs* (S, I) *and* (J, T) *are compatible,*
(iv) *One of* S, I, J *and* T *is continuous.*
(v) *There is an* $x_0 \in X$ *and an associated sequence* $\{x_n\}$ *of* x_0 *relative to these four self maps such that the sequence*

$$Sx_0, Tx_1, Sx_2, Tx_3, \cdots, Sx_{2n}, Tx_{2n+1}, \cdots \tag{2.2}$$

converges to a point $z \in X$.

Then z *is unique common fixed point of* S, I, J *and* T. *Consequently,* z *is also a fixed point of the composition maps* $SI, IS, JT, TJ, SJ, JS, IT$ *and* TI.

Proof. Here too, most of the things are as per requirement in Corollary 3.1 except the contraction condition (3.3), which, as shown in proof of Corollary 3.3, is clearly stronger than (3.1) and the proof follows as a particular case. ∎

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