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LIPSCHITZ CONTINUITY FOR MULTI-SUBLINEAR COMMUTATOR OF SOME INTEGRAL OPERATOR

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Abstract: In this paper, we will study the continuity for some multi-sublinear commutator related to certain integral operator and to a vector Lipschiz function, on Lebesgue spaces, Triebel-Lizorkin spaces, Hardy space and Herz-Hardy spaces.

1. Introduction

Let T be a Calderón-Zygmund operator. A well known result of Coifman, Rochberg and Weiss (see [4]) states that the commutator $[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ (where $b \in BMO$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo (see [2]) proves a similiar result when T is replaced by the fractional operators. In [6][14], Janson and Paluszynski study these result for the Triebel-Lizorkin spaces and the case $b \in Lip_\beta$, where Lip_β is the homogeneous Lipschitz space. In this paper, we will introduce some multi-sublinear commutator related to certain integral operator, then prove the continuity for the multilinear commutator and b on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where $b \in Lip_\beta$. The integral operators include the Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

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2. Definitions and Results

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$, Q will denote a cube of $\mathbb{R}^n$ with side parallel to the axes. Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{Q\ni x}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Denote the Hardy spaces by $H^p(\mathbb{R}^n)$. It is well known that $H^p(\mathbb{R}^n)(0 < p \leq 1)$ has the atomic decomposition characterization (see [10][15]). For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Tribel-Lizorkin space. The Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in\mathbb{R}^n \\ x\neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 1. Let $0 < p,\ q < \infty$, $\alpha \in R$, $B_k = \{x \in \mathbb{R}^n, |x| \leq 2^k\}$, $A_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{Ak}$ for $k \in \mathbf{Z}$, where χ_E denote the characteristic function of the set E.

1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{Loc}^q(\mathbb{R}^n\backslash\{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p};$$

2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{Loc}^q(\mathbb{R}^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p}.$$

Definition 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'^n) : G(f) \in \dot{K}_q^{\alpha,p}(\mathbb{R}^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in S'^n) : G(f) \in K_q^{\alpha,p}(\mathbb{R}^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on $\mathbb{R}^n$ is called a central (α, q)-atom (or a central (a, q)-atom of restricted type), if

1) Supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
3) $\int_{\mathbb{R}^n} a(x)x^\eta dx = 0$ for $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Let $(x, y, t) \to F_t(x, y)$ be a locally integrable function from $\mathbb{R}^n \times \mathbb{R}^n \times [0, +\infty) \to R^1$. Set, for every bounded and compactly supported function f,

$$F_t(f)(x) = \int_{\mathbb{R}^n} F_t(x, y)f(y)dy$$

and

$$F_t^{\vec{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y))F_t(x, y)f(y)dy.$$

Let H be the Banach space $H = \{h : ||h|| < \infty\}$ of real functions defined on $[0, +\infty)$ such that, for fixed f and fixed $x \in \mathbb{R}^n$, $F_t(f)(x)$ and $F_t^{\vec{b}}(f)(x)$ can be viewed as mappings from $[0, +\infty)$ to H.

Definition 4. Suppose b_j $(j = 1, \cdots, m)$ are fixed locally integrable functions on $\mathbb{R}^n$. The multi-sublinear commutator related to F_t is defined by

$$T_{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||,$$

where F_t satisfies: for fixed $\varepsilon > 0$

$$||F_t(x, y)|| \leq C|x - y|^{-n}$$

and

$$||F_t(y, x) - F_t(z, x)|| + ||F_t(x, y) - F_t(x, z)|| \leq C|y - z|^\varepsilon |x - z|^{-n-\varepsilon},$$

if $2|y - z| \leq |x - z|$. We also define that $T(f)(x) = ||F_t(f)(x)||$.

Note that, for fixed $t \in [0, +\infty)$, if $b_1 = ... = b_m$, then $F_t^{\vec{b}}$ is the m order commutator related to F_t (see [1][14]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-4][6-9][11][14][16]). Our main purpose is to establish the boundedness of the multi-sublinear commutator $T_{\vec{b}}$ on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

Given a positive integer m and $1 \le j \le m$, we set $||\vec{b}||_{Lip_\beta} = \prod_{j=1}^m ||b_j||_{Lip_\beta}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \backslash \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_\beta} = ||b_{\sigma(1)}||_{Lip_\beta} \cdots ||b_{\sigma(j)}||_{Lip_\beta}$.

Now we state our theorems as follows.

Theorem 1. *Let $0 < \beta < min(1, \varepsilon/m)$, $1 < p < \infty$, $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in Lip_\beta(\mathbb{R}^n)$ for $1 \le j \le m$. Suppose that $T_{\vec{b}}$ is the multi-sublinear commutator as in Definition 4 such that T is bounded on $L^r(\mathbb{R}^n)$ for any $1 < r < \infty$. Then*

(a) $T_{\vec{b}}$ is bounded from $L^p(\mathbb{R}^n)$ to $\dot{F}_p^{m\beta,\infty}(\mathbb{R}^n)$.

(b) $T_{\vec{b}}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1/p - 1/q = m\beta/n$ and $1/p > m\beta/n$.

Theorem 2. *Let $0 < \beta \le 1$, $\max(n/(n + m\beta), n/(n + m\varepsilon)) < p \le 1$, $1/q = 1/p - m\beta/n$, $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in Lip_\beta(\mathbb{R}^n)$ for $1 \le j \le m$. Suppose that $T_{\vec{b}}$ is the multi-sublinear commutator as in Definition 4 such that T is bounded on $L^r(\mathbb{R}^n)$ for any $1 < r < \infty$. Then $T_{\vec{b}}$ is bounded from $H^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Theorem 3. *Let $0 < \beta \le 1$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = m\beta/n$, $n(1 - 1/q_1) \le \alpha < n(1 - 1/q_1) + m\beta$, $\vec{b} = (b_1, \cdots, b_m)$ with $b_j \in Lip_\beta(\mathbb{R}^n)$ for $1 \le j \le m$. Suppose that $T_{\vec{b}}$ is the multi-sublinear commutator as in Definition 4 such that T is bounded on $L^r(\mathbb{R}^n)$ for any $1 < r < \infty$. Then $T_{\vec{b}}$ is bounded from $H\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)$ to $\dot{K}_{q_2}^{\alpha,p}(\mathbb{R}^n)$.*

3. Proof of Theorems

We will need the following lemmas.

Lemma 1.([14]) *For $0 < \beta < 1$, $1 < p < \infty$, we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left|\left|\sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx\right|\right|_{L^p} \\ &\approx \left|\left|\sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx\right|\right|_{L^p}. \end{aligned}$$

Lemma 2.([14]) *For* $0<\beta<1,\ 1\le p\le\infty$, *we have*

$$\begin{aligned}||f||_{Lip_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x)-f_Q|dx\\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx\right)^{1/p}.\end{aligned}$$

Lemma 3.([2]) *For* $1\le r<\infty$ and $\beta>0$, *let*

$$M_{\beta,r}(f)(x)=\sup_{x\in Q}\left(\frac{1}{|Q|^{1-\beta r/n}}\int_Q |f(y)|^r dy\right)^{1/r},$$

suppose that $r<p<\beta/n$, and $1/q=1/p-\beta/n$, *then*

$$||M_{\beta,r}(f)||_{L^q}\le C||f||_{L^p}.$$

Lemma 4.([14]) *Let* $Q_1\subset Q_2$, *then*

$$|f_{Q_1}-f_{Q_2}|\le C||f||_{Lip_\beta}|Q_2|^{\beta/n}.$$

Lemma 5. ([5],[13]) *Let* $0<p<\infty$, $1<q<\infty$ *and* $\alpha\ge n(1-1/q)$. *A temperate distribution* f *belongs to* $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$*(or* $HK_q^{\alpha,p}(\mathbb{R}^n)$*) if and only if there exist central* (α,q)*-atoms(or central* (α,q)*-atoms of restricted type)* a_j *supported on* $B_j=B(0,2^j)$ *and constants* λ_j, $\sum_j|\lambda_j|^p<\infty$ *such that* $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$ *(or* $f=\sum_{j=0}^{\infty}\lambda_j a_j$*) in the* S'^n *sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(\text{or } ||f||_{HK_q^{\alpha,p}})\sim\left(\sum_j|\lambda_j|^p\right)^{1/p}.$$

Proof of Theorem 1. (a). Fix a cube $Q=(x_0,l)$ and $\tilde{x}\in Q$. Set $\vec{b}_Q=((b_1)_Q,\cdots,(b_m)_Q)$, where $(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy,\ 1\le j\le m$. Write $f=f_1+f_2$, where $f_1=f\chi_{2Q}, f_2=f\chi_{\mathbb{R}^n\setminus 2Q}$, we have

$F_t^{\vec{b}}(f)(x)=\int_{\mathbb{R}^n}\prod_{j=1}^m((b_j(x)-(b_j)_Q)-(b_j(y)-(b_j)_Q))F_t(x,y)f(y)dy=$
$=\sum_{j=o}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b_j(x)-(b_j)Q)_\sigma\int_{\mathbb{R}^n}(b_j(y)-$
$-(b_j)Q)_{\sigma^c}F_t(x,y)f(y)dy=$
$=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(x)+$
$+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(x)+$
$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma\int_{\mathbb{R}^n}(\vec{b}(y)-\vec{b}_Q)_{\sigma^c}F_t(x,y)f(y)dy=$
$=(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(x)+$
$+(-1)^mF_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)+$

$+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)+$
$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)$, then

$$\begin{aligned}
&|T_{\vec{b}}(f)(x)-T(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\
\leq\ & ||F_t^{\vec{b}}(f)(x)-F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)||\\
\leq\ & ||(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(x)||\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)||\\
&+||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)||\\
&\quad+||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||\\
&=I_1(x)+I_2(x)+I_3(x)+I_4(x),
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q|T_{\vec{b}}(f)(x)-T((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|dx\\
\leq\ & \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_1(x)dx+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_2(x)dx+\\
& \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_3(x)dx+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_4(x)dx\\
=\ & I+II+III+IV.
\end{aligned}$$

For I, by using Lemma 2, we have

$$\begin{aligned}
I\ &\leq\ \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\sup_{x\in Q}|b_1(x)-(b_1)_Q|\cdots|b_m(x)-(b_m)_Q|\int_Q|T(f)(x)|dx\\
&\leq\ C||\vec{b}||_{Lip_\beta}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{\frac{m\beta}{n}}\int_Q|T(f)(x)|dx\\
&\leq\ C||\vec{b}||_{Lip_\beta}M(T(f))(\tilde{x}).
\end{aligned}$$

Fix $1<r<p$. For II, let μ,μ' be the integers such that $\mu+\mu'=m$, $0\leq\mu<m$, $0<\mu'\leq m$. By using the Hölder's inequality, the boundedness of T on L^r and Lemma 2, we get
$II\leq\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma||T((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|dx$
$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\left(\int_Q|\vec{b}(x)-\vec{b}_Q|^{r'}dx\right)^{1/r'}\left(\int_Q|T((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|^r dx\right)^{1/r}$

$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\left(\int_Q|\vec{b}(x)-\vec{b}_Q|^{r'}dx\right)^{1/r'}\left(\int_Q|(b(\vec{x})-\vec{b}_Q)_{\sigma^c}f(x)|^r dx\right)^{1/r}$
$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{\frac{\mu\beta}{n}}||\vec{b}_{\sigma^c}||_{Lip_\beta}|Q|^{\frac{\mu'\beta}{n}}|Q|^{\frac{1}{r}}\left(\int_Q|f(x)|^r dx\right)^{1/r}$
$\leq C||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x})$. For III, by Hölder's inequality, we have

$$\begin{aligned} III &\leq C\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\left(\int_{\mathbb{R}^n}|T(\prod_{j=1}^m(b_j-(b_j)_Q)f_1)(x)|dx\right)^{1/r}|Q|^{1-1/r} \\ &\leq C\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{1-1/r}\left(\int_{2Q}|\prod_{j=1}^m(b_j(x)-(b_j)_Q)f(x)|^r dx\right)^{1/r} \\ &\leq C\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{1-1/r}||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}\left(\int_{2Q}|f(x)|^r dx\right)^{1/r} \\ &\leq C||\vec{b}||_{Lip_\beta}M_r(f)(\tilde{x}). \end{aligned}$$

For IV, since $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, by Lemma 4, we have
$I_4(x)\leq\int_{(2Q)^c}||F_t(x,y)-F_t(x_0,y)||f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy$
$\leq C\int_{(2Q)^c}|x_0-x|^\varepsilon|x_0-y|^{-(n+\varepsilon)}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy$
$\leq C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^\varepsilon|x_0-y|^{-(n+\varepsilon)}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy$
$\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|\prod_{j=1}^m(|b_j(y)-(b_j)_{2^{k+1}Q}|+|(b_j)_{2^{k+1}}-(b_j)_Q|)dy$
$\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{\frac{m\beta}{n}}||\vec{b}||_{Lip_\beta}M(f)(\tilde{x})$
$\leq C||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}M(f)(\tilde{x})\sum_{k=1}^\infty 2^{(m\beta-\varepsilon)k}$
$\leq C||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}M(f)(\tilde{x})$, so

$$IV\leq C||\vec{b}||_{Lip_\beta}M(f)(\tilde{x}).$$

We put these estimates together, by using Lemma 1 and taking the supremum over all Q such that $x\in Q$, we obtain

$$||T_{\vec{b}}(f)(x)||_{\dot{F}_p^{m\beta,\infty}}\leq C||\vec{b}||_{Lip_\beta}||f||_{L^p}.$$

This completes the proof of (a).

(b). By some argument as in proof of (a), we have

$$\begin{aligned} & \frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x) - T(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)|dx \\ \le\ & \frac{1}{|Q|}\int_Q I_1(x)dx + \frac{1}{|Q|}\int_Q I_2(x)dx + \frac{1}{|Q|}\int_Q I_3(x)dx + \frac{1}{|Q|}\int_Q I_4(x)dx \\ \le\ & C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(T(f)) + M_{m\beta,r}(f) + M_{m\beta,r}(f) + M_{m\beta,1}(f)). \end{aligned}$$

thus

$$(T_{\vec{b}}(f))^{\#} \le C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(T(f)) + M_{m\beta,r}(f) + M_{m\beta,1}(f)).$$

By using Lemma 3 and the boundedness of T we have

$$\begin{aligned} ||T_{\vec{b}}(f)||_{L^q} &\le C||(T_{\vec{b}}(f))^{\#}||_{L^q} \\ &\le C||\vec{b}||_{Lip_\beta}(||M_{m\beta,1}(T(f))||_{L^q} + ||M_{m\beta,r}(f)||_{L^q} + ||M_{m\beta,1}(f)||_{L^q}) \\ &\le C||f||_{L^p}. \end{aligned}$$

This completes the proofs of (b) and of Theorem 1.

Proof of Theorem 2. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||T_{\vec{b}}(a)||_{L^q} \le C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int_{\mathbb{R}^n} a(x)x^\gamma dx = 0$ for $|\gamma| \le [n(1/p - 1)]$.

Write

$$\begin{aligned} ||T_{\vec{b}}(a)(x)||_{L^q} &\le \left(\int_{|x-x_0|\le 2r} |T_{\vec{b}}(a)(x)|^q dx\right)^{1/q} + \left(\int_{|x-x_0|>2r} |T_{\vec{b}}(a)(x)dx|^q\right)^{1/q} \\ &= I + II. \end{aligned}$$

For I, choose $1 < p_1 < 1/\beta$ and q_1 such that $1/q_1 = 1/p_1 - \beta/n$. By the boundednss of $T_{\vec{b}}$ from $L^{p_1}(\mathbb{R}^n)$ to $L^{q_1}(\mathbb{R}^n)$(see Theorem 1), we get

$$I \le C||T_{\vec{b}}(a)||_{L^{q_1}}\mathbb{R}^n \le C||a||_{L^{q_1}}\mathbb{R}^n \le C.$$

For II, let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \ne 0$. We get $|T_{\vec{b}}(a)(x)| = \left|||\int_Q \prod_{j=1}^m (b_j(x) - b_j(y))F_t(x,y)a(y)dy||\right|$
$\le |b_1(x) - b_1(x_0)|\cdots|b_m(x) - b_m(x_0)|\int_Q ||F_t(x,y) - F_t(x,x_0)|||a(y)|dy$
$+ \sum_{j=1}^m \sum_{\sigma\in C_j^m} |(b(x) - b(x_0))_{\sigma^c}|\int_Q |(b(y) - b(x_0))_\sigma|||F_t(x,y)|||a(y)|dy$
$\le C||\vec{b}||_{Lip_\beta}|x - x_0|^{m\beta}\cdot\int_Q ||F_t(x,y) - F_t(x,x_0)|||a(y)|dy$
$+ C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x - x_0|^{\tau\beta}\int_Q |y - x_0|^{\tau'\beta}||F_t(x,y)|||a(y)|dy$

$\leq C||\vec{b}||_{Lip_\beta}\frac{|x-x_0|^{m\beta}}{|x-x_0|^{n+\varepsilon}}\int_Q|x_0-y|^\varepsilon|a(y)|dy$
$+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x-x_0|^{\tau\beta}}{|x-y|^n}\int_Q|y-x_0|^{\tau'\beta}|a(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta}\frac{|x-x_0|^{m\beta}}{|x-x_0|^{n+\varepsilon}}\int_Q|x_0-y|^\varepsilon|a(y)|dy$
$+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x-x_0|^{\tau\beta}}{|x-x_0|^n}\int_Q|y-x_0|^{\tau'\beta}|a(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta}|x-x_0|^{-(n+\varepsilon)}\cdot r^{m\beta+\varepsilon+n(1-\frac{1}{p})}$
$+C||\vec{b}||_{Lip_\beta}|x-x_0|^{-n}\cdot r^{m\beta+n(1-\frac{1}{p})}$
$\leq C||\vec{b}||_{Lip_\beta}|x-x_0|^{-n}\cdot r^{m\beta+n(1-\frac{1}{p})}$, so

$$\begin{aligned} II &\leq C||\vec{b}||_{Lip_\beta}\cdot r^{m\beta+n(1-\frac{1}{p})}\left(\int_{|x-x_0|>2r}|x-x_0|^{-nq}dx\right)^{1/q} \\ &\leq C||\vec{b}||_{Lip_\beta}. \end{aligned}$$

This completes the proof of Theorem 2.

Proof of Theorem 3. By Lemma 5, let $f\in H\dot{K}_{q_1}^{\alpha,p}(\mathbb{R}^n)$ and $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j,\ supp\ \ a_j\subset B_j=B(0,2^j),\ a_j$ be a central $(\alpha,q)-$atom, and $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$. We have

$$\begin{aligned} ||T_{\vec{b}}(f)||^p_{\dot{K}_{q_2}^{\alpha,p}} &\leq C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p \\ &\quad +C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}}\right)^p \\ &= I+II. \end{aligned}$$

For II, by the boundedness of $T_{\vec{b}}$ (see Theorem 1) on (L^{q_1},L^{q_2}), we get

$$\begin{aligned} II &\leq C||\vec{b}||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=k-1}^{\infty}|\lambda_j|||a_j||_{L^{q_1}})^p \\ &\leq C||\vec{b}||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}(\sum_{j=k-1}^{\infty}|\lambda_j|\cdot 2^{-j\alpha})^p \end{aligned}$$

Hence,

$II\leq C||\vec{b}||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{(k-j)\alpha p},\quad 0<p\leq 1$

and

$II \leq C||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{-j\alpha p/2})(\sum_{j=k-1}^{\infty} 2^{-j\alpha p'/2})^{p/p'}, 1 < p < \infty,$
therefore

$II \leq C||\vec{b}||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$

For I, when $m = 1$, similarly to Theorem 2, we have
$|T_{b_1}(a_j)(x)| \leq |b_1(x) - b_1(0)| \int_{B_j} ||F_t(x,y) - F_t(y)|||a_j(y)|dy$
$+ |\int_{B_j} ||f_t(b_1(y) - b_1(0))|||a_j(y)|dy$
$\leq C||b_1||_{Lip_\beta} \left[\int_{B_j} \frac{|x|^\beta |y|^\varepsilon}{|x|^{n+\varepsilon}} \cdot |a_j(y)|dy + \int_{B_j} \frac{|y|^\beta}{|x-y|^n} \cdot |a_j(y)|dy\right]$
$\leq C||b_1||_{Lip_\beta} \left[\frac{|x|^\beta}{|x|^{n+\varepsilon}} \int_{B_j} |y|^\varepsilon |a_j(y)|dy + \frac{1}{|x|^n} \int_{B_j} |y|^\varepsilon |a_j(y)|dy\right]$
$\leq C||b_1||_{Lip_\beta} \left[|x|^{-(n+\varepsilon)} \cdot |x|^\beta \cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}|x|^{-n} \cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right]$
$\leq C||b_1||_{Lip_\beta}|x|^{-n} \cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}.$
From that we have

$$\begin{aligned} ||T_{b_1}(a_j)\chi_k||_{L^{q_2}} &\leq C||b_1||_{Lip_\beta} \cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)} \left(\int_{B_k} |x|^{-nq_2}dx\right)^{1/q_2} \\ &\leq C||b_1||_{Lip_\beta} \cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)} \cdot 2^{-kn(1-\frac{1}{q_2})} \\ &\leq C||b_1||_{Lip_\beta} \cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}, \end{aligned}$$

so
$I \leq C||b_1||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{\infty} |\lambda_j| \cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^p$

$$\leq C||b_1||^p_{Lip_\beta} \begin{cases} \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, \quad 0 < p \leq 1 \\ \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{\frac{p}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right) \\ \quad \times \left(\sum_{j=-\infty}^{k-2} 2^{\frac{p'}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^{p/p'}, \quad 1 < p < \infty \end{cases}$$

$$\leq C||b_1||^p_{Lip_\beta} \begin{cases} \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, \quad 0 < p \leq 1 \\ \sum_{j=-\infty}^{\infty} |\lambda_j|^p \sum_{k=j+2}^{\infty} 2^{\frac{p}{2}[(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)]}, \quad 1 < p < \infty \end{cases}$$

$\leq C||b_1||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$
Then

$$||T_{b_1}(f)||_{\dot{K}^{\alpha,p}_{q_2}} \leq C||b_1||_{Lip_\beta}(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p} \leq C||f||_{H\dot{K}^{\alpha,p}_{q_1}}.$$

When $m \geq 2$, we have
$|T_{\vec{b}}(a_j)(x)| \leq |(b_1(x) - b_1(0)) \cdots |b_m(x) - b_m(0)| \int_{B_j} ||F_t(x-y) - F_t(x)|||a_j(y)|dy$
$+ \sum_{j=1}^{\infty} \sum_{\sigma \in C_j^m} |(b(x) - b(0))_{\sigma^c}| \int_{B_j} |(b(y) - b(0))_\sigma|||F_t(x,y)|||a_j(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta} \int_{B_j} ||F_t(x,y) - F_t(x)||a_j(y)|dy$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \int_{B_j} |y|^{\tau'\beta}||F_t(x,y)|||a_j(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta} \frac{|x|^{m\beta}}{|x|^{n+\varepsilon}} \int_{B_j} |y|^{\varepsilon}|a_j(y)|dy$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} \frac{|x|^{\tau\beta}}{|x|^n} \int_{B_j} |y|^{\tau'\beta}|a_j(y)|dy$
$\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta}|x|^{-(n+\varepsilon)} \cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}$
$+ C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta}|x|^{-n} \cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}$
$\leq C||\vec{b}||_{Lip_\beta}|x|^{-n} \cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)}$.
Then

$$\begin{aligned} & ||T_{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \\ \leq \ & C||\vec{b}||_{Lip_\beta} \cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)} \cdot \left(\int_{B_j} |x|^{-nq_2} dx \right)^{1/q_2} \\ \leq \ & C||\vec{b}||_{Lip_\beta} \cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}, \end{aligned}$$

so
$I \leq C||\vec{b}||_{Lip_\beta}^p \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]} \right)^p$

$$\leq C||\vec{b}||_{Lip_\beta}^p \begin{cases} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{(j-k)(m\beta+n(1-\frac{1}{q_1})-\alpha)p}, \quad 0 < p \leq 1 \\ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{\frac{p}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]} \right) \\ \quad \times \left(\sum_{j=-\infty}^{k-2} 2^{\frac{p'}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]} \right)^{p/p'}, \quad 1 < p < \infty \end{cases}$$

$\leq C||\vec{b}||_{Lip_\beta}^p \sum_{j=-\infty}^{\infty} |\lambda_j|^p$. From I and II, we have

$$||T_{\vec{b}}(f)||_{\dot{K}_{q_2}^{\alpha,p}} \leq C||\vec{b}||_{Lip_\beta} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C||f||_{H\dot{K}_{q_1}^{\alpha,p}}.$$

This completes the proof of Theorem 3.

4. Applications

Now we give some applications of Theorems in this paper.

Application 1. Littlewood-Paley operator.

Fixed $\varepsilon > 0$. Let ψ be a fixed function which satisfies the following properties:

(1) $\int_{\mathbb{R}^n} \psi(x)dx = 0$,

(2) $|\psi(x)| \le C(1+|x|)^{-(n+1)}$,

(3) $|\psi(x+y) - \psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

The multi-sublinear commutator related to the Littlewood-Paley operator is defined by

$$g_{\psi}^{\vec{b}}(f)(x) = \left(\int_0^{\infty} |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^{m}(b_j(x) - b_j(y))\psi_t(x-y)f(y)dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. Set $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_{\psi}(f)(x) = \left(\int_0^{\infty} |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which are the Littlewood-Paley operator(see [8]). Let H be the space

$$H = \left\{h : ||h|| = \left(\int_0^{\infty} |h(t)|^2 dt/t\right)^{1/2} < \infty\right\},$$

then, for each fixed $x \in \mathbb{R}^n$, $F_t^{\vec{b}}(f)(x)$ may be viewed as the mapping from $[0, +\infty)$ to H, and it is clear that

$$g_{\psi}^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad g_{\psi}(f)(x) = ||F_t(f)(x)||.$$

It is easy to see that g_{ψ} satisfies the conditions of Theorems in the paper(see [7-9]), thus Theorem 1, Theorem 2 and Theorem 3 hold for $g_{\psi}^{\vec{b}}$.

Application 2. Marcinkiewicz operator.

Fixed $0 < \gamma \le 1$. Let Ω be homogeneous of degree zero on $\mathbb{R}^n$ with $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_{\gamma}(S^{n-1})$. The multi-sublinear commutator related to the Marcinkiewicz operator is defined by

$$\mu_\Omega^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \prod_{j=1}^m (b_j(x) - b_j(y)) \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which are the Marcinkiewicz operator(see [16]). Let H be the space

$$H = \left\{ h : ||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty \right\}.$$

Then, it is clear that

$$\mu_\Omega^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||, \quad \mu_\Omega(f)(x) = ||F_t(f)(x)||.$$

It is easy to see that μ_Ω satisfies the conditions of Theorems in the paper(see [8][16]), thus Theorem 1, Theorem 2 and Theorem 3 hold for $\mu_\Omega^{\vec{b}}$.

Application 3. Bochner-Riesz operator.

Let $\delta > (n-1)/2$, $B_t^\delta(f)\hat{}(\xi) = (1 - t^2|\xi|^2)_+^\delta \hat{f}(\xi)$ and $B_t^\delta(z) = t^{-n}B^\delta(z/t)$ for $t > 0$. Set

$$F_{\delta,t}^{\vec{b}}(f)(x) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(y)) B_t^\delta(x-y) f(y)dy.$$

The multi-sublinear commutator related to the maximal Bochner-Riesz operator is defined by

$$B_{\delta,*}^{\vec{b}}(f)(x) = \sup_{t>0} |B_{\delta,t}^{\vec{b}}(f)(x)|.$$

We also define that

$$B_{\delta,*}(f)(x) = \sup_{t>0} |B_t^\delta(f)(x)|$$

which is the maximal Bochner-Riesz operator(see [10]). Let H be the space $H = \{h : ||h|| = \sup_{t>0} |h(t)| < \infty\}$, then

$$B^{\vec{b}}_{\delta,*}(f)(x) = ||B^{\vec{b}}_{\delta,t}(f)(x)||, \quad B^{\delta}_{*}(f)(x) = ||B^{\delta}_{t}(f)(x)||.$$

It is easily to see that $B^{\vec{b}}_{\delta,*}$ satisfies the conditions of Theorems in the paper(see [8]), thus Theorem 1, Theorem 2 and Theorem 3 hold for $B^{\vec{b}}_{\delta,*}$.

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