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COMMON FIXED POINT THEOREMS OF MEIR-KEELER TYPE FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS

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Abstract. We prove common fixed point theorems of Meir-Keeler type for four mappings satisfying implicit relations in metric spaces using the concept of occasionally weakly compatible mappings which generalizes results of [1], [7], [11-17], [19], [21], [28], [34-40], [47-48] and [50].

1. Introduction and Preliminaries

Let A and S be self-mappings of a metric space (X, d). A and S are commuting in X if $SAx = ASx$ for all $x \in X$.

Sessa [52] defined A and S to be weakly commuting in X if for all $x \in X$

$$d(SAx, ASx) \leq d(Ax, Sx). \tag{1.1}$$

Jungck [16] defined A and S to be compatible as a generalization of weakly commuting if

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0 \tag{1.2}$$

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whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. It is easy to show that commuting implies weakly commuting implies compatible and there are examples in the literature verifying that the inclusions are proper, see [16] and [52].

Jungck et al [18] defined A and S to be compatible mappings of type (A) if

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(ASx_n, S^2x_n) = 0 \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Clearly, weakly commuting implies compatible of type (A) and the converse is not true in general, see [18]. Examples are given to show that the two concepts of compatibility are independent, see [18].

Recently, Pathak and Khan [42] defined S and T to be compatible mappings of type (B) as a generalization of compatible mappings of type (A) if

$$\begin{aligned} \lim_{n\to\infty} d(ASx_n, S^2x_n) &\leq \frac{1}{2}[\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, A^2x_n)] \text{ and} \qquad (1.4) \\ \lim_{n\to\infty} d(SAx_n, A^2x_n) &\leq \frac{1}{2}[\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, S^2x_n)] \end{aligned}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Clearly, compatible mappings of type (A) are compatible mappings of type (B), but the converse is not true in general, see [42].

However, compatibility, compatibility of type (A) and compatibility of type (B) are equivalent if S and T are continuous, see [42].

Pathak et al [43] defined A and S to be compatible mappings of type (P) if

$$\lim_{n\to\infty} d(A^2x_n, S^2x_n) = 0 \tag{1.5}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

However, compatibility, compatibility of type (A), compatibility of type (B) and compatibility of type (P) are equivalent if S and T are continuous, see [43].

Pathak et al [44] defined A and S to be compatible mappings of type (C) as a generalization of compatible mappings of type (A) if

(1.6)

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, S^2x_n) + \lim_{n\to\infty} d(At, A^2x_n)] \text{ a}$$

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(SAx_n, St) + \lim_{n\to\infty} d(St, A^2x_n) + \lim_{n\to\infty} d(St, S^2x_n)]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Compatibility of type (B), compatibility of type (P) and compatibility of type (C) are equivalent if S and T are continuous, see [44].

Jungck [20] defined S and T to be weakly compatible if they commute at their coincidence points, i.e., $SAu = ASu$ for all $u \in C(A, S)$, the set of coincidence points of A and S

It is proved in [16], [18], [42], [43] and [44] that if S and T are compatible, or compatible of type (A), or compatible of type (P), or compatible of type (B), or compatible of type (C), then they are weakly compatible. The converse is not true in general, see [8].

In 1994, Pant [30] defined A and S to be pointwise $R-$weakly commuting if for all $x \in X$, there exists an $R > 0$ such that

$$d\,(SAx, ASx) \leq Rd\,(Sx, Ax) \text{ for all } x \in X.$$

It was proved in [31] that pointwise $R-$weakly commuting is equivalent to commutativity at coincidence points. Thus, A and S are pointwise $R-$weakly commuting if and only if they are weakly compatible.

Definition 1.1 [2]. A and S are said to be occasionally weakly compatible (owc) if $SAu = ASu$ for some $u \in C(A, S)$.

Remark 1.2 [2]. If A and S are weakly compatible, then they are occasionally weakly compatible, but the converse is not true in general, see [2].

Lemma 1.3 [22]. If A and S have a unique coincidence point $w = Ax = Sx$, then w is the unique common fixed point of A and S.

In 1969, Meir and Keeler [27] established a fixed point theorem for self-mappings in a metric space (X, d) satisfying the following condition:

For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon < d(x, y) < \epsilon + \delta \text{ implies } d(fx, fy) < \epsilon. \tag{1.7}$$

There exists a vast literature which generalizes the result of Meir and Keeler.

In [26], Maiti and Pal proved a fixed point theorem for a self mapping of a metric space (X, d) satisfying the following condition, which is a generalization of (1.7).

For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon < \max\{d(x,y), d(x,fx), d(y,fy)\} < \epsilon + \delta \Longrightarrow d(fx,fy) < \epsilon.$$

In [41] and [49], Park-Rhoades, respectively, Rao-Rao extended this result for two self-mappings f and g of a metric space (X, d) satisfying the following condition: For every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\begin{aligned} \epsilon &< \max\{d(fx,fy), d(fx,gx), d(fy,gy), \\ \frac{1}{2}[d(fx,gy) + d(fy,gx)]\} &< \epsilon + \delta \Longrightarrow d(gx,gy) < \epsilon. \end{aligned}$$

In 1986, Jungck [16] and Pant [28] extended these results for four self-mappings A, B, S and T of a metric space (X, d).

The following conditions have been used by many authors to prove common fixed point theorems for four mappings.

$$d(Ax, By) \le hM(x,y), \quad 0 \le h < 1, \tag{1}$$

where $M(x,y) = \max\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), [d(Sx,By) + d(Ax,Ty)]/2\}$,

a Meir-Keeler type (ϵ, δ)-contractive condition of the form: given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \le M(x,y) < \epsilon + \delta \Rightarrow d(Ax,By) < \epsilon, \tag{2}$$

a contractive condition of the form

$$d(Ax, By) \le \phi(M(x,y)) \tag{3}$$

involving a contractive gauge function $\phi : \mathbb{R} \to \mathbb{R}$ is such that $\phi(t) < t$ for all $t > 0$.

Clearly, condition (1) is a special case of both conditions (2) and (3).

A ϕ-contractive condition (3) does not guarantee the existence of a fixed point unless some additional condition is assumed. Moreover, a ϕ-contractive condition in general does not imply the contractive conditions (2) or (4), see Pant [34]. Therefore, to ensure the existence of common fixed point under the contractive condition (3), the following conditions on the function ϕ have been introduced and used by many authors.

(i) ϕ is non-decreasing and the function $t \mapsto t/(t - \phi(t))$ is non-increasing (Carbone et al. [6]),

(ii) ϕ is non-decreasing and $\lim_{n\to\infty}\phi^n(t) = 0$ for all $t > 0$ (Jachymski [3]),

(iii) ϕ is upper semi continuous (Boyd and Wong [1], Jachymski [9], Maiti and Pal [26], Pant [32]),

(iv) ϕ is non-decreasing and continuous from the right (Park and Rhoades [41]).

It is now known (e.g., Jachymski [9], Pant et al. [35]) that if any of the conditions (I), (II), (III) or (IV) is assumed on ϕ, then a ϕ-contractive condition (3) implies an analogous (ϵ, δ)-contractive condition (2) and both the contractive conditions hold simultaneously.

An (ϵ, δ)-contractive condition of type (2) neither ensures the existence of a fixed point nor implies an analogous ϕ-contractive condition (3), see Pant et al [35]. Hence, the two types of contractive conditions (2) and (3) are independent of each other.

Thus, to ensure the existence of common fixed point under the contractive condition (2), the following conditions on the function δ have been introduced and used by several authors:

(v) δ is non-decreasing (Pant [29, 32])

(vi) δ is lower semi-continuous (Jungck [16], Jungck et al [17]).

Jachymski [9] has shown that the (ϵ, δ)-contractive condition (2) with a non-decreasing δ implies a ϕ-contractive condition (3). Also, Pant et al.[35] have shown that the (ϵ, δ)-contractive condition (2) with a lower semi-continuous δ, implies a ϕ-contractive condition (3).

Instead of supposing one of the contractive condition (2) or (3) with additional conditions on δ and ϕ, many authors have assumed a contractive condition (2) together with a lipschitz condition analogue of (3), see [11], [37] and [38].

Lemma 1.4 [9]. *Let A, B, S and T be self mappings of a metric space (X, d) such that $A(X) \subset T(X)$ and $B(X) \subset S(X)$. Assume further that given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$*

$$\epsilon < M(x, y) < \epsilon + \delta \Longrightarrow d(Ax, By) \leq \epsilon \tag{4}$$

and

$$d(Ax, By) < M(x, y), \text{ whenever } M(x, y) > 0, \tag{5}$$

where $M(x, y) = \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [d(Sx, By) + d(Ax, Ty)]/2\}$.

Then, for each $x_0 \in X$, the sequence $\{y_n\}$ in X defined by:

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \text{ for } n = 0, 1, 2....$$

is a Cauchy sequence.

Jachymski [9] has shown that the contractive condition (2) implies (4), but the contractive condition (4) does not imply (2).

The following theorem was proved by [5].

Theorem 1.5. Let (X, d) be a complete metric space, $c \in [0, 1)$ and $f : X \to X$ a mapping such that for all $x, y \in X$

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable mapping which is summable and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \varphi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Several authors generalized Theorem 1.5, see [3], [8], [24], [25], [51], [54] and [55].

The study of fixed points of mappings satisfying an implicit relation was initiated in [45] and [46].

2. Implicit relations

Let $\mathcal{F}_6$ be the set of all continuous functions $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

$(F_1) : F(u, 0, u, 0, 0, u) \leq 0$ implies $u = 0$.

$(F_2) : F(u, 0, 0, u, u, 0) \leq 0$ implies $u = 0$.

$(F_3) : F(u, u, 0, 0, u, u) \geq 0$ for all $u > 0$.

Example 2.1.

$F(t_1, ...t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, $a, b, c \geq 0$, $b + c < 1$ and $a + 2c \leq 1$.

Example 2.2.

$F(t_1, ...t_6) = t_1 - \max\{t_2, (t_3 + t_4)/2, k(t_5 + t_6)/2\}$, where $0 < k \leq 1$.

Example 2.3.

$F(t_1, ..., t_6) = t_1 - \max\{k_1 t_2, k_2(t_3 + t_4)/2, (t_5 + t_6)/2\}$, where $0 < k_1 \leq 1$, $1 \leq k_2 < 2$.

Example 2.4.

$F(t_1, ...t_6) = t_1 - \max\{k_1(t_2 + t_3 + t_4), k_2(t_5 + t_6)/2\}$, where $0 \leq k_1 < 1$, $0 \leq k_2 < 1$.

Example 2.5.

$F(t_1, ...t_6) = t_1 - h \max\{t_2, t_3, t_4, t_5, t_6\}$, where $0 \leq h < 1$.

Example 2.6.
$F(t_1, ...t_6) = t_1^2 - at_2^2 - t_3t_4 - bt_5^2 - ct_6^2$, where $a, b, c \leq 0, a+b+c \leq 1$.
Example 2.7.
$F(t_1, ...t_6) = t_1^3 - k(t_2^3 + t_3^3 + t_4^3 + t_5^3 + t_6^3)$, where $0 \leq k \leq \frac{1}{3}$.
Example 2.8.

$$F(t_1, ...t_6) = (1 + pt_2)t_1 - p\max\{t_3t_4, t_5t_6\} \\ -h\max\{t_2, t_3, t_4, t_5, t_6\},\ 0 < h < 1,\ p \geq 0.$$

Example 2.9.
$F(t_1, ...t_6) = t_1^2 - c_1 \max\{t_2^2, t_3^2, t_4^2\} - c_2 \max\{t_3t_6, t_4t_5\} - c_3t_5t_6$, $c_1, c_2, c_3 \geq 0$, $c_1 + c_2 < 1, c_1 + c_3 \leq 1$.
Example 2.10.
$F(t_1, ...t_6) = t_1^3 - at_1^2t_2 - bt_1t_3t_4 - ct_5^2t_6 - dt_5t_6^2$, $a, b, c, d \geq 0$ and $a + c + d \leq 1$.
Example 2.11.

$$F(t_1, ...t_6) = t_1^3 - c\frac{t_3^2t_4^2 + t_5^2t_6^2}{t_2 + t_3 + t_4 + 1},\ 0 < c \leq 1.$$

Example 2.12.
$F(t_1, ...t_6) = t_1 - \phi(\max\{t_2, t_3, t_4, k(t_5 + t_6)/2\})$, where $0 < k \leq 1$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for all $t > 0$.

The following theorem was proved by [47].

Theorem 2.13. *Let f, g, S and T be self-mappings of a metric space (X, d) such that*

$$S(X) \subset g(X) \text{ and } T(X) \subset f(X). \tag{2.1}$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$

$$\begin{aligned} \epsilon &\leq \max\{d(fx, gy), d(fx, Sx), d(gy; Ty), \\ \frac{1}{2}[d(fx, Ty) + d(Sx, gy)]\} &< \epsilon + \delta \\ \text{implies } d(Sx, Ty) &< \epsilon \end{aligned} \tag{2.2}$$

and there exists $F \in F_6$ satisfying

$$F(d(Sx, Ty), d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) < 0$$

for all $x, y \in X$. If one of $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then f and S have a coincidence point and g and T have a coincidence point. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point in X.

Let F_4 be the set of all continuous functions $F(t_1, ..., t_4) : \mathbb{R}_+^4 \to \mathbb{R}$ satisfying the following conditions:

(F_2) : $F(u, 0, u, u) \leq 0$ implies $u = 0$.

$(F_3) : F(u, u, 0, 2u) \geq 0$ for all $u > 0$.

Example 2.14.

$F(t_1, t_2, t_3, t_4) = t_1 - at_2 - bt_3 - ct_4$, $a, b, c \geq 0$, $b + c < 1$ and $a + 2c \leq 1$.

Example 2.15.

$F(t_1, t_2, t_3, t_4) = t_1 - \max\{t_2, t_3/2, kt_4/2\}$, where $0 < k \leq 1$.

Example 2.16.

$F(t_1, t_2, t_3, t_4) = t_1 - \max\{k_1 t_2, k_2 t_3/2, t_4/2\}$, where $0 \leq k_1 \leq 1$, $1 \leq k_2 < 2$.

The following theorem was proved by [48].

Theorem 2.17. *Let f, g, S and T be self-mappings of a metric space (X, d) satisfying* (2.1), (2.2) *and there exists $F \in F_4$ satisfying*

$$F(d(Sx, Ty), d(fx, gy), d(fx, Sx)+d(gy, Ty), d(fx, Ty)+d(Sx, gy)) < 0$$

for all $x, y \in X$. If one of $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then f and S have a coincidence point and g and T have a coincidence point. Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S and T have a unique common fixed point in X.

Theorems 2.13 and 2.17 generalize the results of [11], [37] and [38].

The purpose of this paper is to prove common fixed point theorems of Meir-Keeler type for owc mappings satisfying implicit relations.

3. Main Results

Lemma 3.1. *Let f, g, S and T be self-mappings of a metric space (X, d) satisfying the inequality*

$$F\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt, \int_0^{d(fx,gy)} \varphi(t)dt, \quad (3.1)$$

$$\int_0^{d(fx,Sx)} \varphi(t)dt, \int_0^{d(gy,Ty)} \varphi(t)dt,$$

$$\int_0^{d(fx,Ty)} \varphi(t)dt, \int_0^{d(Sx,gy)} \varphi(t)dt\Big) < 0$$

for all $x, y \in X$, $x \neq y$, where F satisfies (F_6) and φ is as in Theorem 1.4. If there exist $u, v \in X$ such that $fu = Su$ and $gv = Tv$, *then f*

and S have a unique point of coincidnce u and g and T have a unique point of coincidnce v.

Proof. First, we prove that $Su = Tv$. If $Su \neq Tv$. Using (3.1) we have

$$
\begin{aligned}
& F(\int_0^{d(Su,Tv)} \varphi(t)dt, \int_0^{d(fu,gv)} \varphi(t)dt, \int_0^{d(fu,Su)} \varphi(t)dt, \int_0^{d(gv,Tv)} \varphi(t)dt, \\
& \int_0^{d(fu,Tv)} \varphi(t)dt, \int_0^{d(Su,gv)} \varphi(t)dt) \\
= & F(\int_0^{d(Su,Tv)} \varphi(t)dt, \int_0^{d(Su,Tv)} \varphi(t)dt, 0, 0, \\
& \int_0^{d(Su,Tv)} \varphi(t)dt, \int_0^{d(Su,Tv)} \varphi(t)dt) \\
< & \ 0
\end{aligned}
$$

which is a contradiction of F_3 and so $Su = Tv$. Assume that $Sp = Tp$ with $Sp \neq Su$. Then $Sp \neq Tv$ and by (3.1) we get

$$
\begin{aligned}
& F(\int_0^{d(Sp,Tv)} \varphi(t)dt, \int_0^{d(Sp,Tv)} \varphi(t)dt, 0, 0, \\
& \int_0^{d(Sp,Tv)} \varphi(t)dt, \int_0^{d(Sp,Tv)} \varphi(t)dt) \\
< & \ 0
\end{aligned}
$$

which is a contradiction of F_3 and so $Sp = Tv = Su$. Therefore, $z = fu = Su$ is the unique point of coincidnce f and S. Similarly, $z = fv = Sv$ is the unique point of coincidnce g and T.

Theorem 3.2. *Let f, g, S and T be self-mappings of a metric space (X, d) satisfying (2.1) and (2.2) and there exists $F \in F_6$ satisfying* (3.1) *and* φ is as in Theorem 1.4. *If one of $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then f and S have a coincidence point and g and T have a coincidence point. Moreover, if the pairs (f, S) and (g, T) are owc, then f, g, S and T have a unique common fixed point in X.*

Proof. Let x_0 be an arbitrary point in X. By (4.2) we can define inductively a sequence $\{y_n\}$ such that

$y_{2n} = Sx_{2n} = gx_{2n+1}$ and $y_{2n+1} = fx_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2....$

By Lemma 1.4, it follows that $\{y_n\}$ is a Cauchy sequence in X.

Suppose that $g(X)$ is complete. Since the subsequence $\{y_{2n}\} = \{Jx_{2n+1}\} \subset J(X)$ is a Cauchy sequence, it converges to a point $z = gv$ for some $v \in X$. Hence, the subsequence $\{y_{2n+1}\}$ converges also to z.

So, the subsequences $\{fx_{2n+2}\}, \{gx_{2n+1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to z.

If $z \neq Tv$, using (3.1) we obtain

$$\begin{aligned} &F\Big(\int_0^{d(Sx_{2n},Tv)} \varphi(t)dt, \int_0^{d(fx_{2n},gv)} \varphi(t)dt, \int_0^{d(fx_{2n},Sx_{2n})} \varphi(t)dt, \int_0^{d(gv,Tv)} \varphi(t)dt, \\ &\quad \int_0^{d(fx_{2n},Tv)} \varphi(t)dt, \int_0^{d(Sx_{2n},gv)} \varphi(t)dt\Big) \\ &< 0 \end{aligned}$$

Letting $n \to \infty$, we obtain

$$F\Big(\int_0^{d(z,Tv)} \varphi(t)dt, 0, 0, \int_0^{d(z,Tv)} \varphi(t)dt, \int_0^{d(z,Tv)} \varphi(t)dt, 0)\Big) \leq 0$$

By (F_2), we get $z = Tv = gv$. Since $T(X) \subset f(X)$, there exists $u \in X$ such that $z = Tv = fu$.

If $z \neq Su$, using (3.1) we have

$$\begin{aligned} &F\Big(\int_0^{d(Su,Tx_{2n+1})} \varphi(t)dt, \int_0^{d(fu,gx_{2n+1})} \varphi(t)dt, \int_0^{d(fu,Su)} \varphi(t)dt, \int_0^{d(gx_{2n+1},Tx_{2n+1})} \varphi(t)dt, \\ &\quad \int_0^{d(fu,Tx_{2n+1})} \varphi(t)dt, \int_0^{d(Su,gx_{2n+1})} \varphi(t)dt\Big) \\ &< 0 \end{aligned}$$

Letting $n \to \infty$ we get

$$F\Big(\int_0^{d(Su,z)} \varphi(t)dt, 0, \int_0^{d(z,Su)} \varphi(t)dt, 0, 0, \int_0^{d(z,Su)} \varphi(t)dt \leq 0.$$

Using (F_1), we get $z = Su = fu = gv = Tv$. By Lemma 3.1, f and S have a unique point of coincidnce u and g and T have a unique point of coincidnce v and by Lemma 1.3, z is the unique common fixed point of f, g, S and T. Assume that $f(X)$ is complete. Since the subsequence $\{y_{2n+1}\} = \{fx_{2n+2}\} \subset f(X)$ is a Cauchy sequence, it converges to a point $z = fu$ for some $u \in X$. Hence, the sequence $\{y_{2n}\}$ converges also to z. So, the subsequences $\{fx_{2n+2}\}, \{gx_{2n+1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ converge to z. Similarly, we get $z = fu = Su$. Since $S(X) \subset g(X)$, there exists $u \in X$ such that $z = Su = gv$. Similarly, we obtain $z = gv = Tv = fu = Su$ and $z = fz = Sz = gz = Tz$.

Corollary 3.3. Theorem 2.13.

Proof. It follows from Theorem 3.2 for $\varphi(t) = 1$.

Corollary 3.4. *Let f, g, S and T be self-mappings of a metric space (X, d) satisfying* $(2.1), (2.2)$ *and*

$$\begin{aligned} d(Sx, Ty) &< \max\{k_1 d(fx, gy), k_2[d(fx, Sx) + d(gy, Ty)]/2, \\ &\quad [d(fx, Ty) + d(Sx, gy)]/2\} \end{aligned}$$

for all $x, y \in X$, where $0 \le k_1 < 1$ and $1 < k_2 < 2$. If one of $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then f and S have a coincidence point and g and T have a coincidence point. Moreover, if the pairs (f, S) and (g, T) are owc, then f, g, S and T have a unique common fixed point in X.

Proof. It follows from Example 2.3 and Theorem 3.2.

Corollary 3.5. *Let f, g, S and T be self-mappings of a metric space (X, d) satisfying* $(2.1), (2.2)$ *and*

$$\begin{aligned} d(Sx, Ty) &< \max\{d(fx, gy), [d(fx, Sx) + d(gy, Ty)]/2, \\ &\quad k[d(fx, Ty) + d(Sx, gy)]/2\} \end{aligned}$$

for all $x, y \in X$, where $0 < k \le 1$. If one of $f(X)$, $g(X)$, $S(X)$ and $T(X)$ is a complete subspace of X, then f and S have a coincidence point and g and T have a coincidence point. Moreover, if the pairs (f, S) and (g, T) are owc, then f, g, S and T have a unique common fixed point in X.

Proof. It follows from Example 2.2 and Theorem 3.2.

Corollary 3.6. *Let f, g, S and T be self-mappings of a metric space (X, d) satisfying* $(2.1), (2.2)$ *and*

$$\begin{aligned} d(Sx, Ty) &< \max\{k_1(d(fx, gy) + d(fx, Sx) + d(gy, Ty)), \\ &\quad k_2[d(fx, Ty) + d(Sx, gy)]/2\} \end{aligned}$$

for all $x, y \in X$*, where* $0 < k_1 < 1$ *and* $0 < k_2 \leq 1$ *If one of* $f(X)$*,* $g(X)$*,* $S(X)$ *and* $T(X)$ *is a complete subspace of* X*, then* f *and* S *have a coincidence point and* g *and* T *have a coincidence point. Moreover, if the pairs* (f, S) *and* (g, T) *are owc, then* f, g, S *and* T *have a unique common fixed point in* X*.*

Proof. It follows from Example 2.3 and Theorem 3.2.

Remark 3.7. By Examples 2.1, 2.4-2.12, we obtain several Corollaries.

As in Lemma 3.1 and Theorem 3.2, we can prove

Lemma 3.8. *Let* f, g, S *and* T *be self-mappings of a metric space* (X, d) *satisfying the inequality*

$$F\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt, \int_0^{d(fx,gy)} \varphi(t)dt, \int_0^{d(fx,Sx)} \varphi(t)dt + \int_0^{d(gy,Ty)} \varphi(t)dt, \int_0^{d(fx,Ty)} \varphi(t)dt + \int_0^{d(Sx,gy)} \varphi(t)dt\Big) < 0$$

for all x, y *in* X*, where* F *satisfies* (F_4) *and* φ is as in Theorem 1.4. If there exist $u, v \in X$ such that $fu = Su$ and $gv = Tv$*, then* f and S have a unique point of coincidnce u and g and T have a unique point of coincidnce v.

Theorem 3.9. *Let* f, g, S *and* T *be self-mappings of a metric space* (X, d) *satisfying (2.1) and (2.2) and there exists* $F \in F_4$ *satisfying* (3.2) *and* φ *is as in Theorem 1.4. If one of* $f(X)$*,* $g(X)$*,* $S(X)$ *and* $T(X)$ *is a complete subspace of* X*, then* f *and* S *have a coincidence point and* g *and* T *have a coincidence point. Moreover, if the pairs* (f, S) *and* (g, T) *are owc, then* f, g, S *and* T *have a unique common fixed point in* X*.*

Corollary 3.10. Theorem 2.17.

Proof. It follows from Theorem 3.9 for $\varphi(t) = 1$.

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