"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 20 (2010), No. 2, 29 - 38

ON qI-OPEN SETS IN IDEAL BITOPOLOGICAL SPACES

S. JAFARI AND N. RAJESH

Abstract. In this paper, we introduce and study the concept of $q\mathcal{I}$-open set. Based on this new concept, we define new classes of functions, namely $q\mathcal{I}$-continuous functions, $q\mathcal{I}$-open functions and $q\mathcal{I}$-closed functions, for which we prove characterization theorems.

1. Introduction and Preliminaries

A bitopological space (X, τ_1, τ_2) is a nonempty set X equipped with two topologies τ_1 and τ_2 [5]. In a bitopological space (X, τ_1, τ_2), a set $A \subset X$ is said to be quasi-open [7] if $A = U \cup V$ for some $U \in \tau_1$ and $V \in \tau_2$. Clearly, every τ_1-open set as well as τ_2-open set is quasi-open, but not conversely. Any union of quasi-open sets is quasi-open. A set is said to be quasi-closed [7] if its complement is quasi-open. Every τ_1-closed set as well as τ_2-closed set is quasi-closed, but not conversely. Any intersection of quasi-closed sets is quasi-closed [7]. The quasi-closure [7] of a set A, denoted by $q\,\mathrm{Cl}(A)$, is the intersection of all quasi-closed sets containing A. In fact, a set A is quasi-closed if and only if $A = q\,\mathrm{Cl}(A)$. The concept of ideal in topological spaces has been introduced and studied by Kuratowski [4] and Vaidyanathasamy [8]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

Keywords and phrases: Ideal bitopological spaces, quasi-open set, quasi-closed set.
(2000)Mathematics Subject Classification:54C10

Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [8] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\tau,\mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. In this paper, we introduce and study the concept of $q\mathcal{I}$-open set. Based on this new concept, we define new classes of functions, namely $q\mathcal{I}$-continuous functions, $q\mathcal{I}$-open functions and $q\mathcal{I}$-closed functions, for which we prove characterization theorems.

2. Quasi-local functions

Definition 2.1. *Given a bitopological space (X,τ_1,τ_2) with an ideal $\mathcal{I}$ on X, the quasi-local function of A with respect to τ_1, τ_2 and $\mathcal{I}$, denoted by $A^*_q(\tau_1,\tau_2,\mathcal{I})$ is defined as follows $A^*_q(\tau_1,\tau_2,\mathcal{I}) = \{x \in X : A \cap U \notin \mathcal{I}$ for every quasi-open set containing $x\}$. When there is no ambiguity, we will write A^*_q for $A^*_q(\tau_1,\tau_2,\mathcal{I})$.*

Remark 2.2. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be an ideal bitopological space and A a subset of X. Then we have the following:*

1. *$A^*_q \subset A^*(\tau_1,\mathcal{I})$ and $A^*_q \subset A^*(\tau_2,\mathcal{I})$ for every subset A of X.*
2. *$A^*_q(\tau_1,\tau_2,\{\emptyset\}) = q\,\mathrm{Cl}(A)$.*
3. *$A^*_q(\tau_1,\tau_2,\mathcal{P}(X)) = \emptyset$.*
4. *If $A \in \mathcal{I}$, then $A^*_q = \emptyset$.*
5. *Neither $A \subset A^*_q$ nor $A^*_q \subset A$.*

Theorem 2.3. *Let $(X,\tau_1,\tau_2,\mathcal{I})$ be an ideal bitopological space and A, B subsets of X. Then we have the following:*

1. *If $A \subset B$, then $A^*_q \subset B^*_q$.*
2. *$A^*_q = q\,\mathrm{Cl}(A^*_q) \subset q\,\mathrm{Cl}(A)$ and A^*_q is a quasi-closed set in (X,τ_1,τ_2).*
3. *$(A^*_q)^*_q \subset A^*_q$.*
4. *$(A \cup B)^*_q = A^*_q \cup B^*_q$.*
5. *$A^*_q \setminus B^*_q = (A\setminus B)^*_q \setminus B^*_q \subset (A\setminus B)^*_q$*
6. *If $C \in \mathcal{I}$, then $(A\setminus C)^*_q \subset A^*_q = (A \cup C)^*_q$.*

Proof. (1). Suppose that $A \subset B$ and $x \notin B^*_q$. Then there exists a quasi-open set U containing x such that $U \cap B \in \mathcal{I}$. Since $A \subset B$, $U \cap A \in \mathcal{I}$ and $x \notin A^*_q$. This shows that $A^*_q \subset B^*_q$.
(2). We have $A^*_q \subset q\,\mathrm{Cl}(A^*_q)$ in general. Let $x \in q\,\mathrm{Cl}(A^*_q)$. Then $A^*_q \cap U \neq \emptyset$ for every quasi-open set U containing x. Therefore, there exists $y \in A^*_q \cap U$ and quasi-open set U containing y. Since $y \in A^*_q$, $U \cap A \notin \mathcal{I}$ and hence $x \in A^*_q$. Therefore, we have $q\,\mathrm{Cl}(A^*_q) \subset A^*_q$.

Again, let $x \in q\,\mathrm{Cl}(A^*_q) = A^*_q$, then $U \cap A \notin \mathcal{I}$ for every quasi-open set U containing x. This implies $U \cap A \neq \emptyset$ for every quasi-open set U containing x. Therefore, $x \in q\,\mathrm{Cl}(A)$. This proves $A^*_q = q\,\mathrm{Cl}(A^*_q) \subset q\,\mathrm{Cl}(A)$.

(3). Let $x \in (A^*_q)^*_q$. Then for every quasi-open set U containing x, $U \cap A^*_q \notin \mathcal{I}$ and hence $U \cap A^*_q \neq \emptyset$. Let $y \in U \cap A^*_q$. Then there exists a quasi-open set U containing y and $y \in A^*_q$. Hence we have $U \cap A \notin \mathcal{I}$ and $x \in A^*_q$. This shows that $(A^*_q)^*_q \subset A^*_q$.

(4). By (1), we have $A^*_q \cup B^*_q \subset (A \cup B)^*_q$. For the reverse inclusion, let $x \in (A \cup B)^*_q$. Then for every quasi-open set U containing x, $(U \cap A) \cup (U \cap B) = U \cap (A \cup B) \notin \mathcal{I}$. Therefore, $U \cap A \notin \mathcal{I}$ or $U \cap B \notin \mathcal{I}$. This implies that $x \in A^*_q$ or $x \in B^*_q$. Hence $x \in A^*_q \cup B^*_q$.

(5). We have $A^*_q = (A \backslash B)^*_q \cup (B \cap A)^*_q$; thus $A^*_q \backslash B^*_q = A^*_q \cap (X \backslash B^*_q) = (A \backslash B)^*_q \cup (B \cap A)^*_q \cap (X \backslash B^*_q) = ((A \backslash B)^*_q \cap (X \backslash B^*_q)) \cup ((B \cap A)^*_q \cap (X \backslash B^*_q)) = ((A \backslash B)^*_q \backslash B^*_q) \cup \emptyset \subset (A \backslash B)^*_q$.

(6). Since $A \backslash C \subset A$, by (1), $(A \backslash C)^*_q \subset A^*_q$. By (4) and Remark 2.2 (4), $(A \cup C)^*_q = A^*_q \cup C^*_q = A^*_q \cup \emptyset = A^*_q$. Therefore, we obtain $(A \backslash C)^*_q \subset A^*_q$. Therefore, $(A \backslash C)^*_q \subset A^*_q = (A \cup C)^*_q$. □

Remark 2.4. *Let $\tau = \tau_1 = \tau_2$. Then by Theorem 2.3 we obtain the results for a topological space $(X, \tau, \mathcal{I})$ established in Theorem 2.3 of* [3].

Theorem 2.5. *Let (X, τ_1, τ_2) be a bitopological space with ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ on X and A a subset of X. Then we have the following:*

(1) *If $\mathcal{I}_1 \subset \mathcal{I}_2$, then $A^*_q(\mathcal{I}_2) \subset A^*_q(\mathcal{I}_1)$.*
(2) *$A^*_q(\mathcal{I}_1 \cap \mathcal{I}_2) = A^*_q(\mathcal{I}_1) \cup A^*_q(\mathcal{I}_2)$.*

Proof. (1). Let $\mathcal{I}_1 \subset \mathcal{I}_2$ and $x \in A^*_q(\mathcal{I}_2)$. Then $A \cap U \notin \mathcal{I}_2$ for every quasi-open set U containing x. By hypothesis, $A \cap U \notin \mathcal{I}_1$; hence $x \in A^*_q(\mathcal{I}_1)$. Therefore, we have $A^*_q(\mathcal{I}_2) \subset A^*_q(\mathcal{I}_1)$.

(2). Let $x \in A^*_q(\mathcal{I}_1 \cap \mathcal{I}_2)$. Then, for every quasi-open set U containing x, $A \cap U \notin (\mathcal{I}_1 \cap \mathcal{I}_2)$; hence $A \cap U \notin \mathcal{I}_1$ or $A \cap U \notin \mathcal{I}_2$. This shows that $x \in A^*_q(\mathcal{I}_1)$ or $x \in A^*_q(\mathcal{I}_2)$. Therefore, we have $x \in A^*_q(\mathcal{I}_1) \cup A^*_q(\mathcal{I}_2)$; hence $A^*_q(\mathcal{I}_1 \cap \mathcal{I}_2) \subset A^*_q(\mathcal{I}_1) \cup A^*_q(\mathcal{I}_2)$. By Theorem 2.3 (1), we have $A^*_q(\mathcal{I}_1) \cup A^*_q(\mathcal{I}_2) \subset A^*_q(\mathcal{I}_1 \cap \mathcal{I}_2)$. Thus, $A^*_q(\mathcal{I}_1 \cap \mathcal{I}_2) = A^*_q(\mathcal{I}_1) \cup A^*_q(\mathcal{I}_2)$. □

Definition 2.6. *The quasi-$*$-closure of $A \subset X$, denoted by $q\,\mathrm{Cl}^*(A)$, is defined by $q\,\mathrm{Cl}^*(A) = A \cup A^*_q$.*

Proposition 2.7. *The set operator $q\,\mathrm{Cl}^*$ satisfies the following:*

(1) *$A \subset q\,\mathrm{Cl}^*(A)$.*

(2) $q\,\mathrm{Cl}^*(\emptyset) = \emptyset$ *and* $q\,\mathrm{Cl}^*(X) = X$.
(3) *If* $A \subset B$, *then* $q\,\mathrm{Cl}^*(A) \subset q\,\mathrm{Cl}^*(B)$.
(4) $q\,\mathrm{Cl}^*(A) \cup q\,\mathrm{Cl}^*(B) \subset q\,\mathrm{Cl}^*(A \cup B)$.

Proof. The proof follows from the Definition 2.6. □

Remark 2.8. *If* $\mathcal{I} = \{\emptyset\}$, *then* $q\,\mathrm{Cl}^*(A) = q\,\mathrm{Cl}(A)$ *for* $A \subset X$.

Definition 2.9. *A subset* A *of an ideal bitopological space* $(X, \tau_1, \tau_2, \mathcal{I})$ *is said to be* $q\mathcal{I}$*-open if* $A \subset q\,\mathrm{Int}(A_q^*)$*.The complement of a* $q\mathcal{I}$*-open set is called a* $q\mathcal{I}$*-closed set. The family of all* $q\mathcal{I}$*-open (resp.* $q\mathcal{I}$*-closed) sets of* $(X, \tau_1, \tau_2, \mathcal{I})$ *is denoted by* $Q\mathcal{I}O(X)$ *(resp.* $Q\mathcal{I}C(X)$*). The family of all* $q\mathcal{I}$*-open sets of* $(X, \tau_1, \tau_2, \mathcal{I})$ *containing the point* x *is denoted by* $Q\mathcal{I}O(X, x)$.

Definition 2.10. *A subset* A *of an ideal bitopological space* $(X, \tau_1, \tau_2, \mathcal{I})$ *is said to be:*
(1) $(1, 2)$*-preopen if* $A \subset q\,\mathrm{Int}(q\,\mathrm{Cl}(A))$.
(2) $(1, 2)$*-semiclosed if* $q\,\mathrm{Int}(q\,\mathrm{Cl}(A)) \subset A$.

Proposition 2.11. *Every* $q\mathcal{I}$*-open set is* $(1, 2)$*-preopen.*

Proof. Let $A \in Q\mathcal{I}O(X)$. Then $A \subset q\,\mathrm{Int}(A_q^*)$. By Theorem 2.3 (2), $A \subset q\,\mathrm{Int}(q\,\mathrm{Cl}(A))$. This shows that A is an $(1, 2)$-preopen set. □

The following example shows that the converse of Proposition 2.11 is not true in general.

Example 2.12. *Let* $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{c\}, X\}$, $\tau_2 = \{\emptyset, \{a, b\}, X\}$ *and* $\mathcal{I} = \{\varnothing, \{a\}\}$. *Then the set* $\{d\}$ *is* $(1, 2)$*-preopen but not* $q\mathcal{I}$*-open.*

Remark 2.13. *For an ideal bitopological space* $(X, \tau_1, \tau_2, \mathcal{I})$, *we have the following:*
(1) X *needs not be a* $q\mathcal{I}$*-open set.*
(2) *If* $\mathcal{I} = \mathcal{P}(X)$, *then only the empty set is* $q\mathcal{I}$*-open.*
(3) $q\mathcal{I}$*-openness and quasi-openness are independent concepts.*
(4) *If* $\mathcal{I} = \{\emptyset\}$, $q\mathcal{I}$*-openness and quasi-openness are equivalent.*

Proposition 2.14. *If* A *is* $q\mathcal{I}$*-open, then* $A_q^* = (q\,\mathrm{Int}(A_q^*))_q^*$.

Proof. Since A is $q\mathcal{I}$-open, $A \subset q\,\mathrm{Int}(A_q^*)$. Then $A_q^* \subset (q\,\mathrm{Int}(A_q^*))_q^*$. Also we have $q\,\mathrm{Int}(A_q^*) \subset A_q^*$, $(q\,\mathrm{Int}(A_q^*))^* \subset (A_q^*)_q^* \subset A_q^*$. Hence we have, $A_q^* = (q\,\mathrm{Int}(A_q^*))_q^*$. □

Proposition 2.15. *Any union of* $q\mathcal{I}$*-open sets is* $q\mathcal{I}$*-open.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be a family of $q\mathcal{I}$-open sets of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$. Then $U_\alpha \subset q\operatorname{Int}((U_\alpha)^*_q)$, for every $\alpha \in \Delta$. Thus, $\underset{\alpha\in\Delta}{\cup} U_\alpha \subset \underset{\alpha\in\Delta}{\cup}(q\operatorname{Int}((U_\alpha)^*_q))) \subset q\operatorname{Int}(\underset{\alpha\in\Delta}{\cup}(U_\alpha)^*_q) \subset q\operatorname{Int}(\underset{\alpha\in\Delta}{\cup}(U_\alpha)^*_q)$. □

Proposition 2.16. *If A is $q\mathcal{I}$-open and $(1,2)$-semiclosed, then $A = q\operatorname{Int}(A^*_q)$.*

Proof. Let A be $q\mathcal{I}$-open. Then $A \subset q\operatorname{Int}(A^*_q)$. Since A is $(1,2)$-semiclosed, $q\operatorname{Int}(A^*_q) \subset q\operatorname{Int}(q\operatorname{Cl}(A)) \subset A$. Thus $q\operatorname{Int}(A^*_q) \subset A$. Hence we have, $A = q\operatorname{Int}(A^*_q)$. □

Definition 2.17. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x a point of X. Then*

(i) *x is called a $q\mathcal{I}$-interior point of S if there exists $V \in Q\mathcal{I}O(X)$ such that $x \in V \subset S$.*
ii) *the set of all $q\mathcal{I}$-interior points of S is called the $q\mathcal{I}$-interior of S and is denoted by $q\mathcal{I}\operatorname{Int}(S)$.*

Theorem 2.18. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

(1) $q\mathcal{I}\operatorname{Int}(A) = \cup\{T : T \subset A$ *and* $A \in Q\mathcal{I}O(X)\}$.
(2) $q\mathcal{I}\operatorname{Int}(A)$ *is the largest $q\mathcal{I}$-open subset of X contained in A.*
(3) *A is $q\mathcal{I}$-open if and only if $A = q\mathcal{I}\operatorname{Int}(A)$.*
(4) $q\mathcal{I}\operatorname{Int}(q\mathcal{I}\operatorname{Int}(A)) = q\mathcal{I}\operatorname{Int}(A)$.
(5) *If $A \subset B$, then $q\mathcal{I}\operatorname{Int}(A) \subset q\mathcal{I}\operatorname{Int}(B)$.*
(6) $q\mathcal{I}\operatorname{Int}(A) \cup q\mathcal{I}\operatorname{Int}(B) \subset q\mathcal{I}\operatorname{Int}(A \cup B)$.
(7) $q\mathcal{I}\operatorname{Int}(A \cap B) \subset q\mathcal{I}\operatorname{Int}(A) \cap q\mathcal{I}\operatorname{Int}(B)$.

Proof. (1). Let $x \in \cup\{T : T \subset A$ and $A \in Q\mathcal{I}O(X)\}$. Then, there exists $T \in Q\mathcal{I}O(X, x)$ such that $x \in T \subset A$ and hence $x \in q\mathcal{I}\operatorname{Int}(A)$. This shows that $\cup\{T : T \subset A$ and $A \in Q\mathcal{I}O(X)\} \subset q\mathcal{I}\operatorname{Int}(A)$. For the reverse inclusion, let $x \in q\mathcal{I}\operatorname{Int}(A)$. Then there exists $T \in Q\mathcal{I}O(X, x)$ such that $x \in T \subset A$. We obtain $x \in \cup\{T : T \subset A$ and $A \in Q\mathcal{I}O(X)\}$. This shows that $q\mathcal{I}\operatorname{Int}(A) \subset \cup\{T : T \subset A$ and $A \in Q\mathcal{I}O(X)\}$. Therefore, we obtain $q\mathcal{I}\operatorname{Int}(A) = \cup\{T : T \subset A$ and $A \in Q\mathcal{I}O(X)\}$.
The proof of (2)-(5) are obvious.
(6). Clearly, $q\mathcal{I}\operatorname{Int}(A) \subset q\mathcal{I}\operatorname{Int}(A \cup B)$ and $q\mathcal{I}\operatorname{Int}(B) \subset q\mathcal{I}\operatorname{Int}(A \cup B)$. Then we obtain $q\mathcal{I}\operatorname{Int}(A) \cup q\mathcal{I}\operatorname{Int}(B) \subset q\mathcal{I}\operatorname{Int}(A \cup B)$.
(7). Since $A \cap B \subset A$ and $A \cap B \subset B$, by (5), we have $q\mathcal{I}\operatorname{Int}(A \cap B)$

$\subset q\mathcal{I}\operatorname{Int}(A)$ and $q\mathcal{I}\operatorname{Int}(A \cap B) \subset q\mathcal{I}\operatorname{Int}(B)$. Then $q\mathcal{I}\operatorname{Int}(A \cap B) \subset q\mathcal{I}\operatorname{Int}(A) \cap q\mathcal{I}\operatorname{Int}(B)$. □

Definition 2.19. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then*

(1) *x is called a $q\mathcal{I}$-cluster point of S if $V \cap S \neq \emptyset$ for every $V \in Q\mathcal{I}O(X, x)$.*
(2) *the set of all $q\mathcal{I}$-cluster points of S is called the $q\mathcal{I}$-closure of S and is denoted by $q\mathcal{I}\operatorname{Cl}(S)$.*

Theorem 2.20. *Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:*

(1) *$q\mathcal{I}\operatorname{Cl}(A) = \cap\{F : A \subset F$ and $F \in Q\mathcal{I}C(X)\}$.*
(2) *$q\mathcal{I}\operatorname{Cl}(A)$ is the smallest $q\mathcal{I}$-closed subset of X containing A.*
(3) *A is $q\mathcal{I}$-closed if and only if $A = q\mathcal{I}\operatorname{Cl}(A)$.*
(4) *$q\mathcal{I}\operatorname{Cl}(q\mathcal{I}\operatorname{Cl}(A) = q\mathcal{I}\operatorname{Cl}(A)$.*
(5) *If $A \subset B$, then $q\mathcal{I}\operatorname{Cl}(A) \subset q\mathcal{I}\operatorname{Cl}(B)$.*
(6) *$q\mathcal{I}\operatorname{Cl}(A \cup B) = q\mathcal{I}\operatorname{Cl}(A) \cup q\mathcal{I}\operatorname{Cl}(B)$.*
(7) *$q\mathcal{I}\operatorname{Cl}(A \cap B) \subset q\mathcal{I}\operatorname{Cl}(A) \cap q\mathcal{I}\operatorname{Cl}(B)$.*

Proof. (1). Suppose that $x \notin q\mathcal{I}\operatorname{Cl}(A)$. Then there exists $F \in Q\mathcal{I}O(X)$ such that $F \cap A = \emptyset$. Since $X\backslash F$ is $q\mathcal{I}$-closed set containing A and $x \notin X\backslash F$, we obtain $x \notin \cap\{F : A \subset F$ and $F \in Q\mathcal{I}C(X)\}$. Then there exists $F \in Q\mathcal{I}C(X)$ such that $A \subset F$ and $x \notin F$. Since $X\backslash V$ is $q\mathcal{I}$-closed set containing x, we obtain $(X\backslash F) \cap A = \emptyset$. This shows that $x \notin q\mathcal{I}\operatorname{Cl}(A)$. Therefore, we obtain $q\mathcal{I}\operatorname{Cl}(A) = \cap\{F : A \subset F$ and $F \in Q\mathcal{I}C(X)$.
Proofs of the rest of statements are obvious. □

Theorem 2.21. *Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A \subset X$. Then the following properties hold:*

(1) *$q\mathcal{I}\operatorname{Cl}(X\backslash A) = X\backslash q\mathcal{I}\operatorname{Int}(A)$;*
(2) *$q\mathcal{I}\operatorname{Int}(X\backslash A) = X\backslash q\mathcal{I}\operatorname{Cl}(A)$.*

Proof. (1). Since $W \subset A$ if and only if $X\backslash A \subset X\backslash W$, W is $q\mathcal{I}$-open if and only if $q\mathcal{I}$-closed. Thus, $q\mathcal{I}\operatorname{Cl}(A) = \cap\{X\backslash W : W \in Q\mathcal{I}O(X)$ and $W \subset A\} = X\backslash \cup \{W \in Q\mathcal{I}O(X)$ and $W \subset A\} = X\backslash q\mathcal{I}\operatorname{Int}(A)$.
(2). Follows from (1). □

Definition 2.22. *A subset B_x of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be a $q\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists a $q\mathcal{I}$-open set U such that $x \in U \subset B_x$.*

Theorem 2.23. *A subset of an ideal bitopological space* $(X,\tau_1,\tau_2,\mathcal{I})$ *is* $q\mathcal{I}$*-open if and only if it is a* $q\mathcal{I}$*-neighbourhood of each of its points.*

Proof. Let G be a $q\mathcal{I}$-open set of X. Then by definition, it is clear that G is a $q\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is $q\mathcal{I}$-open. Conversely, suppose G is a $q\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in Q\mathcal{I}O(X)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is $q\mathcal{I}$-open and arbitrary union of $q\mathcal{I}$-open sets is $q\mathcal{I}$-open, G is $q\mathcal{I}$-open in $(X,\tau_1,\tau_2,\mathcal{I})$. □

3. $q\mathcal{I}$-continuous functions

Definition 3.1. *A function* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2)$ *is called* $q\mathcal{I}$*-continuous if* $f^{-1}(V)$ *is* $q\mathcal{I}$*-open in* X *for every quasi-open set* V *of* Y *or eqivalently,* $f^{-1}(V)$ *is* $q\mathcal{I}$*-closed in* X *for every quasi-closed set* V *of* Y*.*

Definition 3.2. *A function* $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ *is called* $(1,2)$*-*$\mathcal{I}$*-continuous if* $f^{-1}(V)$ *is* $(1,2)$*-preopen in* X *for every quasi-open set* V *of* Y *or eqivalently,* $f^{-1}(V)$ *is* $(1,2)$*-preclosed in* X *for every quasi-closed set* V *of* Y*.*

It is clear that every $q\mathcal{I}$-continuous function is $(1,2)$-precontinuous. But the converse is not true in general.

Example 3.3. *Let* $(X,\tau_1,\tau_2,\mathcal{I})$ *be as in Example 2.12,* $\sigma_1 = \{\varnothing, \{d\}, X\}$ *and* $\sigma_2 = \{\varnothing, \{a,d\}, X\}$*. Then the identity function* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2)$ *is* $(1,2)$*-precontinuous but not* $q\mathcal{I}$*-continuous.*

Theorem 3.4. *For a function* $f : (X,\tau_1,\tau_2,\mathcal{I}) \to (Y,\sigma_1,\sigma_2)$*, the following statement are equivalent:*

(1) *f is $q\mathcal{I}$-continuous.*
(2) *For each $x \in X$ and every quasi-open set V containing $f(x)$, there exists $W \in Q\mathcal{I}O(X,x)$ such that $f(W) \subset V$.*
(3) *For each $x \in X$ and each quasi-open set V containing $f(x)$, $f^{-1}(V)^*_q$ is a $q\mathcal{I}$-neighborhood of x.*

Proof. (1) ⇒ (2) Let $x \in X$ and V be a quasi-open set of Y containing $f(x)$. Since f is $q\mathcal{I}$-continuous, $f^{-1}(V)$ is a $q\mathcal{I}$-open set. Putting $W = f^{-1}(V)$, we have $f(W) \subset V$.
(2)⇒ (1) Let A be a quasi-open set in Y. If $f^{-1}(A) = \emptyset$, then $f^{-1}(A)$ is clearly a $q\mathcal{I}$-open set. Assume that $f^{-1}(A) \neq \emptyset$. Let $x \in f^{-1}(A)$. Then

$f(x) \in A$, which implies that there exists a $q\mathcal{I}$-open W containing x such that $f(W) \subset A$. Thus $W \subset f^{-1}(A)$. Since W is a $q\mathcal{I}$-open, $x \in W \subset q\,\mathrm{Int}(W_q^*) \subset q\,\mathrm{Int}((f^{-1}(A)_q^*))$ and so $f^{-1}(A) \subset q\,\mathrm{Int}(f^{-1}(A)_q^*)$. Hence $f^{-1}(A)$ is a $q\mathcal{I}$-open set and so f is $q\mathcal{I}$-continuous.
(2) $\Rightarrow$ (3) Let $x \in X$ and V be a quasi-open set of Y containing $f(x)$. Then there exist a $q\mathcal{I}$-open set W containing x such that $f(W) \subset V$. It follows that $W \subset f^{-1}(f(W)) \subset f^{-1}(V)$. Since W is a $q\mathcal{I}$-open set, $x \in W \subset q\,\mathrm{Int}(W^*) \subset q\,\mathrm{Int}(f^{-1}(V)_q^*) \subset f^{-1}(V)^*$. Hence $f^{-1}(V)_q^*$ is a $q\mathcal{I}$-neighborhood of x.
(3) $\Rightarrow$ (1) Obvious. □

Remark 3.5. *Let $\tau = \tau_1 = \tau_2$ and $\sigma = \sigma_1 = \sigma_2$. Then by Theorem 3.4 we obtain the results for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ established in Theorem 3.1 of* [1].

Definition 3.6. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ is said to be:*

(1) *$q\mathcal{I}$-open if $f(U)$ is a $q\mathcal{I}$-open set of Y for every quasi-open set U of X.*
(2) *$q\mathcal{I}$-closed if $f(U)$ is a $q\mathcal{I}$-closed set of Y for every quasi-closed set U of X.*

Theorem 3.7. *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$, the following statements are equivalent:*

(1) *f is $q\mathcal{I}$-open;*
(2) *$f(q\,\mathrm{Int}(U)) \subset q\mathcal{I}\,\mathrm{Int}(f(U))$ for each subset U of X;*
(3) *$q\,\mathrm{Int}(f^{-1}(V)) \subset f^{-1}(q\mathcal{I}\,\mathrm{Int}(V))$ for each subset V of Y.*

Proof. (1) $\Rightarrow$ (2): Let U be any subset of X. Then $q\,\mathrm{Int}(U)$ is a quasi-open set of X. Then $f(q\,\mathrm{Int}(U))$ is a $q\mathcal{I}$-open set of Y. Since $f(q\,\mathrm{Int}(U)) \subset f(U)$, $f(q\,\mathrm{Int}(U)) = q\mathcal{I}\,\mathrm{Int}(f(q\,\mathrm{Int}(U))) \subset q\mathcal{I}\,\mathrm{Int}(f(U))$.
(2) $\Rightarrow$ (3): Let V be any subset of Y. Then $f^{-1}(V)$ is a subset of X. Hence $f(q\,\mathrm{Int}(f^{-1}(V))) \subset q\mathcal{I}\,\mathrm{Int}(f(f^{-1}(V))) \subset q\mathcal{I}\,\mathrm{Int}(V))$. Then $q\,\mathrm{Int}(f^{-1}(V)) \subset f^{-1}(f(q\,\mathrm{Int}(f^{-1}(V)))) \subset f^{-1}(\mathcal{I}\,\mathrm{Int}(V))$.
(3) $\Rightarrow$ (1): Let V be any quasi-open set of X. Then $q\,\mathrm{Int}(V) = V$ and $f(U)$ is a subset of Y. Now, $V = q\,\mathrm{Int}(V) \subset q\,\mathrm{Int}(f^{-1}(f(V))) \subset f^{-1}(q\mathcal{I}\,\mathrm{Int}(f(V)))$. Then $f(V) \subset f(f^{-1}(q\mathcal{I}\,\mathrm{Int}(f(V)))) \subset q\mathcal{I}\,\mathrm{Int}(f(V))$ and $q\mathcal{I}\,\mathrm{Int}(f(V)) \subset f(V)$. Hence $f(V)$ is a $q\mathcal{I}$-open set of Y; hence f is $q\mathcal{I}$-open. □

Theorem 3.8. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a $q\mathcal{I}$-open function. If V is a subset of Y and U is a quasi-closed subset of X containing $f^{-1}(V)$, then there exists a $q\mathcal{I}$-closed set F of Y containing V such that $f^{-1}(F) \subset U$.*

Proof. Let V be any subset of Y and U a quasi-closed subset of X containing $f^{-1}(V)$, and let $F = Y\backslash(f(X\backslash V))$. Then $f(X\backslash V) \subset f(f^{-1}(X\backslash V)) \subset X\backslash V$ and $X\backslash U$ is a quasi-open set of X. Since f is $q\mathcal{I}$-open, $f(X\backslash U)$ is a $q\mathcal{I}$-open set of Y. Hence F is a $q\mathcal{I}$-closed set of Y and $f^{-1}(F) = f^{-1}(Y\backslash(f(X\backslash U)) \subset U$. □

Remark 3.9. *Let $\tau = \tau_1 = \tau_2$ and $\sigma = \sigma_1 = \sigma_2$. Then by Theorem 3.8 we obtain the results for a function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ established in Theorem 4.2 of* [1].

Theorem 3.10. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a function. Then f is a $q\mathcal{I}$-closed function if and only if for each subset V of X, $q\mathcal{I}\,\mathrm{Cl}(f(V)) \subset f(q\,\mathrm{Cl}(V))$.*

Proof. Let f be an $q\mathcal{I}$-closed function and V any subset of X. Then $f(V) \subset f(q\,\mathrm{Cl}(V))$ and $f(q\,\mathrm{Cl}(V))$ is a $q\mathcal{I}$-closed set of Y. We have $q\mathcal{I}\,\mathrm{Cl}(f(V)) \subset q\mathcal{I}\,\mathrm{Cl}(f(q\,\mathrm{Cl}(V))) = f(q\,\mathrm{Cl}(V))$. Conversely, let V be a quasi-closed set of X. Then $f(V) \subset q\mathcal{I}\,\mathrm{Cl}(f(V)) \subset f(q\,\mathrm{Cl}(V)) = f(V)$; hence $f(V)$ is a $q\mathcal{I}$-closed subset of Y. Therefore, f is a $q\mathcal{I}$-closed function. □

Theorem 3.11. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a function. Then f is a $q\mathcal{I}$-closed function if and only if for each subset V of Y, $f^{-1}(q\mathcal{I}\,\mathrm{Cl}(V)) \subset q\,\mathrm{Cl}(f^{-1}(V))$.*

Proof. Let V be any subset of Y. Then by Theorem 3.10, $q\mathcal{I}\,\mathrm{Cl}(V) \subset f(q\,\mathrm{Cl}(f^{-1}(V)))$. Since f is bijection,
$f^{-1}(q\mathcal{I}\,\mathrm{Cl}(V)) = f^{-1}(q\mathcal{I}\,\mathrm{Cl}(f(f^{-1}(V)))) \subset f^{-1}(f(q\,\mathrm{Cl}(f^{-1}(V)))) = q\,\mathrm{Cl}(f^{-1}(V))$.
Conversely, let U be any subset of X. Since f is bijection, $q\mathcal{I}\,\mathrm{Cl}(f(U)) = f(f^{-1}(q\mathcal{I}\,\mathrm{Cl}(f(U))) \subset f(q\,\mathrm{Cl}(f^{-1}(f(U)))) = f(q\,\mathrm{Cl}(U))$. Therefore, by Theorem 3.10, f is an $q\mathcal{I}$-closed function. □

Theorem 3.12. *Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ be a $q\mathcal{I}$-closed function. If V is a subset of Y and U is a quasi-open subset of X containing $f^{-1}(V)$, then there exists a $q\mathcal{I}$-open set F of Y containing V such that $f^{-1}(F) \subset U$.*

Proof. The proof is similar to Theorem 3.8. □

Remark 3.13. *Let $\tau = \tau_1 = \tau_2$ and $\sigma = \sigma_1 = \sigma_2$. Then by Theorem 3.12 we obtain the results for a function $f : (X, \tau) \to (Y, \sigma, \mathcal{I})$ established in Theorem 4.2 of* [1].

References

[1] M. E. Abd. El-Monsef, E. F. Lashien and A. A. Nasef, **On $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions**, Kyungpook Math. J., 32(1)(1992), 21-30.
[2] B.P. Dvalishvili, **Bitopological spaces: theory, relations with generalized algebraic structures, and applications**, North-Holland Mathematics Studies, 2005.
[3] D. Jankovic and T. R. Hamlett, **New Topologies fron old via ideals**, Amer. Math. Monthly, 97(1990), 295-310.
[4] K. Kuratowski,**Topology**, Academic Press, New York, 1966.
[5] B.P. Dvalishvili, **Bitopological spaces: theory, relations with generalized algebraic structures, and applications**, North-Holland Mathematics Studies, 2005.
[6] R. L. Newcomb, **Topologies which are compact modulo an ideal**, Ph.D. Thesis, University of California, USA(1967).
[7] M. S. Sarsak, **Quasi separation axioms**, Turk. J. Math., 30(2006), 25-31.
[8] R. Vaidyanathaswamy, **The localisation theory in set topology**, Proc. Indian Acad. Sci., 20(1945), 51-61.

Department of Mathematics
College of Vestsjaelland South, Herrestraede
11, 4200 Slagelse
Denmark jafaripersia@gmail.com

Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005
Tamilnadu, India. nrajesh_topology@yahoo.co.in