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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 20 (2010), No. 2, 39 - 48

COMMON FIXED POINTS OF TWO MAPS IN COMPLETE G-METRIC SPACES

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Abstract. In this paper, we present some common fixed point theorems for contraction type and expansion type mappings in complete G- metric spaces.

1. Introduction and preliminaries

Recently, Mustafa and Sims [2] introduced a new structure of generalized metric spaces, which are called G-metric spaces as generalization of metric spaces (X, d). Some authors [1, 2, 3, 4, 5, 6, 7] have proved some fixed points theorems for mappings satisfying different contractive conditions in this new structure.

The main purpose of this paper is to present some common fixed point theorems for contraction type and expansion type mappings in complete G-metric spaces.

We now recall the definitions of G-metric spaces and some their properties.

Throughout this paper, we denote by $\mathbf{N}$ the set of positive integers.

Definition 1.1 ([2]). Let X be a nonempty set and let $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions:

Keywords and phrases: Common fixed point; Contraction mapping; Expansion mapping; G-metric space.
(2000)Mathematics Subject Classification:47H10, 54H25.

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$, with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, z with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

Note that if $G(x, y, z) = 0$, then $x = y = z$ (see [2]).

Example1.2 Let (X, d) be a usual metric space. Then (X, G_s) and (X, G_m) are G-metric spaces, where

$$\begin{aligned} G_s(x, y, z) &= d(x, y) + d(y, z) + d(x, z), \\ G_m(x, y, z) &= \max\{d(x, y), d(z, y), d(x, z)\} \end{aligned}$$

for all $x, y, z \in X$.

Definition 1.3 ([2]). Let (X, G) be a G-metric space. A squence $\{x_n\}$ in X is said to be:

(a) a G-convergent sequence if, for each $\varepsilon > 0$, there is an $x \in X$ and $N \in \mathbf{N}$ such that for all $n, m \geq N$, $G(x, x_n, x_m) < \varepsilon$,

(b) a G-Cauchy sequence if, for each $\varepsilon > 0$, there is an $N \in \mathbf{N}$ such that for all $n, m, l \geq N$, $G(x_n, x_m, x_l) < \varepsilon$.

A G-metric spaces (X, G) is said to be complete if every G-Cauchy sequence is convergent in X.

Proposition 1.4 ([2]). *Let (X, G) be a G-metric space. Then the following are equivalent.*

(1) $\{x_n\}$ *is G-convergent to* x,
(2) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$,
(3) $G(x_n, x, x) \to 0$, *as* $n \to \infty$,
(4) $G(x_m, x_n, x) \to 0$, *as* $m, n \to \infty$.

Proposition 1.5 ([2]). *If (X, G) is a G-metric space, then the following are equivalent.*

(1) *The sequence* $\{x_n\}$ *is G-Cauchy.*

(2) *For every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.*

Definition 1.6 ([2]). Let (X, G) and (X', G') be two G-metric spaces and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-continuous at a point $a \in X$ if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$, and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \varepsilon$.

A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.7 ([2]). *Let (X, G), (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.*

It is easy to prove the following the lemma:

Lemma 1.8 *Let (X, d) be a metric space and (X, G_m) a G-metric space defined by metric d on X as in Example 1.2. Then (X, d) is complete if and only if the G-metric space (X, G_m) is complete.*

2. Fixed points for contraction type mappings

In this section we will prove a common fixed point theorem for contraction type mappings in complete G-metric spaces.

Theorem 2.1 *Let (X, G) be a complete G-metric space and let f and g be self mappings of X satisfying the following inequalities;*

$$G(fgx, gx, gx) \leq \varphi(G(gx, x, x)), \tag{1}$$

$$G(gfx, fx, fx) \leq \varphi(G(fx, x, x)) \tag{2}$$

for all x in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function , with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If either f or g is continuous, then f and g have a common fixed point.

We note that the function φ satisfies $\varphi(0) = 0$.

Proof. Let x_0 be an arbitrary point in X and define sequence $\{x_n\}$ inductively by

$$x_{2n+1} = fx_{2n} \text{ , } x_{2n+2} = gx_{2n+1}$$

for $n = 0, 1, 2, ...$

Note that if $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and g. Indeed, if $x_{2n} = x_{2n+1}$ for some $n \geq 0$, then x_{2n} is a fixed point of

f. On the other hand, we have from inequality (2) that

$$\begin{aligned} G(x_{2n+2}, x_{2n+1}, x_{2n+1}) &= G(gx_{2n+1}, fx_{2n}, fx_{2n}) = G(gfx_{2n}, fx_{2n}, fx_{2n}) \\ &\leq \varphi(G(fx_{2n}, x_{2n}, x_{2n})) = \varphi(G(x_{2n+1}, x_{2n}, x_{2n})) \\ &= \varphi(0) = 0 \end{aligned}$$

which implies $G(x_{2n+2}, x_{2n+1}, x_{2n+1}) = 0$ and so $x_{2n+1} = x_{2n+2}$. Thus x_{2n} is a common fixed point of f and g. If $x_{2n+1} = x_{2n+2}$ for some $n \geq 0$, similarly by using inequality (1) leads to x_{2n+1} is a common fixed point of f and g.

Now we suppose that $x_n \neq x_{n+1}$ for all n. Using inequality (1), we have

$$\begin{aligned} G(x_{2n+3}, x_{2n+2}, x_{2n+2}) &= G(fx_{2n+2}, gx_{2n+1}, gx_{2n+1}) \\ &= G(fgx_{2n+1}, gx_{2n+1}, gx_{2n+1}) \\ &\leq \varphi(G(gx_{2n+1}, x_{2n+1}, x_{2n+1})) \\ &= \varphi(G(x_{2n+2}, x_{2n+1}, x_{2n+1})). \end{aligned} \tag{3}$$

Similarly using inequality (2), we have

$$\begin{aligned} G(x_{2n+2}, x_{2n+1}, x_{2n+1}) &= G(gx_{2n+1}, fx_{2n}, fx_{2n}) \\ &= G(gfx_{2n}, fx_{2n}, fx_{2n}) \\ &\leq \varphi(G(fx_{2n}, x_{2n}, x_{2n})) \\ &= \varphi(G(x_{2n+1}, x_{2n}, x_{2n})). \end{aligned} \tag{4}$$

Then from inequalities (3) and (4), we obtain

$$G(x_{n+2}, x_{n+1}, x_{n+1}) \leq \varphi(G(x_{n+1}, x_n, x_n))$$

for $n = 0, 1, 2, ...$, and in general

$$G(x_{n+2}, x_{n+1}, x_{n+1}) \leq \varphi^{n+1}(G(x_1, x_0, x_0))$$

for $n = 0, 1, 2, ...$

Let $n > m$. Then from rectangle inequality of G, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \\ &\leq G(x_n, x_{n-1}, x_{n-1}) + G(x_{n-1}, x_{n-2}, x_{n-2}) + ... + G(x_{m+1}, x_m, x_m) \\ &\leq \varphi^{n-1}(G(x_1, x_0, x_0)) + \varphi^{n-2}(G(x_1, x_0, x_0)) + ... + \varphi^m(G(x_1, x_0, x_0)) \\ &= (\varphi^{n-1} + \varphi^{n-2} + ... + \varphi^m)(G(x_1, x_0, x_0)) \\ &= \sum_{k=m}^{n-1} \varphi^k(G(x_1, x_0, x_0)) \leq \sum_{k=1}^{n-1} \varphi^k(G(x_1, x_0, x_0)). \end{aligned}$$

Take any $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$, we can choose a sufficiently large natural number such that

$$\sum_{k=1}^{n-1} \varphi^k(G(x_1, x_0, x_0)) < \varepsilon,$$

for all $n > m \geq N$, and it follows from Proposition 1.5 that $\{x_n\}$ is a G-Cauchy sequence in the complete G-metric space (X, G) and so has a limit z in X.

Now we suppose that f is continuous. Since $x_{2n+1} = fx_{2n}$, it follows from Proposition 1.7 that

$$z = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n} = fz$$

and so z is fixed point of f.

Using inequality (2) we have

$$\begin{aligned} G(gz, z, z) &= G(gfz, fz, fz) \\ &\leq \varphi(G(fz, z, z)) = \varphi(G(z, z, z)) = \varphi(0) = 0 \end{aligned}$$

which implies $G(gz, z, z) = 0$. Hence $gz = z$. We have therefore proved that z is a common fixed point of f and g.

Similarly, considering the continuity of g, it can be seen that f and g have common fixed point and this completes the proof.

Putting $f = g$ in Theorem 2.1, then we get the following corollary;

Corollary 2.2 *Let (X, G) be a complete G-metric space and let f be a self-mapping of X satisfying the following inequality;*

$$G(f^2x, fx, fx) \leq \varphi(G(fx, x, x))$$

for all x in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecrasing function , with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If f is continuous, then f has a fixed point.

Corollary 2.3 *Let (X, d) be a complete metric space and let f and g be a self-mappings of X satisfying the following inequalities;*

$$d(fgx, gx) \leq \varphi(d(gx, x)), \tag{5}$$

$$d(gfx, fx) \leq \varphi(d(fx, x)) \tag{6}$$

for all x in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecrasing function , with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If either f or g continuous, then f and g have a common fixed point.

Proof. We consider the G_m-metric defined by metric

$$G_m(x,y,z) = \max\{d(x,y), d(z,y), d(x,z)\}$$

on X as in Example 1.2. Then from Lemma 1.8, (X, G_m) is complete. If f is continuous on (X,d), then f also continuous on (X, G_m).

From inequalities (5) and (6) and definition of G_m, we get

$$G_m(fgx, gx, gx) = d(fgx, gx) \leq \varphi(d(gx, x)) = \varphi(G_m(gx, x, x))$$

and

$$G_m(gfx, fx, fx) = d(gfx, fx) \leq \varphi(d(fx, x)) = \varphi(G_m(fx, x, x)).$$

Thus f and g satisfy the inequalities (1) and (2). Hence, from Theorem 2.1, f and g have a common fixed point in X.

3. Fixed points for expansion type mapping

In this section we consider expansion type mapping.

Theorem 3.1 *Let (X, G) be a complete G-metric space and let f and g be surjective self-mappings of X satisfying the following inequalities;*

$$\varphi(G(fgx, gx, gx)) \geq G(gx, x, x) \tag{7}$$

$$\varphi(G(gfx, fx, fx)) \geq G(fx, x, x) \tag{8}$$

for all x in X, where $\varphi : [0,\infty) \to [0,\infty)$ is a nondecreasing function , with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If either f or g is continuous, then f and g have a common fixed point.

Proof. Let x_0 be an arbitrary point in X. Since f and g are surjective mappings, there exist points $x_1 \in f^{-1}(x_0)$ and $x_2 \in g^{-1}(x_1)$. Continuing in this way, we obtain the sequence $\{x_n\}$ with $x_{2n+1} \in f^{-1}(x_{2n})$ and $x_{2n+2} \in g^{-1}(x_{2n+1})$.

Note that if $x_n = x_{n+1}$ for some n, then x_n is a fixed point of f and g. Indeed, if $x_{2n} = x_{2n+1}$ for some $n \geq 0$ then x_{2n} is a fixed point of f. On the other hand, we have (7) that

$$\begin{aligned} 0 = \varphi(0) &= \varphi(G(x_{2n}, x_{2n+1}, x_{2n+1})) \\ &= \varphi(G(fx_{2n+1}, gx_{2n+2}, gx_{2n+2})) \\ &= \varphi(G(fgx_{2n+2}, gx_{2n+2}, gx_{2n+2})) \\ &\geq G(x_{2n+1}, x_{2n+2}, x_{2n+2}) \end{aligned}$$

which implies $G(x_{2n+1}, x_{2n+2}, x_{2n+2}) = 0$ and so $x_{2n+1} = x_{2n+2}$. Thus x_{2n} is a common fixed point of f and g.

If $x_{2n+1} = x_{2n+2}$ for some $n \geq 0$, similarly by using inequality (8) leaads to x_{2n+1} is a common fixed point of f and g.

Now we suppose that $x_n \neq x_{n+1}$ for all n. Using inequality (8), we have

$$\begin{aligned} \varphi(G(x_{2n+1}, x_{2n+2}, x_{2n+2})) &= \varphi(G(gfx_{2n+3}, fx_{2n+3}, fx_{2n+3})) \\ &\geq G(x_{2n+2}, x_{2n+3}, x_{2n+3}). \end{aligned} \tag{9}$$

Similarly, using inequality (7), we have

$$\begin{aligned} \varphi(G(x_{2n}, x_{2n+1}, x_{2n+1})) &= \varphi(G(fgx_{2n+2}, gx_{2n+2}, gx_{2n+2})) \\ &\geq G(x_{2n+1}, x_{2n+2}, x_{2n+2}). \end{aligned} \tag{10}$$

Then form inequalities (9) and (10), we obtain

$$\varphi(G(x_n, x_{n+1}, x_{n+1})) \geq G(x_{n+1}, x_{n+2}, x_{n+2})$$

for $n = 0, 1, 2, ...$ and it follows that

$$\varphi^n(G(x_0, x_1, x_1)) \geq G(x_n, x_{n+1}, x_{n+1})$$

for $n = 0, 1, 2, ...$

Let $m > n$. Then from rectangle inequality of G, we have

$$\begin{aligned} G(x_n, x_m, x_m) &\leq \\ &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + ... + G(x_{m-1}, x_m, x_m) \\ &\leq \varphi^n(G(x_0, x_1, x_1)) + \varphi^{n+1}(G(x_0, x_1, x_1)) + ... + \varphi^{m-1}(G(x_0, x_1, x_1)) \\ &= \sum_{k=n}^{m-1} \varphi^k(G(x_0, x_1, x_1)) \leq \sum_{k=1}^{m-1} \varphi^k(G(x_0, x_1, x_1)). \end{aligned}$$

Take any $\varepsilon > 0$. Since $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$, we can choose a sufficiently large natural number such that $\sum_{k=n}^{m-1} \varphi^k(G(x_0, x_1, x_1)) < \varepsilon$ for all $m > n \geq N$, and it follows from Proposition 1.5 that $\{x_n\}$ is a G-Cauchy sequence in the complete G-metric space (X, G) and so has a limit z in X.

Now we suppose that f is continuous. Since $x_{2n} = fx_{2n+1}$, it follows from Proposition 1.7 that

$$z = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} fx_{2n+1} = fz$$

and so z is fixed point of f. Since g is surjective, there exists y such that $gy = z$. Thus, using inequality (7) we have

$$\begin{aligned} 0 = \varphi(G(fz, gy, gy)) &= \varphi(G(fgy, gy, gy) \geq G(gy, y, y) \\ &= G(z, y, y)) \end{aligned}$$

which implies $G(z, y, y) = 0$ and so $y = z$. Thus $z = gz$. We have therefore proved that z is a common fixed point of f and g.

Similarly, considering the continuity of g, it can be seen that f and g have a common fixed point and this completes the proof.

Putting $f = g$ in Theorem 3.1, then we get the following corollary.

Corollary 3.2 *Let (X, G) be a complete G-metric space and let f be a surjective self-mapping of X satisfying the following inequality;*

$$\varphi(G(f^2x, fx, fx)) \geq G(fx, x, x)$$

for all x in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecrasing function , with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If f is continuous, then f has a fixed point.

Using the same procedure as in the proof of Corollary 2.3, we obtain the following corollary.

Corollary 3.3 *Let (X, d) be a complete metric space and let f and g be self-mappings of X satisfying the following inequalities;*

$$\begin{aligned} \varphi(d(fgx, gx)) &\geq d(gx, x) \\ \varphi(d(gfx, fx)) &\geq d(fx, x) \end{aligned}$$

for all x in X, where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecrasing function , with $\sum_{n=1}^{\infty} \varphi^n(t) < \infty$ for all $t > 0$. If either f or g continuous, then f and g have a common fixed point.

Putting $f = g$ and define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{1}{k}t$ where $k > 1$ in Corollary 3.3. Then we obtain the following corollary.

Corollary 3.4 ([8]) *Let (X, d) be a complete metric space and let f be a self-mapping of X satisfying the following inequalities;*

$$d(f^2x, fx) \geq kd(fx, x)$$

for all x in X, where $k > 1$. If f is continuous, then f has a fixed point.

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