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ACCURATE BOUNDS FOR A CONVERGENCE TO GAMMA CONSTANT

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Abstract. The aim of this paper is to improve the result of Negoi [A zarter convergence to the constant of Euler Gaz. Matem. Ser. A 15 (1997) 111–113] about the convergence speed of a sequence convergent to Euler-Mascheroni constant.

1. **Introduction**

It is of general knowledge that the sequence

$$D_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

is convergent to a limit denoted $\gamma = 0.577215\ldots$ now known as Euler-Mascheroni constant.

Many authors have given bounds for $D_n - \gamma$, see [1,12-18], which show the fact that D_n converges slowly, as n^{-1}. Quicker approximations to the Euler-Mascheroni constant were established in the recent past and we mention here the sequences

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right) \qquad ([3])$$

and

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$$T_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2} + \frac{1}{24n}\right) \qquad ([11]),$$

satisfying for every integer $n \geq 1$,

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}$$

and

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}. \tag{1.1}$$

2. **The Results**

Inequalities (1.1) produce approximations formulas for γ of the form

$$\gamma \approx T_n + \frac{a}{(n+b)^3} \tag{2.1}$$

where $a = \frac{1}{48}$ and $b = 0$, or $b = 1$.

We improve the bounds for $\gamma - T_n$ in the following way, which show us that the best approximation of type (2.1) is obtained for $a = \frac{1}{48}$ and $b = \frac{83}{360}$.

Theorem 2.1. For every integer $n \geq 1$, it holds

$$\frac{1}{48\left(n + \frac{83}{360} + \frac{4909}{64800n}\right)^3} < \gamma - T_n < \frac{1}{48\left(n + \frac{83}{360}\right)^3}.$$

Proof. The sequences

$$a_n = \gamma - T_n - \frac{1}{48\left(n + \frac{83}{360}\right)^3}, \quad b_n = \gamma - T_n - \frac{1}{48\left(n + \frac{83}{360} + \frac{4909}{64800n}\right)^3}$$

converge to zero and we prove that a_n is strictly increasing, while b_n is strictly decreasing.

In this sense, we have $a_{n+1} - a_n = f(n)$, $b_{n+1} - b_n = g(n)$, where

$$f(x) = -\frac{1}{x+1} + \ln\left(x + 1 + \frac{1}{2} + \frac{1}{24(x+1)}\right) - \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right)$$
$$+ \frac{1}{48\left(x + \frac{83}{360}\right)^3} - \frac{1}{48\left(x + 1 + \frac{83}{360}\right)^3}$$

and

$$g(x) = -\frac{1}{x+1} + \ln\left(x + 1 + \frac{1}{2} + \frac{1}{24(x+1)}\right) - \ln\left(x + \frac{1}{2} + \frac{1}{24x}\right)$$

$$+\frac{1}{48\left(x+\frac{83}{360}+\frac{4909}{64800x}\right)^3}-\frac{1}{48\left(x+1+\frac{83}{360}+\frac{4909}{64800(x+1)}\right)^3}.$$

We have $f'(x)=-\frac{P(x)}{x(x+1)^2(24x^2+12x+1)(24x^2+60x+37)(360x+83)^4(360x+443)^4}$, where $P(x)>0$, for every $x\geq 1$, since
$P(x)=$
$= 5672\,643\,926\,062\,610\,449\,496\,158\,(x-1) + 12\,732\,497\,919\,066\,452\,434\,047\,216\,(x-1)^2$
$+16\,068\,904\,633\,211\,798\,843\,573\,760\,(x-1)^3 + 12\,479\,859\,545\,691\,357\,275\,443\,200\,(x-1)^4$
$+6112\,058\,983\,164\,612\,569\,088\,000\,(x-1)^5 + 1844\,735\,671\,639\,207\,526\,400\,000\,(x-1)^6$
$+313\,941\,813\,277\,483\,008\,000\,000\,(x-1)^7 + 23\,081\,380\,892\,835\,840\,000\,000\,(x-1)^8$
$+1087\,307\,210\,044\,441\,359\,719\,625$ Then

$$g'(x)=\frac{Q(x)}{x(x+1)^2\,(24x^2+12x+1)\,(24x^2+60x+37)\,U^4(x)}\ ,$$

where $U(x)=(64800x^2+144540x+84649)(64800x^2+14940x+4909)$ and

$$\begin{aligned}Q(x) \;=\;& 26\,771\,478\,859\,493\,011\,892\,076\,242\,676\,092\,371\,394x\\
&+348\,885\,068\,818\,902\,937\,863\,469\,966\,659\,757\,801\,104x^2\\
&+3265\,904\,802\,976\,624\,394\,962\,075\,841\,092\,287\,361\,920x^3\\
&+24\,284\,886\,346\,727\,756\,010\,131\,506\,523\,992\,132\,185\,600x^4\\
&+138\,041\,160\,473\,871\,425\,526\,166\,462\,225\,060\,582\,272\,000x^5\\
&+567\,885\,725\,474\,982\,630\,218\,262\,119\,830\,609\,566\,720\,000x^6\\
&+1660\,928\,443\,478\,105\,240\,966\,737\,358\,122\,783\,027\,200\,000x^7\\
&+3457\,404\,001\,342\,515\,820\,083\,602\,245\,533\,877\,248\,000\,000x^8\\
&+5141\,962\,753\,000\,217\,911\,708\,434\,748\,133\,591\,040\,000\,000x^9\\
&+5456\,518\,497\,659\,377\,824\,777\,025\,648\,492\,953\,600\,000\,000x^{10}\\
&+4084\,620\,321\,401\,962\,876\,540\,481\,171\,816\,448\,000\,000\,000x^{11}\\
&+2098\,735\,133\,357\,560\,633\,848\,230\,846\,791\,680\,000\,000\,000x^{12}\\
&+700\,781\,633\,710\,803\,256\,330\,998\,566\,092\,800\,000\,000\,000x^{13}\\
&+135\,940\,882\,712\,765\,931\,653\,645\,205\,504\,000\,000\,000\,000x^{14}\\
&+11\,492\,194\,762\,471\,633\,096\,204\,615\,680\,000\,000\,000\,000x^{15}\\
&+1103\,217\,596\,360\,716\,040\,009\,228\,783\,034\,478\,357.\end{aligned}$$

Now f is strictly decreasing, g is strictly increasing on $[1,\infty)$, with $f(\infty)=g(\infty)=0$, so $f(x)>0$ and $g(x)<0$ for every $x\in[1,\infty)$.

Hence a_n is strictly increasing, and b_n is strictly decreasing and the conclusion follows.■

Finally we propose the following much better approximation formula than (2.1)

$$\gamma \approx T_n+\frac{1}{48\left(n+\frac{83}{360}+\frac{4909}{64800n}-\frac{11976997}{489888000n^2}\right)^3},$$

which follows from

Theorem 2.2. For every integer $n\geq 1$, it holds

$$\frac{1}{48\left(n+\frac{83}{360}+\frac{4909}{64800n}-\frac{11976997}{489888000n^2}\right)^3}<$$

$$\gamma-T_n<\frac{1}{48\left(n+\frac{83}{360}+\frac{4909}{64800n}-\frac{11976997}{489888000n^2}-\frac{2864050703}{70\ 543872000n^3}\right)^3}.$$

The proof is similar with the previous one, but the computations become more difficult, so we omit it for sake of simplicity.

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