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A CARISTI SELECTION THEOREM FOR A MULTIFUNCTION SATISFYING AN IMPLICIT CONTRACTIVE CONDITION OF LATIF-BEG TYPE

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Abstract. We prove a general Caristi selection theorem for a multifunction which satisfies an implicit contractive condition of Latif-Beg type which generalize the result from [13, Theorem 2.1].

1. INTRODUCTION

J. Caristi's fixed point theorem [1] states that each operator f from a complete metric space (X,d) into itself satisfying the condition:
there exists a lower semi-continuous mapping $g : X \to R_+$ such that

$$d(x, f(x)) \leq g(x) - g(f(x)),$$

for each $x \in X$, has at least a fixed point.
For the multivalued functions, there exists several result involving multivalued Caristi type conditions (see for example [2],[5],[6]). There are several extensions and generalizations of these important principle of nonlinear analysis (see for example the references listed in [9],[11]).
Let (X,d) be a metric space and P(X) the space of all nonempty subsets of X. We denote by Pcl(X) the space of all nonempty closed sets of X. If X,Y are non-empty sets and $F : X \to P(Y)$ is a multifunction, then a selection of F is a single valued function $f : X \to Y$ such that $f(x) \in F(x)$, for each $x \in X$.

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First result for Caristi selection was established by J.R.Jachymski [3] for a multifunction with closed valued. Other results are recently obtained in [9]-[13]. In a recent paper [13] S$\hat{\imath}$ntămărean proved a selection theorem for a multivalued operator which satisfies a contractive condition of Latif-Beg type [4].

Theorem 1.1 [13] Let (X, d) be a metric space and $T : (X, d) \to P(X)$ a multivalued operator with the property that there exist $a_1, ..., a_5 \in R_+$, with $a_1 + a_2 + a_3 + 2a_4 < 1$ such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that

$$d(u_x, u_y) \leq a_1 d(x, y) + a_2 d(x, u_x) + a_3 d(y, u_y) + a_4 d(x, u_y) + a_5 d(y, u_x)$$

Then there exist $t : X \to X$ a selection of T and a functional $g : X \to R_+$ so that

$$d(x, t(x)) \leq g(x) - g(t(x))$$

for all $x \in X$.

We denote $D(x, A) = Inf\{d(x, y) : y \in A\}$.

In [7], [8] is introduced the study of fixed points for mappings satisfying implicit relations. The purpose of this paper is to prove a general Caristi selection theorem for multifunction satisfying and implicit relations of Latif-Beg type generalizing Theorem 1.1.

2. IMPLICIT RELATIONS

Let $\mathcal{F}$ be the set of all real function $F(t_1, ..., t_6) : R_+^6 \to R$ satisfying the following conditions:

(F1): F is non-increasing in variable t_5;

(F2): there exists $h \in (0, 1)$ such that for every $u \geq 0$, $v \geq 0$ with $F(u, v, v, u, u + v, 0) \leq 0$ we have $u \leq hv$.

Example 2.1 $F(t_1, ..., t_6) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6$, where $a_1, ..., a_5 \in R_+$, $a_1 + a_2 + a_4 > 0$ and $a_1 + a_2 + a_3 + 2a_4 < 1$.

(F1): Obviously.

(F2): Let $F(u, v, v, u, u + v, 0) = u - a_1 v - a_2 v - a_3 v - a_4(u + v) \leq 0$. Then $u \leq hv$, where $0 < h = \frac{a_1 + a_2 + a_4}{1 - a_3 - a_4} < 1$.

Example 2.2 $F(t_1, ..., t_6) = t_1 - kmax\left\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\right\}$, where $k \in (0, 1)$.

(F1): Obviously.

(F2): Let $F(u, v, v, u, u + v, 0) = u - kmax\left\{u, v, \frac{1}{2}(u + v)\right\} \leq 0$. If

$u > 0$ and $u \geq v$, then $u(1-k) \leq 0$, a contradiction. Hence $u < v$ which implies $u \leq hv$, where $0 < h = k < 1$. If $u = 0$, then $u \leq hv$.
Example 2.3 $F(t_1, ..., t_6) = t_1^2 + at_1t_2 - bt_3t_4 - ct_5t_6$, where $a, c \geq 0$ and $0 < b < 1$.
(F1): Obviously.
(F2): Let $F(u, v, v, u, u+v, 0) = u^2 + auv - buv \leq 0$ which implies $u^2 - buv \leq 0$. If $u > 0$, then $u \leq hv$, where $0 < h = b < 1$. If $u = 0$, then $u \leq hv$.

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Example 2.4. $F(t_1, ..., t_6) = max\{t_1, t_2, t_4\} - min\{t_1, t_5, t_6\} - qt_3$, where $0 < q < 1$.
(F_1): Obviously.
(F_2): Let $F(u, v, v, u, u+v, 0) = max\{u, v\} - min\{u, u+v, 0\} - qv \leq 0$. If $u > 0$ and $u > v$, then $u(1-q) \leq 0$ a contradiction. Hence $u < v$ and $u \leq hv$ where $0 < h = q < 1$. If $u = 0$, then $u \leq hv$.

Example 2.5. $F(t_1, ..., t_6) = t_1^2 + \frac{t_1}{1+t_5t_6} - (at_2^2 + bt_3^2 + ct_4^2)$, where $a, b, c \geq 0$, $a + b > 0$ and $a + b + c < 1$.
(F_1): Obviously.
(F_2): Let $F(u, v, v, u, u+v, 0) = u^2 + u - (av^2 + bv^2 + cu^2) \leq 0$ which implies $u^2 - (av^2 + bv^2 + cu^2) \leq 0$. Hence $u \leq hv$, where $0 < h = \sqrt{\frac{a+b}{1-c}} < 1$

3. MAIN RESULT

Theorem 3.1. Let (X, d) be a metric surface and $T : (X, d) \to Pcl(X)$ a multifunction such that for each $x \in X$, any $u_x \in T(x)$ and for all $y \in X$, there exists $u_y \in T(y)$ so that $F(d(u_x, u_y), d(x, y), d(x, u_x), d(y, u_y), d(x, u_y), d(y, u_x)) \leq 0$, where $F \in \mathcal{F}$. Then there exists $t : X \to X$ a Caristi selection of F.
Proof: Let $\varepsilon = \frac{1-h}{2} < 1$ and $g(x) = \frac{1}{\varepsilon} D(x, T(x))$. Then $\varepsilon + h = \frac{1+h}{2} < 1$
Since $\frac{1}{\varepsilon + h} > 1$, for each $x \in X$ we can choose $t(x) \in T(x)$ such that

$$d(x, t(x)) \leq \frac{1}{\varepsilon + h} D(x, T(x)).$$

For $x \in X$, taking into account that $t(x) \in T(x)$, there exists $u_{t(x)} \in T(t(x))$ such that

$F(d(t(x), u_{t(x)}), d(x, t(x)), d(x, t(x)), d(t(x), u_{t(x)}), d(x, u_{t(x)}), d(t(x), t(x)))$ ≤ 0
which implies by $(F1)$ that
$F(d(t(x), u_{t(x)}), d(x, t(x)), d(x, t(x)), d(t(x), u_{t(x)}), d(x, t(x)) + d(t(x), u_{t(x)}), 0) \leq 0$
Since $F \in \mathcal{F}$, we have by $(F2)$ that

$$d(t(x), u_{t(x)}) \leq hd(x, t(x)).$$

Since $D(t(x), T(t(x))) \leq d(t(x), u_t(x))$ we obtain

$$D(t(x), T(t(x))) \leq hd(x, t(x)).$$

We prove now that $t(x)$ is a Caristi type operator. Indeed, for each $x \in X$ we have

$d(x, t(x)) = \frac{1}{\varepsilon} \cdot [(\varepsilon + hd(x, t(x))] \leq \frac{1}{\varepsilon} \cdot [D(x, T(x)) - D(t(x), T(t(x))] = g(x) - g(t(x))$.

Remark 3.1 If the multifunction $T : X \to Pcl(X)$ from Theorem 3.1 is upper semi-continuous, then the functional $g : X \to R_+$ is lower semi-continuous.

Corollary 3.1 Theorem 1.1
Proof The proof it follows from Theorem 3.1 and Example 2.1.

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