"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 20 (2010), No. 2, 61 - 66

TORSEFORMING VECTOR FIELDS IN A 3-DIMENSIONAL PARA SASAKIAN MANIFOLD

D.G. PRAKASHA

Abstract. The object of the present paper is to study a torseforming vector field in a 3-dimensional para-Sasakian manifold. Here we prove that the torseforming vector field in a 3-dimensional para-Sasakian manifold is a concircular vector field.

1. Introduction

The contact manifolds are odd dimensional manifolds with specified contact structure. One can obtain different structures like Sasakian, Quasi-Sasakian, para-Sasakian, Kenmotsu and trans-Sasakian by providing additional conditions. The geometry of these manifolds is extensively studied by [1] to [11]. Now the torseforming vector field in a Riemannian manifold has been introduced by K. Yano in 1994 [13]. In the present paper we consider a torseforming vector field in a 3-dimensional para-Sasakian manifold and have shown that such a vector field is a concircular vector field.

2. Preliminaries

An n-dimensional differentiable manifold M^n is called an almost para-contact manifold if it admits an almost para-contact structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ξ , 1-form η and compatible Riemannian metric g satisfying [11]

Keywords and phrases: Torseforming vector field, Concircular vector field, para-Sasakian manifold.
(2000)Mathematics Subject Classification: 53D10, 53D15.

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta\phi = 0 \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$g(X,\xi) = \eta(X), \quad g(X,\phi Y) = -g(\phi X, Y). \tag{2.3}$$

An almost para-contact Riemannian manifold is called a Para-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \forall X, Y \in TM \tag{2.4}$$

where ∇ is Levi-Civita connection of the Riemannian metric. From the above equation it follows that

$$\begin{aligned} \nabla_X \xi &= \phi X, \\ (\nabla_X \eta)(Y) &= g(\phi X, Y) \end{aligned} \tag{2.5}$$

$\forall X \in TM$. In an n-dimensional para-Sasakian manifold M, the curvature tensor R, the Ricci tensor S, and the Ricci operator Q satisfies

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$R(\xi,X)\xi = X - \eta(X)\xi, \tag{2.7}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.8}$$

$$Q\xi = -(n-1)\xi \tag{2.9}$$

$\forall X, Y \in TM$. Since the conformal curvature tensor vanishes in a 3-dimensional Riemannian manifold, therefore we get

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &\quad -\frac{r}{2}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.10}$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$ and r is the scalar curvature of the manifold.

Using (2.10)the expressions for Ricci tensor and the scalar curvature in a 3-dimensional para-Sasakian manifold are given respectively by [1]

$$S(X,Y) = \left[\frac{r}{2} + 1\right] g(X,Y) - \left[\frac{r}{2} + 3\right] \eta(X)\eta(Y) \tag{2.11}$$

and

$$\begin{aligned} R(X,Y)Z &= \left[\frac{r}{2} + 2\right] [g(Y,Z)X - g(X,Z)Y] \\ &\quad - \left[\frac{r}{2} + 3\right] [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi] \\ &\quad - \left[\frac{r}{2} + 3\right] [\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{2.12}$$

3. Torseforming Vector Field In A 3-Dimensional para-Sasakian Manifold

Definition 3.1. *A vector field ρ defined by $g(X,\rho) = \omega(X)$ for any vector field X is said to be a torseforming vector field(*[12],[13]*) if*

$$(\nabla_X \omega)(Y) = kg(X,Y) + \pi(X)\omega(Y) \tag{3.1}$$

where k is a non-zero scalar and π is a non-zero 1-form.

We consider a unit torseforming vector field $\widetilde{\rho}$ corresponding to the vector field ρ.Suppose $g(X,\widetilde{\rho}) = T(X)$.then

$$T(X) = \frac{\omega(X)}{\sqrt{\omega(\rho)}}. \tag{3.2}$$

From (3.1) we get

$$\frac{(\nabla_X \omega)(Y)}{\sqrt{\omega(\rho)}} = \frac{k}{\sqrt{\omega(\rho)}} g(X,Y) + \frac{\pi}{\sqrt{\omega(\rho)}}\omega(Y). \tag{3.3}$$

Using (3.2) in the above, we obtain

$$(\nabla_X T)(Y) = \lambda g(X,Y) + \pi(X)T(Y) \tag{3.4}$$

where $\lambda = \frac{k}{\sqrt{\omega(\rho)}}$.
Putting $Y = \widetilde{\rho}$ in (3.4),we obtain

$$(\nabla_X T)(\widetilde{\rho}) = \lambda g(X,\widetilde{\rho}) + \pi(X)T(\widetilde{\rho}). \tag{3.5}$$

As $T(\widetilde{\rho}) = g(\widetilde{\rho},\widetilde{\rho}) = 1$,equation (3.5) reduces to

$$\pi(X) = -\lambda T(X) \tag{3.6}$$

and hence (3.4) can be written in the form

$$(\nabla_X T)(Y) = \lambda[g(X,Y) - T(X)T(Y)] \tag{3.7}$$

which implies T is closed.
Taking covariant differentiation of (3.7) and using Ricci identity we get

$$\begin{aligned} -T(R(X,Y)Z) &= (X\lambda)[g(Y,Z) - T(Y)T(Z)] - (Y\lambda)[g(X,Z) - T(X)T(Z)] \\ &\quad + \lambda^2[g(Y,Z)T(X) - g(X,Z)T(Y)] \end{aligned} \tag{3.8}$$

Putting $Z = \xi$ in (3.8) and using (2.6), we obtain

$$\begin{aligned} T[\eta(Y)X - \eta(X)Y] &= (X\lambda)[\eta(Y) - T(Y)T(\xi)] \\ &\quad -(Y\lambda)[\eta(X) - T(X)T(\xi)] \\ &\quad +\lambda^2[\eta(Y)T(X) - \eta(X)T(Y)]. \end{aligned} \tag{3.9}$$

Since $T(\xi) = g(\xi, \widetilde{\rho}) = \eta(\widetilde{\rho})$, (3.9) reduces to

$$\begin{aligned}[\lambda^2 - 1][\eta(Y)T(X) - \eta(X)T(Y)] \\ +[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] \\ +\eta(\widetilde{\rho})[(Y\lambda)T(X) - (X\lambda)T(Y)] = 0.\end{aligned} \tag{3.10}$$

Putting $X = \widetilde{\rho}$ in (3.10) and as $T(\widetilde{\rho}) = g(\widetilde{\rho}, \widetilde{\rho}) = 1$, we get

$$[\lambda^2 - 1 + (\widetilde{\rho}\lambda)][\eta(Y) - \eta(\widetilde{\rho})T(Y)] = 0. \tag{3.11}$$

Thus we have the following:

Lemma 3.1. *If a 3-dimensional para-Sasakian manifold admits a torseforming vector field, then the following cases occur:*

$$\eta(Y) = \eta(\widetilde{\rho})T(Y)] \tag{3.12}$$

$$\widetilde{\rho}\lambda = 1 - \lambda^2. \tag{3.13}$$

We derive the following results from the Lemma 3.1.
Now $Y = \xi$ in (3.12) implies $1 = (\eta(\widetilde{\rho}))^2$ and thus $\eta(\widetilde{\rho}) = \pm 1$.So

$$\eta(Y) = \pm T(Y). \tag{3.14}$$

Using (3.14) in (2.5) and in view of (3.7) we have

$$g(\phi X, Y) = \pm\lambda[g(X, Y) - T(X)T(Y)].$$

This implies that $\lambda = \pm C$, where C is constant(say). Hence (3.6) reduces to

$$\pi(X) = \pm C T(X). \tag{3.15}$$

Since T is closed, π is also closed. Hence we can state:

Lemma 3.2. *The equation (3.12) implies that the vector field $\widetilde{\rho}$ is a concircular vector field.*

We next assume the case (3.13). Then

$$\eta(Y) - \eta(\widetilde{\rho})T(Y) \neq 0 \tag{3.16}$$

From (3.8), we get

$$-T(QX) = (X\lambda) + (\widetilde{\rho}\lambda)T(X) + 2\lambda^2 T(X). \tag{3.17}$$

where $g(QX, Y) = S(X, Y)$.
Put $X = \xi$ in (3.17) and using (2.9), we obtain

$$\xi\lambda = -\eta(\widetilde{\rho})[\lambda^2 - 1]. \tag{3.18}$$

Putting $Y = \xi$ in (3.10) and by virtue of (3.18) and $T(\xi) = \eta(\widetilde{\rho})$ we get

$$X\lambda = -[\lambda^2 - 1]T(X). \tag{3.19}$$

From (3.19) it follows that

$$Y\pi(X) = -[(Y\lambda)T(X) + \lambda(YT(X))].$$

Using (3.19) in the above equation, we get

$$Y\pi(X) = -[-(\lambda^2 - 1)T(Y)T(X) + \lambda(YT(X))]. \tag{3.20}$$

Also

$$X\pi(Y) = -[-(\lambda^2 - 1)T(X)T(Y) + \lambda(XT(Y))] \tag{3.21}$$

and

$$\pi([X, Y]) = -\lambda T([X, Y]). \tag{3.22}$$

From (3.20), (3.21) and (3.22), we obtain

$$d\pi(X, Y) = -\lambda[(dT)(X, Y)]. \tag{3.23}$$

Since T is closed, π is also closed. Thus we have

Lemma 3.3. *The equation (3.13) implies that the vector field $\widetilde{\rho}$ is a concircular vector field.*

When $\widetilde{\rho}$ is a concircular vector field, putting $\widetilde{\rho} = f\rho$, where f is a scalar function, we can easily verify that $\widetilde{\rho}$ is a concircular one. Again since $\beta \neq 0$, then $\alpha \neq 0$ from (3.7). Thus $\widetilde{\rho}$ is a proper concircular vector field [13]. Thus from Lemma 3.2 and Lemma 3.3, we can state the following:

Theorem 3.1. *A torseforming vector field in a 3-dimensional para-Sasakian manifold is a concircular vector field.*

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.D.G. Prakasha
Department of Mathematics,
Karnatak University, Dharwad - 580 003, INDIA.
E-mail : prakashadg@gmail.com