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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 20 (2010), No. 2, 67 - 78

ON THE CURVATURE TENSOR FIELD ASSOCIATED TO HOMOGENEOUS METRICAL STRUCTURES ON T^2M

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Abstract. The main goal of this paper is to compute the curvature tensor field of the geometrical model determined by the second order prolongation of a Riemannian space endowed with certain homogeneous structure.

1. Introduction

The generalized Lagrange geometry of second order was defined and studied by R. Miron [6, 7] and represents the geometry of generalized Lagrangians modeled on the second order tangent bundle (T^2M, p, M). These spaces are useful in the study of the geometry of higher-order Lagrangians [6, 7], for the prolongation of Riemannian, Finslerian and Lagrangian structures [6, 7], for the study of stationary curves [9], and for the development of a gauge theory having the second order tangent bundle as the geometrical model [3, 10, 12]. The term "homogeneity" has been discussed in Miron's papers [4, 5] where new geometrical models on Riemannian spaces and on Finslerian spaces are also introduced, respectively. In [12, 14] an extension of Miron's theory of homogeneity to the second order tangent bundle is presented.

Keywords and phrases: Second order tangent bundle, homogeneous metrical structure, torsion field, curvature field.
(2000)Mathematics Subject Classification:53B40, 53C60, 58B20.

This paper is a continuation of our previous works [11, 14]. The curvature tensor field is a powerful tool in differential geometry and its applications. The aim of this paper is to study the curvature tensor field within the framework of the second order tangent bundle endowed with the homogeneous metrical structure. The basic concepts and the notations are the same with those from [11, 14].

2. Homogeneous Sasaki lift of a (M,γ) Riemannian space to the manifold T^2M

Consider $\mathfrak{R}^n = (M,\gamma)$ a Riemannian space generated by a real, differentiable, n–dimensional manifold M and by a Riemannian metric γ on M, given by the local components $\big(\gamma_{ij}(x)\big)$, $x \in U \subset M$. It is possible to extend γ to $p^{-1}(U) \subset E = T^2M$ by:

$$(1) \qquad \big(\gamma_{ij} \circ p\big)(u) = \gamma_{ij}(x), \quad u \in p^{-1}(U), \quad p(u) = x.$$

In this case $\gamma_{ij} \circ p$ are the local components of a tensor field on E. Usually, we write these local components with γ_{ij} as well. Furthermore, with $\gamma^k_{ij}(x)$ we will denote the Christoffel symbols of the second species of the metric γ and with $r^k_{ijh}(x)$ we will denote the local components of the curvature tensor field of the metric γ. It is possible to introduce on E a nonlinear connection determined only by this metric, cf. [6]. Moreover, the coefficients of connection are determined by the following relations (see also [11]):

$$(2) \qquad \overset{(0)}{N}{}^{i}_{(1)j}\big(x, y^{(1)}\big) = \gamma^i_{j0},$$

$$(3) \qquad \overset{(0)}{N}{}^{i}_{(2)j}\big(x, y^{(1)}, y^{(2)}\big) = \frac{1}{2}\left(\frac{\partial \gamma^i_{j0}}{\partial x^p} y^{(1)p} + \gamma^i_{0m} \cdot \gamma^m_{j0}\right) + \gamma^i_{j\bar{0}},$$

where the notation "0" stands for the contraction by $\big(y^{(1)}\big)$ and the notation "$\bar{0}$" stands for the contraction by $\big(y^{(2)}\big)$. In the next section, we will partially avoid this particular nonlinear connection and we will use one, more general, determined in the following result.

Theorem 2.1. *If $\overset{(0)}{N}{}^{i}_{(1)j}$ and $\overset{(0)}{N}{}^{i}_{(2)j}$ are the local components of the nonlinear connection determined only by Riemannian metric γ, and X^i_j and Y^i_j are the local components of any d- tensor field of type $(1,1)$ on E, then the functions*

$$(4) \qquad N^i_{(1)j} = \overset{(0)}{N}{}^{i}_{(1)j} + X^i_j,$$

$$(5) \qquad N^i_{(2)j} = \overset{(0)}{N}{}^{i}_{(2)j} + \big(Y^i_m - \gamma^i_{m0}\big) X^m_j,$$

are the local components of a nonlinear connection N on E.

The nonlinear connection N assures the existence of a basis $\left(d_k, d_k^{(1)}, d_k^{(2)}\right)$ adapted to the tangent space T_uE. The vector fields of the adapted basis are defined with the help of the following relations:

$$d_k = \frac{\partial}{\partial x^k} - N^i_{(1)k}\frac{\partial}{\partial y^{(1)i}} - N^i_{(2)k}\frac{\partial}{\partial y^{(2)i}}, \tag{6}$$

$$d_k^{(1)} = \frac{\partial}{\partial y^{(1)i}} - N^i_{(1)k}\frac{\partial}{\partial y^{(2)i}}, \quad d_k^{(2)} = \frac{\partial}{\partial y^{(2)k}}. \tag{7}$$

For further developments, we need the following result.

Theorem 2.2. *The Lie brackets of the vector fields of the adapted basis $\left(d_k, d_k^{(1)}, d_k^{(2)}\right)$ are given by:*

$$[d_j, d_k] = R^i_{(01)jk}\cdot d_i^{(1)} + R^i_{(02)jk}\cdot d_i^{(2)}, \tag{8}$$

$$\left[d_j, d_k^{(1)}\right] = B^i_{(11)jk}\cdot d_i^{(1)} + B^i_{(12)jk}\cdot d_i^{(2)}, \tag{9}$$

$$\left[d_j, d_k^{(2)}\right] = B^i_{(21)jk}\cdot d_i^{(1)} + B^i_{(22)jk}\cdot d_i^{(2)}, \tag{10}$$

$$\left[d_j^{(1)}, d_k^{(1)}\right] = R^i_{(12)jk}\cdot d_i^{(2)}, \quad \left[d_j^{(1)}, d_k^{(2)}\right] = B^i_{(21)jk}\cdot d_i^{(2)}, \tag{11}$$

where:

$$R^i_{(01)jk} = R^{(0)i}_{(01)jk} + X^i_{jk}, \quad R^i_{(02)jk} = R^{(0)i}_{(02)jk} + (XY)^i_{jk}, \tag{12}$$

$$R^i_{(12)jk} = R^{(0)i}_{(12)jk} + X^{(1)i}_{[jk]}, \tag{13}$$

$$B^i_{(11)jk} = B^{(0)i}_{(11)jk} + X^{(1)i}_{jk}, \quad B^i_{(12)jk} = B^{(0)i}_{(12)jk} + (XY)^i_{(12)jk}, \tag{14}$$

$$B^i_{(21)jk} = B^{(0)i}_{(21)jk} + X^{(2)i}_{jk}, \quad B^i_{(22)jk} = B^{(0)i}_{(22)jk} + (XY)^i_{(22)jk}, \tag{15}$$

with the following notations:

$$X^i_{jk} = \frac{dX^i_j}{dx^k} - \frac{dX^i_k}{dx^j}, \quad X^{(1)i}_{jk} = \frac{dX^i_j}{dy^{(1)k}}, \quad X^{(2)i}_{jk} = \frac{dX^i_j}{dy^{(2)k}}, \tag{16}$$

$$(XY)^i_{jk} = N^{(0)i}_{(1)m} \cdot X^m_{jk} + X^i_m \cdot R^{(0)m}_{(01)jk} + X^i_m \cdot X^m_{jk} + \left(\frac{d(Y^i_q \cdot X^q_j)}{dx^k} - \frac{d(Y^i_q \cdot X^q_k)}{dx^j} \right) - \left(\frac{d(\gamma^i_{q0} \cdot X^q_j)}{dx^k} - \frac{d(\gamma^i_{q0} \cdot X^q_k)}{dx^j} \right) \tag{17}$$

$$(XY)^i_{(12)jk} = N^{(0)i}_{(1)m} \cdot X^{(1)m}_{jk} + X^i_m \cdot B^{(0)m}_{(11)jk} + X^i_m \cdot X^{(1)m}_{jk} + \frac{dY^i_j}{dy^{(1)k}} - \frac{dY^i_k}{dy^{(1)j}}, \tag{18}$$

$$(XY)^i_{(22)jk} = N^{(0)i}_{(1)m} \cdot X^{(2)m}_{jk} + X^i_m \cdot B^{(0)m}_{(21)jk} + X^i_m \cdot X^{(2)m}_{jk} + \frac{dY^i_j}{dy^{(2)k}} - \frac{dY^i_k}{dy^{(2)j}}, \tag{19}$$

$$X^{(1)i}_{[jk]} = X^{(1)i}_{jk} - X^{(1)i}_{kj} \tag{20}$$

It is known from [6] that the pair $Prol^{(2)}\mathbb{R}^n = \left(\tilde{T}^2M, G\right)$, where:

$$G = \gamma_{ij}(x) \cdot dx^i \otimes dx^j + \gamma_{ij}(x) \cdot dy^{(1)i} \otimes dy^{(1)j} + \gamma_{ij}(x) \cdot dy^{(2)i} \otimes dy^{(2)j}, \tag{21}$$

is a Riemannian space of dimension $3n$, with the metrical structure G depending only on the Riemannian structure which is apriori given on the Riemannian space $(\mathbb{R}^n, \gamma)$. We say that G is the Sasaki lift of the Riemannian structure γ. Define the omotety $h_t : (x, y^1, y^2) \to (x, ty^1, t^2y^2)$, $t \in \mathbb{R}\backslash\{0\}$ on the fibres of T^2M. Mention that G is transformed according to:

$$G \circ h_t \left(x, y^{(1)}, y^{(2)}\right) = \gamma_{ij}(x) \cdot dx^i \otimes dx^j + t^2 \cdot \gamma_{ij}(x) \cdot dy^{(1)i} \otimes dy^{(1)j} + t^4 \cdot \gamma_{ij}(x) \cdot dy^{(2)i} \otimes dy^{(2)j}. \tag{22}$$

The above remark makes us affirm that the Sasaki lift G is non–homogeneous on the space T^2M. In the following part we concentrate upon a new lift of Sasaki type, called the homogeneous Sasaki lift and denoted by $G^{(0)}$:

$$G \circ h_t\left(x, y^{(1)}, y^{(2)}\right) = \gamma_{ij}(x) \cdot dx^i \otimes dx^j + \frac{1}{F^2} \cdot \gamma_{ij}(x) \cdot dy^{(1)i} \otimes dy^{(1)j} + \frac{1}{F^4} \cdot \gamma_{ij}(x) \cdot dy^{(2)i} \otimes dy^{(2)j}. \tag{23}$$

where $F^2 = \gamma_{ij} \cdot y^{(1)i} y^{(1)j}$. Clearly, the following properties hold true:

- The pair $\left(TT^2M, G^{(0)}\right)$ is a Riemannian space;
- The metric $G^{(0)}$ depends only on the Riemannian metric $\gamma(x)$;
- The distributions N, V_1, V_2 are orthogonal each other with respect to $G^{(0)}$.

Definition 2.3. *A linear connection D on $\widetilde{T}^2M$ is said to be the (0)–metrical connection with respect to $G^{(0)}$ if $DG^{(0)} = 0$ and D preserves by parallelism the horizontal distribution N.*

With respect to the adapted basis $\left(d_k, d_k^{(1)}, d_k^{(2)}\right)$, any linear connection D on E can be represented as follows

$$D_{d_k} d_j = L_{jk}^{(H)i} \cdot d_i + L_{jk}^{(1)i} \cdot d_i^{(1)} + L_{jk}^{(2)i} \cdot d_i^{(2)} \tag{24}$$

$$D_{d_k} d_j^{(1)} = L_{jk}^{(3)i} \cdot d_i + L_{jk}^{(v_1)i} \cdot d_i^{(1)} + L_{jk}^{(4)i} \cdot d_i^{(2)} \tag{25}$$

$$D_{d_k} d_j^{(2)} = L_{jk}^{(5)i} \cdot d_i + L_{jk}^{(6)i} \cdot d_i^{(1)} + L_{jk}^{(v_2)i} \cdot d_i^{(2)} \tag{26}$$

$$D_{d_k^{(1)}} d_j = F_{jk}^{(H)i} \cdot d_i + F_{jk}^{(1)i} \cdot d_i^{(1)} + F_{jk}^{(2)i} \cdot d_i^{(2)} \tag{27}$$

$$D_{d_k^{(1)}} d_j^{(1)} = F_{jk}^{(3)i} \cdot d_i + F_{jk}^{(v_1)i} \cdot d_i^{(1)} + F_{jk}^{(4)i} \cdot d_i^{(2)} \tag{28}$$

$$D_{d_k^{(1)}} d_j^{(2)} = F_{jk}^{(5)i} \cdot d_i + F_{jk}^{(6)i} \cdot d_i^{(1)} + F_{jk}^{(v_2)i} \cdot d_i^{(2)} \tag{29}$$

$$D_{d_k^{(2)}} d_j = C_{jk}^{(H)i} \cdot d_i + C_{jk}^{(1)i} \cdot d_i^{(1)} + C_{jk}^{(2)i} \cdot d_i^{(2)} \tag{30}$$

$$D_{d_k^{(2)}} d_j^{(1)} = C_{jk}^{(3)i} \cdot d_i + C_{jk}^{(v_1)i} \cdot d_i^{(1)} + C_{jk}^{(4)i} \cdot d_i^{(2)} \tag{31}$$

$$D_{d_k^{(2)}} d_j^{(2)} = C_{jk}^{(5)i} \cdot d_i + C_{jk}^{(6)i} \cdot d_i^{(1)} + C_{jk}^{(v_2)i} \cdot d_i^{(2)} \tag{32}$$

The set consisting of the functions $L_{jk}^{(H)i}$,..., $C_{jk}^{(v_2)i}$ represents the set of the coefficients of the linear connection D. Concerning the notion of (0)–metrical connection, there can be proved the following result.

Theorem 2.4. *There exist* (0)*–metrical connections* D *on* $\widetilde{T}^2M$*, which depend only on the Riemannian tensor field* γ*. One of these connections has its coefficients given by:*

$$L_{jk}^{(1)i} = L_{jk}^{(2)i} = L_{jk}^{(3)i} = L_{jk}^{(4)i} = L_{jk}^{(5)i} = L_{jk}^{(6)i} = 0, \tag{33}$$

$$F_{jk}^{(1)i} = F_{jk}^{(2)i} = F_{jk}^{(3)i} = F_{jk}^{(4)i} = F_{jk}^{(5)i} = F_{jk}^{(6)i} = 0, \tag{34}$$

$$C_{jk}^{(1)i} = C_{jk}^{(2)i} = C_{jk}^{(3)i} = C_{jk}^{(4)i} = C_{jk}^{(5)i} = C_{jk}^{(6)i} = 0, \tag{35}$$

$$L_{jk}^{(H)i} = \gamma_{jk}^i, \quad L_{jk}^{(v_1)i} = \gamma_{jk}^i + \frac{1}{F^2}\theta_{jk}^i, \quad L_{jk}^{(v_2)i} = \gamma_{jk}^i + \frac{2}{F^2}\theta_{jk}^i, \tag{36}$$

$$F_{jk}^{(H)i} = 0, \quad F_{jk}^{(v_1)i} = -\frac{1}{F^2}\Lambda_{jk}^i, \quad F_{jk}^{(v_2)i} = -\frac{2}{F^2}\Lambda_{jk}^i, \tag{37}$$

$$C_{jk}^{(H)i} = C_{jk}^{(v_1)i} = C_{jk}^{(v_2)i} = 0, \tag{38}$$

with the following notations

$$\theta_{jk}^i = \left(X_k^t \cdot d_j^i + X_j^t \cdot d_k^i - X_s^t \cdot \gamma_{jk} \cdot \gamma^{is}\right) \cdot y_t^{(1)}, \tag{39}$$

$$\Lambda_{jk}^i = d_j^i \cdot y_k^{(1)} + d_k^i \cdot y_j^{(1)} - \gamma_{jk} \cdot y^{(1)i} \tag{40}$$

Theorem 2.5. *The set of all* (0)*–metrical connections is given by the coefficients* $L_{jk}^{(H,*)i}$*,...,* $C_{jk}^{(v_2,*)i}$ *whose expressions are given by the following relations*

$$L_{jk}^{(H,*)i} = L_{jk}^{(H)i} + O_{rj}^{ih} \cdot I_{hk}^{(H)r}, \quad L_{jk}^{(v_a,*)i} = L_{jk}^{(v_a)i} + O_{rj}^{ih} \cdot I_{hk}^{(v_a)r}, \quad a = 1, 2, \tag{41}$$

$$F_{jk}^{(H,*)i} = F_{jk}^{(H)i} + O_{rj}^{ih} \cdot J_{hk}^{(H)r}, \quad F_{jk}^{(v_a,*)i} = F_{jk}^{(v_a)i} + O_{rj}^{ih} \cdot J_{hk}^{(v_a)r}, \quad a = 1, 2, \tag{42}$$

$$C_{jk}^{(H,*)i} = C_{jk}^{(H)i} + O_{rj}^{ih} \cdot H_{hk}^{(H)r}, \quad C_{jk}^{(v_a,*)i} = C_{jk}^{(v_a)i} + O_{rj}^{ih} \cdot H_{hk}^{(v_a)r}, \quad a = 1, 2, \tag{43}$$

where

$$O_{rj}^{ih} = \frac{1}{2}\left(d_r^i \cdot d_j^h - \gamma_{rj} \cdot \gamma^{ih}\right), \tag{44}$$

and $I_{hk}^{(H)r}$*,* $I_{hk}^{(v_1)r}$*,* $I_{hk}^{(v_2)r}$*,* $J_{hk}^{(H)r}$*,* $J_{hk}^{(v_1)r}$*,* $J_{hk}^{(v_2)r}$*,* $H_{hk}^{(H)r}$*,* $H_{hk}^{(v_1)r}$*,* $H_{hk}^{(v_2)r}$ *are arbitrary* d*–tensor fields.*

3. The curvature theory of a linear connection on the manifold T^2M

For the definition of the curvature tensor field in higher order geometry we refer to [6, 7]. Using the adapted basis $\left(d_k, d_k^{(1)}, d_k^{(2)}\right)$, the curvature tensor field of a (0)–metrical connection D can be expressed as

(45) $R(d_k, d_j)d_h = R_{hjk}^{(1)i} \cdot d_i, \quad R(d_k, d_j)d_h^{(a)} = R_{hjk}^{(a+1)i} \cdot d_i^{(a)}, \quad a = 1, 2$

(46) $R(d_k^{(1)}, d_j)d_h = P_{hjk}^{(1)i} \cdot d_i, \quad R(d_k^{(1)}, d_j)d_h^{(a)} = P_{hjk}^{(a+1)i} \cdot d_i^{(a)}, \quad a = 1, 2$

(47) $R(d_k^{(2)}, d_j)d_h = Q_{hjk}^{(1)i} \cdot d_i, \quad R(d_k^{(2)}, d_j)d_h^{(a)} = Q_{hjk}^{(a+1)i} \cdot d_i^{(a)}, \quad a = 1, 2$

(48) $R(d_k^{(1)}, d_j^{(1)})d_h = S_{hjk}^{(1)i} \cdot d_i, \quad R(d_k^{(1)}, d_j^{(1)})d_h^{(a)} = S_{hjk}^{(a+1)i} \cdot d_i^{(a)}, \quad a = 1, 2$

(49) $R(d_k^{(2)}, d_j^{(1)})d_h = O_{hjk}^{(1)i} \cdot d_i, \quad R(d_k^{(2)}, d_j^{(1)})d_h^{(a)} = O_{hjk}^{(a+1)i} \cdot d_i^{(a)}, \quad a = 1, 2$

(50) $R(d_k^{(2)}, d_j^{(2)})d_h = Z_{hjk}^{(1)i} \cdot d_i, \quad R(d_k^{(2)}, d_j^{(2)})d_h^{(a)} = Z_{hjk}^{(a+1)i} \cdot d_i^{(a)}, \quad a = 1, 2$

Theorem 3.1. *The local components of the curvature tensor field of a (0)–metrical connection D are given by the following relations:*

$$\begin{aligned}
(51) \quad R_{hjk}^{(1)i} &= d_k L_{hj}^{(H)i} - d_j L_{hk}^{(H)i} + L_{hj}^{(H)m} \cdot L_{mk}^{(H)i} - L_{hk}^{(H)m} \cdot L_{mj}^{(H)i} \\
&\quad - R_{(01)kj}^{m} \cdot F_{hm}^{(H)i} - R_{(01)kj}^{m} \cdot C_{hm}^{(H)i},
\end{aligned}$$

$$\begin{aligned}
(52) \quad R_{hjk}^{(a+1)i} &= d_k L_{hj}^{(v_a)i} - d_j L_{hk}^{(v_a)i} + L_{hj}^{(v_a)m} \cdot L_{mk}^{(v_a)i} - L_{hk}^{(v_a)m} \cdot L_{mj}^{(v_a)i} \\
&\quad - R_{(01)kj}^{m} \cdot F_{hm}^{(v_a)i} - R_{(01)kj}^{m} \cdot C_{hm}^{(v_a)i}, \\
a &= 1, 2,
\end{aligned}$$

$$\begin{aligned}
(53) \quad P_{hjk}^{(1)i} &= d_k^{(1)} L_{hj}^{(H)i} - d_j F_{hk}^{(H)i} + L_{hj}^{(H)m} \cdot F_{mk}^{(H)i} - F_{hk}^{(H)m} \cdot L_{mj}^{(H)i} \\
&\quad + B_{(11)kj}^{m} \cdot F_{hm}^{(H)i} + B_{(12)kj}^{m} \cdot C_{hm}^{(H)i},
\end{aligned}$$

$$\begin{aligned}
(54) \quad P_{hjk}^{(a+1)i} &= d_k^{(1)} L_{hj}^{(v_a)i} - d_j F_{hk}^{(v_a)i} + L_{hj}^{(v_a)m} \cdot F_{mk}^{(v_a)i} \\
&\quad - F_{hk}^{(v_a)m} \cdot L_{mj}^{(v_a)i} + B_{(11)kj}^{m} \cdot F_{hm}^{(v_a)i} + B_{(12)kj}^{m} \cdot C_{hm}^{(v_a)i}, \\
a &= 1, 2,
\end{aligned}$$

$$(55)\ Q^{(1)i}_{hjk} = d^{(2)}_k L^{(H)i}_{hj} - d_j C^{(H)i}_{hk} + L^{(H)m}_{hj} \cdot C^{(H)i}_{mk} - C^{(H)m}_{hk} \cdot L^{(H)i}_{mj} + B^m_{(21)kj} \cdot F^{(H)i}_{hm} + B^m_{(22)kj} \cdot C^{(H)i}_{hm},$$

$$(56)\quad \begin{aligned} Q^{(a+1)i}_{hjk} &= d^{(2)}_k L^{(v_a)i}_{hj} - d_j C^{(v_a)i}_{hk} + L^{(v_a)m}_{hj} \cdot C^{(v_a)i}_{mk} \\ &\quad - C^{(v_a)m}_{hk} \cdot L^{(v_a)i}_{mj} + B^m_{(21)kj} \cdot F^{(v_a)i}_{hm} + B^m_{(22)kj} \cdot C^{(v_a)i}_{hm}, \\ a &= 1, 2, \end{aligned}$$

$$(57)\quad \begin{aligned} S^{(1)i}_{hjk} &= d^{(1)}_k F^{(H)i}_{hj} - d^{(1)}_j F^{(H)i}_{hk} + F^{(H)m}_{hj} \cdot F^{(H)i}_{mk} \\ &\quad - F^{(H)m}_{hk} \cdot F^{(H)i}_{mj} - R^m_{(21)kj} \cdot C^{(H)i}_{hm}, \end{aligned}$$

$$(58)\quad \begin{aligned} S^{(a+1)i}_{hjk} &= d^{(1)}_k F^{(v_a)i}_{hj} - d^{(1)}_j F^{(v_a)i}_{hk} + F^{(v_a)m}_{hj} \cdot F^{(v_a)i}_{mk} \\ &\quad - F^{(v_a)m}_{hk} \cdot F^{(v_a)i}_{mj} - R^m_{(22)kj} \cdot C^{(v_a)i}_{hm}, \\ a &= 1, 2, \end{aligned}$$

$$(59)\quad \begin{aligned} O^{(1)i}_{hjk} &= d^{(2)}_k F^{(H)i}_{hj} - d^{(1)}_j C^{(H)i}_{hk} + F^{(H)m}_{hj} \cdot C^{(H)i}_{mk} \\ &\quad - C^{(H)m}_{hk} \cdot F^{(H)i}_{mj} + B^m_{(21)kj} \cdot C^{(H)i}_{hm}, \end{aligned}$$

$$(60)\quad \begin{aligned} O^{(a+1)i}_{hjk} &= d^{(2)}_k F^{(v_a)i}_{hj} - d^{(1)}_j C^{(v_a)i}_{hk} + F^{(v_a)m}_{hj} \cdot C^{(v_a)i}_{mk} \\ &\quad - C^{(v_a)m}_{hk} \cdot F^{(v_a)i}_{mj} + B^m_{(22)kj} \cdot C^{(v_a)i}_{hm}, \\ a &= 1, 2, \end{aligned}$$

$$(61)\quad Z^{(1)i}_{hjk} = d^{(2)}_k C^{(H)i}_{hj} - d^{(2)}_j C^{(H)i}_{hk} + C^{(H)m}_{hj} \cdot C^{(H)i}_{mk} - C^{(H)m}_{hk} \cdot C^{(H)i}_{mj},$$

$$\begin{aligned} Z^{(a+1)i}_{hjk} &= d^{(2)}_k C^{(v_a)i}_{hj} - d^{(2)}_j C^{(v_a)i}_{hk} + C^{(v_a)m}_{hj} \cdot C^{(v_a)i}_{mk} - C^{(v_a)m}_{hk} \cdot C^{(v_a)i}_{mj}, \\ a &= 1, 2. \end{aligned}$$

Theorem 3.2. *Assume that D is the (0)- connection determined in Theorem 2.3. Then the non-zero local components of its curvature tensor field are given by the following relations*

$$(62)\quad R^{(1)i}_{hjk} = r^i_{hjk},$$

$$(63)\quad R^{(2)i}_{hjk} = r^i_{hjk} + \frac{2}{F^4} \cdot A^{(1)i}_{hjk} + \frac{1}{F^2} \cdot A^{(2)i}_{hjk} + \frac{1}{F^4} \cdot A^{(3)i}_{hjk},$$

$$(64)\quad R^{(3)i}_{hjk} = r^i_{hjk} + \frac{4}{F^4} \cdot A^{(1)i}_{hjk} + \frac{2}{F^2} \cdot A^{(2)i}_{hjk} + \frac{2}{F^4} \cdot A^{(3)i}_{hjk},$$

$$P^{(2)i}_{hjk} = \frac{2}{F^4} \cdot A^{(4)i}_{hjk} + \frac{1}{F^2} \cdot A^{(5)i}_{hjk} + \frac{1}{F^4} \cdot A^{(6)i}_{hjk}, \tag{65}$$

$$P^{(3)i}_{hjk} = \frac{4}{F^4} \cdot A^{(4)i}_{hjk} + \frac{2}{F^2} \cdot A^{(5)i}_{hjk} + \frac{2}{F^4} \cdot A^{(6)i}_{hjk}, \tag{66}$$

$$Q^{(2)i}_{hjk} = -\frac{1}{F^2} \cdot A^{(7)i}_{hjk}, \quad Q^{(3)i}_{hjk} = -\frac{2}{F^2} \cdot A^{(7)i}_{hjk} \tag{67}$$

$$S^{(2)i}_{hjk} = -\frac{1}{F^2} \cdot A^{(8)i}_{hjk} + \frac{2}{F^4} \cdot A^{(9)i}_{hjk} + \frac{1}{F^4} \cdot A^{(10)i}_{hjk}, \tag{68}$$

$$S^{(3)i}_{hjk} = -\frac{2}{F^2} \cdot A^{(8)i}_{hjk} + \frac{4}{F^4} \cdot A^{(9)i}_{hjk} + \frac{2}{F^4} \cdot A^{(10)i}_{hjk}, \tag{69}$$

with the following notations

$$A^{(1)i}_{hjk} = \left(X^p_k \cdot \theta^i_{hj} - X^p_j \cdot \theta^i_{hk}\right) \cdot y^{(1)}_p, \tag{70}$$

$$\begin{aligned} A^{(2)i}_{hjk} \;&=\; d_k\theta^i_{hj} - d_j\theta^i_{hk} + \gamma^m_{hj} \cdot \theta^i_{mk} + \theta^m_{hj} \cdot \gamma^i_{mk} \\ &\quad -\gamma^m_{hk} \cdot \theta^i_{mj} - \theta^m_{hk} \cdot \gamma^i_{mj} - R^m_{(01)jk} \cdot \Lambda^i_{mk}, \end{aligned} \tag{71}$$

$$A^{(3)i}_{hjk} = \theta^m_{hj} \cdot \theta^i_{mk} - \theta^m_{hk} \cdot \theta^i_{mj}, \tag{72}$$

$$A^{(4)i}_{hjk} = X^p_j \cdot y^{(1)}_p \cdot \Lambda^i_{hk} - \theta^i_{hj} \cdot y^{(1)}_k, \tag{73}$$

$$A^{(5)i}_{hjk} = d^{(1)}_k \theta^i_{hj} + d_j \Lambda^i_{hk} - \lambda^m_{hj} \cdot \Lambda^i_{mk} + \lambda^i_{mj} \cdot \Lambda^m_{hk} - B^m_{(11)jk} \cdot \Lambda^i_{hm}, \tag{74}$$

$$A^{(6)i}_{hjk} = \theta^i_{mj} \cdot \Lambda^m_{hk} - \theta^m_{hj} \cdot \Lambda^i_{mk}, \tag{75}$$

$$A^{(7)i}_{hjk} = B^m_{(21)jk} \cdot \Lambda^i_{hm} - d^{(2)}_k \theta^i_{hj}, \tag{76}$$

$$A^{(8)i}_{hjk} = d^{(1)}_k \Lambda^i_{hj} - d^{(1)}_j \Lambda^i_{hk}, \tag{77}$$

$$A^{(9)i}_{hjk} = \Lambda^i_{hj} \cdot y^{(1)}_k - \Lambda^i_{hk} \cdot y^{(1)}_j, \tag{78}$$

$$A^{(10)i}_{hjk} = \Lambda^m_{hj} \cdot \Lambda^i_{mk} - \Lambda^m_{hk} \cdot \Lambda^i_{mj}. \tag{79}$$

4. Conclusions

This paper discusses the second order prolongation of a Riemannian space. The basic concepts were introduced by Radu Miron in [6, 7]. On the mentioned geometrical model, in [12] we introduced and studied the notion of (α, β, γ)–Sasaki lift of a Riemannian space (M, γ) to T^2M and then determined the (α, β, γ) corresponding metrical linear connection. For the canonical metrical connection we also determined the local components of the tensor fields of curvature and torsion. We introduced and studied the notion of almost $2 - \pi$ structure on T^2M and dealt with the linear connection compatible with such a structure, as well as with certain necessary conditions for normality. Moreover, it was considered also the d- gauge linear connections on T^2M, preparing the basis for the determination of the second order generalized EYM equations, and the gravitational field equations as well. In a particular case, the (α, β, γ)- Sasaki lift of a Riemannian space (M, γ) to T^2M becomes the so-called homogeneous metrical structure on T^2M. Few steps in the study of the second order homogeneous model of a Riemannian space have been done in [12, 13, 14]. This paper can be view as a continuation of our previous work with respect to the above mentioned model in order to obtain a good "gauge" model for the theory of physical fields.

5. Acknowledgements

This work was supported by AM-POSDRU, project number: POSDRU/89/1.5/S/49944.

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