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STRONGLY $\theta-$PRE-I-CONTINUOUS FUNCTIONS

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Abstract. The topology τ of a space is enlarged to a topology τ^* using an ideal I whose members are disjoint with the members of τ. Many relations between topological concepts with respect to τ and $\tau^*$are obtained. The aim of this paper is to define a new type of functions called strongly θ-pre-I-continuous functions and to obtain

1. Introduction and Preliminaries

The subject of ideals in topological spaces has been studied by Kuratowski [10]. In 1990, Jankovic and Hamlett [5] developed the study in local and systematic fashion and offered some new results, improvements of known results, and some applications. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. In 1996, Dontchev [8] defined pre-I-open sets and pre-I-continuity. In 2002, Hatir and Noiri [6] defined semi-I-open sets and observed a decomposition of continuity. Yuksel and Acikgoz defined other new types of continuity in ideal topological spaces [1,2,3,15,16,17,18].

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Throughout the present paper $Cl(A)$ and $Int(A)$ denote the closure and the interior of A, respectively. Let (X, τ) be a topological space and I be an ideal of subsets of X. An ideal I is defined as a nonempty collection of subsets of X satisfying the following two conditions: (1) If $A \in I$ and $B \subset A$, then $B \in I$; (2) If $A \in I$ and $B \in I$, then $A \cup B \in I$. If I is an ideal on X, then (X, τ, I) is called an ideal topological space. For a subset $A \subset X$, $A^*(\tau, I) = \{x \in X : U \cap A \notin I$ for each open neighborhood U of $x\}$ is called the local function of A with respect to I and τ [10]. We simply write A^* instead of $A^*(\tau, I)$ in case there is no chance for confusion. For every ideal topological space (X, τ, I), there exists a topology $\tau^*(I)$, finer than τ, generated by $\beta(\tau, I) = \{U - J : U \in \tau$ and $J \in I\}$, but in general $\beta(\tau, I)$ is not a topology [5]. Additionally, $Cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for $\tau^*(I)$.

Lemma 1. *[5] Let (X, τ, I) be an ideal topological space and A,B subsets of X. Then the following properties hold:*

(1) If $A \subset B$, then $A^* \subset B^*$.
(2) $A^* = Cl(A^*) \subset Cl(A)$.
(3) $(A^*)^* \subset A^*$.
(4) $(A \cup B)^* = A^* \cup B^*$.

Definition 1. *A subset A of a topological space is said to be preopen [4] if $A \subset Int(Cl(A))$. The complement of a preopen set is said to be preclosed. The intersection of all preclosed sets containing A is called the preclosure [11] of A and is denoted by $pCl(A)$. The preinterior of A is defined by the union of all preopen sets contained in A and is denoted by $pInt(A)$. The family of all preopen sets of X is denoted by $PO(X)$. We set $PO(X, x) = \{U : x \in U$ and $U \in PO(X)\}$.*

Definition 2. *A subset A of an ideal topological space is said to be pre-I-open [8] (resp. semi-I-open [6]) if $A \subset Int(Cl^*(A))$ (resp. $A \subset Cl^*(Int(A))$). The family of all pre-I-open subsets of (X, τ, I) containing a point $x \in X$ is denoted by $PIO(X, x)$. A subset A of (X, τ, I) is said to be pre-I-closed [8] if its complement is pre-I-open. The intersection of all pre-I-closed sets containing A is called the pre-I-closure of A and is denoted by $p\imath Cl(A)$ [17].*

Lemma 2. *[17] Let (X, τ, I) be an ideal topological space and A a subset of X.*

1. $x \in p\imath Cl(A)$ if and only if $U \cap A \neq \emptyset$ for every pre-I-open set U of X containing x.

2. A is pre-I-closed if and only if $A = p\imath Cl(A)$.

Definition 3. *Let (X, τ, I) be an ideal topological space and $A \subset X$. A point x of X is called a θ-I-cluster point (resp. pre-θ-I-cluster point [3]) of A if $Cl^*(U) \cap A \neq \emptyset$ (resp. $p\imath Cl(U) \cap A \neq \emptyset$) for every open (resp. pre-I-open) set U of X containing x. The set of all θ-I-cluster (resp. pre-θ-I-cluster) point of A is called the θ-I-closure (resp. pre-θ-I-closure) of A and is denoted by $Cl_{\theta\imath}(A)$ (resp. $p\imath Cl_\theta(A)$).*

Definition 4. *A function $f : (X, \tau, I) \to (Y, \vartheta, I_1)$ is said to be pre-I-continuous [8] (resp. strongly θ-I-continuous [15] , weakly I-precontinuous, weakly I-continuous [1]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a pre-I-open (resp. open, pre-I-open,open) set U of X containing x such that $f(U) \subset V$ (resp. $f(Cl^*(U)) \subset V$, $f(U) \subset Cl^*(V)$, $f(U) \subset Cl^*(V)$).*

Definition 5. *A function $f : (X, \tau) \to (Y, \vartheta)$ is said to be precontinuous [4] (resp. strongly θ-continuous [13] , strongly θ-precontinuous [14]) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a preopen (resp. open, preopen) set U of X containing x such that $f(U) \subset V$ (resp. $f(Cl(U)) \subset V$, $f(pCl(U)) \subset V$).*

Definition 6. *A subset A of an ideal topological space (X, τ, I) is *-perfect [7] (resp. I-locally closed [9]) if $A^* = A$ (resp. $A = G \cap V$, where G is open and V is *-perfect).*

Lemma 3. *[1] An ideal topological space (X, τ, I) is an RI-space if for each $x \in X$ and each open neighborhood V of x, there exists an open neighborhood U of x such that $x \in U \subset Cl^*(U) \subset V$.*

Lemma 4. *A subset U of an ideal topological space X is pre-θ-I-open in X if and only if for each $x \in U$, there exists a pre-I-open set W containing x such that $p\imath Cl\,(W) \subset U$.*

Proof. Suppose that U is pre-θ-I-open in X. Then, $X - U$ is pre-θ-I-closed. Hence there exists a point $x \in U$ but $x \notin p\imath Cl_\theta(X - U)$. There exists $W \in PIO(X, x)$ such that $p\imath Cl(W) \cap (X - U) = \emptyset$. Thus, we get $p\imath Cl(W) \subset U$. Conversely, assume that U is not pre-θ-I-open in X. Then, $X - U$ is not pre-θ-I-closed.So, there exists a point x such that $x \in p\imath Cl_\theta(X - U)$ but $x \notin X - U$. Since $x \in U$, by hypothesis there exists a pre-I-open set W containing x such that $p\imath Cl(W) \subset U$. This is a contradiction since $x \in p\imath Cl_\theta(X - U)$. ∎

Lemma 5. *[16] Let A and B be subsets of (X, τ, I) and $Int^*(A)$ denote the interior of A with respect to τ^*. Then the following properties hold:*

(1) If $A \subset B$, then $Int^*(A) \subset Int^*(B)$.
(2) $Int^*(A) \subset A$.
(3) $Int(A) \subset Int^*(A)$.
(4) If A is open in (X, τ, I), then $Int^*(A) = A$.
(5) $Int^*(A \cap B) = Int^*(A) \cap Int^*(B)$.
(6) $Int^*(A) \cup Int^*(B) \subset Int^*(A \cup B)$.
(7) If F is closed in (X, τ, I) , then $Int^*(A \cup F) \subset Int^*(A) \cup F$.

Lemma 6. *Let (X, τ, I) be an ideal topological space and $A \subset X$. Then, $p\imath Cl(A) = A \cup Cl(Int^*(A))$.*

Proof. By using the above lemma, we obtain

$$Cl(Int^*(A \cup Cl(Int^*(A)))) \subset Cl(Int^*(A)) \subset A \cup Cl(Int^*(A)).$$

This shows that $A \cup Cl(Int^*(A))$ is pre-I-closed. Since $A \subset A \cup Cl(Int^*(A))$, we have $p\imath Cl(A) \subset A \cup Cl(Int^*(A))$. It is obvious that $Cl(Int^*(A)) \subset Cl(Int^*(p\imath Cl(A)))$. Since $p\imath Cl(A)$ is pre-I-closed, we have $Cl(Int^*(A)) \subset p\imath Cl(A)$ and hence $A \cup Cl(Int^*(A)) \subset p\imath Cl(A)$. ∎

2. Some Characterizations

Definition 7. *A function $f : (X, \tau, I) \to (Y, \vartheta)$ is said to be strongly θ-pre-I-continuous (briefly str.θ.p.I.c.) if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a pre-I-open set U of X containing x such that $f(p\imath Cl(U)) \subset V$.*

Remark 1. *By the above definitions and Theorem 1. below, we have the following implications and none of these implications is reversible as seen in examples.*

$$\begin{tikzcd}
 & \text{s. } \theta\text{-I-c.} \arrow[rr] & & \text{continuous} \arrow[r] \arrow[d] & \text{weakly I-c.} \arrow[d] \\
\text{s. } \theta\text{-c.} \arrow[ur] \arrow[dr] \arrow[rr] & & \textbf{str.}\boldsymbol{\theta}\textbf{.p.I.c.} \arrow[r] \arrow[dl] & \text{p.-I-c.} \arrow[r] \arrow[d] & \text{weakly I-prec.} \arrow[d] \\
 & \text{st.}\theta\text{.p.c.} \arrow[rr] & & \text{precontinuous} \arrow[r] & \text{weakly precontinuous}
\end{tikzcd}$$

Example 1. *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b, c\}\}, Y = \{b, c\}, \vartheta = \{\emptyset, Y, \{c\}\}$ and $I = \{\emptyset, \{a\}\}$. Define a function $f : (X, \tau, I) \to (Y, \vartheta)$ as follows $f(a) = f(c) = b$ and $f(b) = c$. Then, f is str.θ.p.I.c. but it is not strongly θ-continuous.*

Example 2. *Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}, \{a, b\}\}, Y = \{b, c\}, \vartheta = \{\emptyset, Y, \{b\}\}$ and $I = \{\emptyset, \{a\}\}$. Define a function $f : (X, \tau, I) \to (Y, \vartheta)$ as follows $f(a) = f(c) = c$ and $f(b) = b$. Then, f is pre-I-continuous but it is not str.θ.p.I.c.*

Example 3. *In Example 1. the function is str.θ.p.I.c. but it is not continuous.*

Example 4. *In Example 2. the function is continuous but it is not str.θ.p.I.c.*

Example 5. *Let* $X = \{a, b, c\}, \tau = \{\emptyset, X, \{b\}\}, Y = \{a, c\}, \vartheta = \{\emptyset, Y, \{c\}\}$ *and* $I = \{\emptyset, \{b\}\}$. *Define a function* $f : (X, \tau, I) \to (Y, \vartheta)$ *as follows* $f(a) = f(c) = a$ *and* $f(b) = c$. *Then,* f *is strongly* θ*-I-continuous but it is not str.θ.p.I.c.*

Example 6. *In Example 1. the function is str.θ.p.I.c. but it is not strongly θ-I-continuous.,*

Example 7. *Let* $X = \{a, b, c\}, \tau = \{\emptyset, X, \{c\}\}, \vartheta = \{\emptyset, X, \{a\}\}$ *and* $I = \{\emptyset, \{c\}\}$. *Define a function* $f : (X, \tau, I) \to (X, \vartheta, I)$ *as follows* $f(a) = b, f(b) = c$, $f(c) = a$. *Then,* f *is weakly I-continuous but it is not str.θ.p.I.c.*

Example 8. *Let* $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, b\}\}, \vartheta = \{\emptyset, X, \{b\}, \{a, c\}\}$ *and* $I = \{\emptyset, \{c\}\}$. *Then, the identity function* $f : (X, \tau, I) \to (X, \vartheta, I)$ *is str.θ.p.I.c. but it is not weakly I-continuous.*

Example 9. *Let* $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a, b\}\}, \vartheta = \{\emptyset, X, \{a, c\}\}$ *and* $I = \{\emptyset, \{b\}\}$. *Then, the identity function* $f : (X, \tau, I) \to (X, \vartheta)$ *is st.θ.p.c. but it is not str.θ.p.I.c.*

Theorem 1. *The following properties are equivalent for a function* $f : (X, \tau, I) \to (Y, \vartheta)$:

1. f *is str.θ.p.I.c.*
2. $f^{-1}(V)$ *is pre-θ-I-open in* X *for every open set* V *of* Y,
3. $f^{-1}(F)$ *is pre-θ-I-closed in* X *for every closed set* F *of* Y,
4. $f\left(p\imath Cl_\theta\left(A\right)\right) \subset Cl\left(f\left(A\right)\right)$ *for every subset* A *of* X,
5. $p\imath Cl_\theta\left(f^{-1}\left(B\right)\right) \subset f^{-1}\left(Cl\left(B\right)\right)$ *for every subset* B *of* Y.

Proof. (1)$\Rightarrow$(2) Let V be any open set of Y. Suppose that $x \in f^{-1}\left(V\right)$. Since f is str.θ.p.I.c, there exists $U \in PIO(X, x)$ such that $f\left(p\imath Cl(U)\right) \subset V$. Therefore, we have $x \in U \subset p\imath Cl\left(U\right) \subset f^{-1}\left(V\right)$. So, $f^{-1}\left(V\right)$ is pre-θ-I-open in X.

(2)$\Rightarrow$(3) Obvious.

(3)$\Rightarrow$(4) Let A be any subset of X. Since $Cl\left(f\left(A\right)\right)$ is closed in Y, by (3) we obtain, $f^{-1}\left(Cl\left(f\left(A\right)\right)\right)$ is pre-θ-I-closed. Therefore, we have

$$p\imath Cl_\theta\left(A\right) \subset p\imath Cl_\theta\left(f^{-1}\left(f\left(A\right)\right)\right) \subset p\imath Cl_\theta(f^{-1}(Cl(f(A)))) \subset f^{-1}\left(Cl\left(f\left(A\right)\right)\right)$$

Thus, we obtain $f\left(p\imath Cl_{\theta}\left(A\right)\right)\subset Cl\left(f\left(A\right)\right)$.

(4)⇒(5) Let B be any subset of Y. By (4), we obtain $f(p\imath Cl_{\theta}(f^{-1}(B)))\subset Cl(B)$ and hence $p\imath Cl_{\theta}(f^{-1}(B))\subset f^{-1}(Cl(B))$.

(5)⇒(1) Let $x\in X$ and V be any open neighborhood of $f(x)$. Since $Y-V$ is closed in Y, we have $p\imath Cl_{\theta}(f^{-1}(Y-V))\subset f^{-1}(Cl(Y-V))=f^{-1}(Y-V)$. Hence, $f^{-1}(Y-V)$ is pre-θ-I-closed in X. And $f^{-1}(V)$ is a pre- θ-I-open set containing x. By Lemma 4, there exists a pre-I-open set U such that $x\in U\subset p\imath Cl(U)\subset f^{-1}(V)$. Therefore we obtain $f(p\imath Cl(U))\subset V$. ∎

Definition 8. *A net $\{x_{\lambda}\}_{\lambda\in D}$ in X is said to be pI-convergent to $x\in X$ if for each pre-I-open set U containing x the net is eventually in $p\imath Cl(U)$ and it is denoted by $x_{\lambda}\longrightarrow_{p\imath}x$.*

Theorem 2. *A function $f:(X,\tau,I)\to(Y,\vartheta)$ is str.θ.p.I.c. if and only if for each $x\in X$ and each net $x_{\lambda}\longrightarrow_{p\imath}x$, then the net $f(x_{\lambda})\to f(x)$.*

Theorem 3. *Let Y be a regular space. Then the following properties are equivalent for a function $f:(X,\tau,I)\to(Y,\vartheta)$:*

1. *f is weakly I-precontinuous,*
2. *f is pre-I-continuous,*
3. *f is str.θ.p.I.c.*

Proof. (1)⇒(2) Let $x\in X$ and V be any open neighborhood of $f(x)$. Since Y is regular, there exists an open set W such that $f(x)\in W\subset Cl(W)\subset V$. Since f is weakly I-precontinuous, there exists $U\in PIO(X,x)$ such that $f(U)\subset Cl^{*}(W)$. Therefore, we obtain $f(U)\subset V$. This shows that f is pre-I-continuous

(2)⇒(3) Let $x\in X$ and V be any open neighborhood of $f(x)$ .Since Y is regular, there exists an open set W such that $f(x)\in W\subset Cl(W)\subset V$. Since f is pre-I-continuous, there exists $U\in PIO(X,x)$ such that $f(U)\subset W$. We shall show that $f(p\imath Cl(U))\subset Cl(W)$. Suppose that $y\notin Cl(W)$. There exists an open neighborhood S of y such that $S\cap W=\emptyset$. Since f is pre-I-continuous, $f^{-1}(S)\in PIO(X)$. $f^{-1}(S)\cap U=\emptyset$ and hence $f^{-1}(S)\cap p\imath Cl(U)=\emptyset$. Therefore, we obtain $S\cap f(p\imath Cl(U))=\emptyset$, $y\notin f(p\imath Cl(U))$ and so $f(p\imath Cl(U))\subset Cl(W)\subset V$. This shows that f is str.θ.p.I.c.

(3)⇒(1) It can be seen from Remark 1. ∎

Corollary 1. *Let Y be an RI-space. Then $f:(X,\tau,I)\to(Y,\vartheta,I_{1})$ is weakly I- precontinuous if and only if f is pre-I-continuous.*

Definition 9. *An ideal topological space (X, τ, I) is said to be p-I-regular if for each closed set F of X and each point $x \notin F$, there exist disjoint pre-I-open sets U and V such that $x \in U$ and $F \subset V$.*

Lemma 7. *An ideal topological space (X, τ, I) is p-I-regular if and only if each point $x \in X$ and each open neighborhood U of x, there exists $W \in PIO(X, x)$ such that $x \in W \subset p\imath Cl(W) \subset U$.*

Proof. $\Longrightarrow$: Let $x \in X$ and U be an open neighborhood of x. So, $X - U$ is closed and $x \notin X - U$. Since (X, τ, I) is a p-I-regular space, there exist disjoint pre-I-open sets W and V such that $x \in W$, $X - U \subset V$. In [17] Yuksel et al. show that $p\imath Int(Y - A) = Y - p\imath Cl(A)$ and $p\imath Cl(Y - A) = Y - p\imath Int(A)$. Hence $V \subset X - W$ and $V = p\imath Int(V) \subset p\imath Int(X - W) = X - p\imath Cl(W)$. Therefore, we obtain $p\imath Cl(W) \subset X - V \subset U$.

$\Longleftarrow$:Let F be a closed set of X and $x \notin F$. Then $X - F$ is open and $x \in X - F$. By hypothesis, there exists $V \in PIO(X, x)$ such that $x \in V \subset p\imath Cl(V) \subset X - F$. Therefore, we obtain $F \subset X - p\imath Cl(V)$. Consequently, X is a p-I-regular space. ∎

Theorem 4. *A continuous function $f : (X, \tau, I) \to (Y, \vartheta)$ is str.θ.p.I.c. if and only if X is a p-I-regular space.*

Proof. $\Longrightarrow$:Since f is continuous, for any open neighbourhood V of $f(x)$, $f^{-1}(V)$ is an open neighbourhood of x. And since f is str.θ.p.I.c, $f^{-1}(V)$ is pre-θ-I-open set. Hence, there exists $W \in PIO(X, x)$ such that $p\imath Cl(W) \subset f^{-1}(V)$. Therefore, X is a p-I-regular space.

$\Longleftarrow$: Suppose that $f : (X, \tau, I) \to (Y, \vartheta)$ is continuous and X is a p-I-regular space. For any $x \in X$ and any open neighborhood V of $f(x)$, $f^{-1}(V)$ is an open set of X containing x. Since X is a p-I-regular space, there exists $U \in PIO(X, x)$ such that $p\imath Cl(U) \subset f^{-1}(V)$. This shows that f is str.θ.p.I.c. ∎

Definition 10. *[3] An ideal topological space (X, τ, I) is said to be pre-I-regular if for each pre-I-closed set F of X and each point $x \notin F$, there exist disjoint pre-I-open sets U and V such that $x \in U$ and $F \subset V$.*

Lemma 8. *[3] An ideal topological space (X, τ, I) is pre-I-regular if and only if each point $x \in X$ and each pre-I-open neighborhood U of x , there exists $V \in PIO(X, x)$ such that $x \in V \subset p\imath Cl(V) \subset U$.*

Theorem 5. *Let (X, τ, I) be a pre-I-regular space. Then $f : (X, \tau, I) \to (Y, \vartheta)$ is str.θ.p.I.c. if and only if f is pre-I-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. By the pre-I-continuity of f, we have $f^{-1}(V) \in PIO(X,x)$ and since X is a pre-I-regular space, there exists $U \in PIO(X,x)$ such that $p\imath Cl(U) \subset f^{-1}(V)$. Therefore, we have $f(p\imath Cl(U)) \subset V$. This shows that f is str.θ.p.I.c. ∎

Definition 11. *[2] An ideal topological space (X, τ, I) is called an I-submaximal space if every subset of X is I-local closed.*

Lemma 9. *[2] If (X, τ, I) is an I-submaximal space, then $PIO(X) = \tau$.*

Theorem 6. *Let (X, τ, I) be an I-submaximal space. If $f : (X, \tau, I) \to (Y, \vartheta)$ is str.θ.p.I.c. then f is strongly θ-I-continuous.*

Proof. Suppose that f is str.θ.p.I.c. Let $x \in X$ and V be any open set of Y containing $f(x)$. Since f is str.θ.p.I.c, there exists $U \in PIO(X,x)$ such that $f(p\imath Cl(U)) \subset V$. Since X is an I-submaximal space, $p\imath Cl(U) = Cl(U)$ and therefore, $f(Cl^*(U)) \subset V$. This shows that f is strongly θ-I-continuous. ∎

3. Some Properties

Theorem 7. *Let $f : (X, \tau, I) \to (Y, \vartheta, I_1)$ be a function and $g : (X, \tau, I) \to X \times Y$ the graph function of f. Then, the following properties hold:*

1. If g is str.θ.p.I.c, then f is str.θ.p.I.c. and X is a p-I-regular space.

2. If f is str.θ.p.I.c. and X is a pre-I-regular space, then g is str.θ.p.I.c.

Proof. **(1).** Let $x \in X$ and V be any open set of Y containing $f(x)$. Then, $X \times V$ is an open set of $X \times Y$ containing $g(x)$. Since g is str.θ.p.I.c, there exists $U \in PIO(X,x)$ such that $g(p\imath Cl(U)) \subset X \times V$. Therefore, we have $f(p\imath Cl(U)) \subset V$ and f is str.θ.p.I.c. Let U be any open neighborhood of x. Since $g(x) \in U \times Y$ and $U \times Y$ is open in $X \times Y$, there exists $W \in PIO(X,x)$ such that $g(p\imath Cl(W)) \subset U \times Y$. Hence, we have $p\imath Cl(W) \subset U$ and X is a p-I-regular space.

(2). Let $x \in X$ and W be any open set of $X \times Y$ containing $g(x)$. Then, there exist open sets $U \subset X$, $V \subset Y$ such that $g(x) = (x, f(x)) \in U \times V \subset W$. Since f is str.θ.p.I.c, there exists $G \in PIO(X,x)$ such that $f(p\imath Cl(G)) \subset V$ and also, $U \cap G$ is a pre-I-open neighborhood of x. Since X is a pre-I-regular space, there exists $T \in PIO(X,x)$ such that $x \in T \subset p\imath Cl(T) \subset U \cap G$. Therefore, we obtain $g(p\imath Cl(T)) \subset U \times f(p\imath Cl(G)) \subset U \times V \subset W$. This shows that g is str.θ.p.I.c. ∎

Corollary 2. *Let (X, τ, I) be a pre-I-regular space. Then a function $f : (X, \tau, I) \to (Y, \vartheta)$ is str.θ.p.I.c. if and only if the graph function $g : (X, \tau, I) \to X \times Y$ is str.θ.p.I.c.*

Theorem 8. *[17] Let (X, τ, I) be an ideal topological space and let A and X_0 be subsets of X.*

1. *If $A \in PIO(X)$ and $X_0 \in SIO(X)$, then $A \cap X_0 \in PIO(X_0)$.*
2. *If $A \in PIO(X_0)$ and $X_0 \in PIO(X)$, then $A \in PIO(X)$.*

Theorem 9. *[17] Let (X, τ, I) be an ideal topological space and $A \subset X_0 \subset X$ and $p\imath Cl_{X_0}(A)$ denotes the pre-I-closure of A in X_0.*

1. *If X_0 is semi-I-open in X, then $p\imath Cl_{X_0}(A) \subset p\imath Cl(A)$.*
2. *If $A \in PIO(X_0)$ and $X_0 \in PIO(X)$, then $p\imath Cl(A) \subset p\imath Cl_{X_0}(A)$.*

Theorem 10. *If $f : (X, \tau, I) \to (Y, \vartheta)$ is str.θ.p.I.c. and X_0 is a semi-I-open subset of X, then $f/X_0 : (X_0, \tau/X_0, I/X_0) \to (Y, \vartheta)$ is str.θ.p.I.c.*

Proof. Let $x \in X_0$ and V be any open neighborhood of $f(x)$. Since f is str.θ.p.I.c, there exists $U \in PIO(X, x)$ such that $f(p\imath Cl(U)) \subset V$. Put $U_0 = U \cap X_0$, then by Theorems 8. and 9, $U_0 \in PIO(X_0)$ and $p\imath Cl_{X_0}(U_0) \subset p\imath Cl(U_0)$. Therefore, we obtain

$$(f/X_0)(p\imath Cl_{X_0}(U_0)) = f(p\imath Cl_{X_0}(U_0)) \subset f(p\imath Cl(U_0)) \subset f(p\imath Cl(U)) \subset V \quad .$$

This shows that f/X_o is str.θ.p.I.c. ∎

Theorem 11. *A function $f : (X, \tau, I) \to (Y, \vartheta)$ is str.θ.p.I.c. if for each $x \in X$ there exists $X_0 \in PIO(X, x)$ such that $f/X_0 : (X_0, \tau/X_0, I/X_0) \to (Y, \vartheta)$ is str.θ.p.I.c.*

Proof. Let $x \in X$ and V be any open neighborhood of $f(x)$. Since f/X_o is str.θ.p.I.c, there exists $U \in PIO(X_0, x)$ such that $(f/X_0)(p\imath Cl_{X_0}(U)) = f(p\imath Cl_{X_0}(U)) \subset V$. By Theorem 9, $f(p\imath Cl(U)) \subset f(p\imath Cl_{X_0}(U)) \subset V$. This shows that f is str.θ.p.I.c. ∎

Definition 12. *[17] A function $f : (X, \tau, I) \to (Y, \vartheta, I_1)$ is said to be pre-I-irresolute if the inverse image of every pre-I-open subset of Y is pre-I-open in X.*

Theorem 12. *Let $f : (X, \tau, I) \to (Y, \vartheta, I_1)$ be a function. Then the following properties are equivalent:*

1. *f is pre-I-irresolute,*

2. For every subset A of Y, $f^{-1}(p\imath Int(A)) \subset p\imath Int(f^{-1}(A))$,
3. For every subset A of Y, $p\imath Cl(f^{-1}(A)) \subset f^{-1}(p\imath Cl(A))$.

Proof. (1)⇒(2) Let $A \subset Y$. Then $p\imath Int(A)$ is a pre-I-open set in Y. Since f is pre-I-irresolute, $f^{-1}(p\imath Int(A))$ is pre-I-open in X. Therefore, we obtain

$$f^{-1}(p\imath Int(A)) = p\imath Int(f^{-1}(p\imath Int(A))) \subset p\imath Int(f^{-1}(A)).$$

(2)⇒(3) Let $A \subset Y$. Then, $X - f^{-1}(p\imath Cl(A)) = f^{-1}(p\imath Int(Y - A)) \subset p\imath Int(f^{-1}(Y - A)) = X - p\imath Cl(f^{-1}(A))$. Hence, we obtain $p\imath Cl(f^{-1}(A)) \subset f^{-1}(p\imath Cl(A))$.

(3)⇒(1) Let F be a pre-I-closed subset of Y. By (3), $p\imath Cl(f^{-1}(F)) \subset f^{-1}(p\imath Cl(F)) = f^{-1}(F)$. This shows that f is pre-I-irresolute. ∎

Lemma 10. *If a function $f : (X, \tau, I) \to (Y, \vartheta, I_1)$ is pre-I-irresolute, then the inverse image of every pre-θ-I-open subset of Y is pre-θ-I-open in X.*

Proof. Let V be a pre-θ-I-open subset of Y and $x \in f^{-1}(V)$. Then, there exists $W \in PIO(Y)$ such that $f(x) \in W \subset p\imath Cl(W) \subset V$ and $f^{-1}(W) \in PIO(X)$. By using above theorem we obtain, $x \in f^{-1}(W) \subset p\imath Cl(f^{-1}(W) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is pre-θ-I-open. ∎

Theorem 13. *Let $f : (X, \tau, I) \to (Y, \vartheta, I_1)$ and $g : (Y, \vartheta, I_1) \to (Z, \phi, I_2)$ be functions.*
1. If f is str.θ.p.I.c. and g is continuous, then $gof : (X, \tau, I) \to (Z, \phi, I_2)$ is str.θ.p.I.c.
2. If f is pre-I-irresolute and g is str.θ.p.I.c., then $gof : (X, \tau, I) \to (Z, \phi, I_2)$ is str.θ.p.I.c.

Proof. 1. This is obvious from Theorem 1.
2. This follows immediately from Theorem 1. and Lemma 10. ∎

4. Separation Axioms

Definition 13. *An ideal topological space (X, τ, I) is said to be pre-I-T_2 if for any pair of distinct points x and y in X, there exist $U \in PIO(X, x)$ and $V \in PIO(X, y)$ such that $U \cap V = \emptyset$.*

Definition 14. *An ideal topological space (X, τ, I) is said to be pre-I-Urysohn if for any pair of distinct points x and y in X, there exist $U \in PIO(X, x)$ and $V \in PIO(X, y)$ such that $p\imath Cl(U) \cap p\imath Cl(V) = \emptyset$.*

Remark 2. *Every pre-I-Urysohn space is a pre-I-T_2 space.*

Theorem 14. *If $f:(X,\tau,I)\to(Y,\vartheta)$ is a str.θ.p.I.c. injection and Y is a T_0 space, then X is a pre-I-T_2 space.*

Proof. Let x and y be any distinct points of X. By hypothesis, $f(x)\neq f(y)$ and there exists an open set U in Y such that $f(x)\in U$, $f(y)\notin U$ or $f(y)\in U$, $f(x)\notin U$. If the first case holds, there exists $W\in PIO(X,x)$ such that $f(p\imath Cl(W))\subset U$ and $y\notin p\imath Cl(W)$. Hence, $X-p\imath Cl(W)$ is a pre-I-open neighborhood of y . If the second case holds, we obtain a similar result. Therefore, X is a pre-I-T_2 space. ∎

Theorem 15. *If $f:(X,\tau,I)\to(Y,\vartheta)$ is a str.θ.p.I.c. injection and Y is a T_2 space, then X is a pre-I-Urysohn space.*

Proof. Let x and y be any distinct points of X. By hypothesis $f(x)\neq f(y)$ and there exist disjoint open sets U and V in Y containing $f(x)$ and $f(y)$, respectively. Since f is str.θ.p.I.c, there exist $G\in PIO(X,x)$ and $H\in PIO(X,y)$ such that $f(p\imath Cl(G))\subset U$ and $f(p\imath Cl(H))\subset V$. It follows that $f(p\imath Cl(G))\cap f(p\imath Cl(H))=\emptyset$. This shows that X is a pre-I-Urysohn space. ∎

Lemma 11. *[18] Let (X,τ,I) be an ideal topological space and A,B subsets of X. Then, $Cl^*(A)\times Cl^*(B)\subset Cl^*(A\times B)$.*

Lemma 12. *Let (X,τ,I) be an ideal topological space and A,B subsets of X. If $A,B\in PIO(X)$, then $A\times B\in PIO(X\times X)$.*

Proof. $A\times B\subset Int_X(Cl^*_X(A))\times Int_X(Cl^*_X(B))=Int_{X\times X}(Cl^*_X(A)\times Cl^*_X(B))\subset Int_{X\times X}(Cl^*_{X\times X}(A\times B))$. This shows that $A\times B\in PIO(X\times X)$. ∎

Lemma 13. *Let (X,τ,I) be an ideal topological space and $A,B\subset X$. $p\imath Cl(A\times B)\subset p\imath Cl(A)\times p\imath Cl(B)$.*

Proof. Let $(x,y)\in X\times X$ and $(x,y)\in p\imath Cl(A\times B)$. Then for any $U\in PIO(X,x)$ and $V\in PIO(X,y)$, by Lemma 12. we have, $(x,y)\in U\times V\in PIO(X\times X)$ and $(A\cap U)\times(B\cap V)=(A\times B)\cap(U\times V)\neq\emptyset$. Hence $x\in p\imath Cl(A)$ and $y\in p\imath Cl(B)$ and $(x,y)\in p\imath Cl(A)\times p\imath Cl(B)$. ∎

Theorem 16. *If $f:(X,\tau,I)\to(Y,\vartheta)$ is str.θ.p.I.c. and Y is a T_2 space, then the subset $A=\{(x,y):f(x)=f(y)\}$ is pre-θ-I-closed in $X\times X$.*

Proof. Suppose that $(x,y)\notin A$. It follows that $f(x)\neq f(y)$ and there exist disjoint open sets V,W in Y containing $f(x)$ and $f(y)$, respectively. Since f is str.θ.p.I.c, there exist $U \in PIO(X,x)$ and $G \in PIO(X,y)$ such that $f(p_I Cl(U)) \subset V$ and $f(p_I Cl(G)) \subset W$. By Lemma 12, we get $(x,y) \in U \times G \in PIO(X \times X)$ and also, $(p_I Cl(U) \times p_I Cl(G)) \cap A = \emptyset$. By using Lemma 13, we obtain $(x,y) \in U \times G \subset p_I Cl(U \times G) \subset (X \times X) - A$ and so A is pre-θ-I-closed set. ∎

Definition 15. *The graph $G(f)$ of a function $f : (X,\tau,I) \to (Y,\vartheta)$ is said to be strongly pre-I-closed if for each $(x,y) \in (X \times Y) - G(f)$, there exist a pre-I-open neighborhood U of x and an open neighborhood V of y such that $(p_I Cl(U) \times V) \cap G(f) = \emptyset$.*

Lemma 14. *The graph $G(f)$ of a function $f : (X,\tau,I) \to (Y,\vartheta)$ is strongly pre-I-closed in $X \times Y$ if and only if for each point $(x,y) \in (X \times Y) - G(f)$, there exist a pre-I-open neighborhood U of x and an open neighborhood V of y such that $f(p_I Cl(U)) \cap V = \emptyset$.*

Theorem 17. *If $f : (X,\tau,I) \to (Y,\vartheta)$ is str.θ.p.I.c. and Y is a T_2 space, then $G(f)$ is strongly pre-I-closed in $X \times Y$.*

Proof. Let $(x,y) \in (X \times Y) - G(f)$. It follows that $f(x) \neq y$ and there exist disjoint open sets V and W in Y containing $f(x)$ and y, respectively. Since f is str.θ.p.I.c, there exists $U \in PIO(X,x)$ such that $f(p_I Cl(U)) \subset V$. Therefore, $f(p_I Cl(U)) \cap W = \emptyset$ and $G(f)$ is strongly pre-I-closed in $X \times Y$. ∎

5. Preservation Properties

Definition 16. *An ideal topological space X is said to be pre-θ-I connected if we can not write X as the union of two nonempty disjoint pre-θ-I-open sets.*

Theorem 18. *If $f : (X,\tau,I) \to (Y,\vartheta)$ is a str.θ.p.I.c. surjection and X is a pre-θ-I connected space, then Y is connected.*

Proof. Suppose that Y is not connected. Then, there exist two nonempty disjoint open sets V_1, V_2 such that $Y = V_1 \cup V_2$. Since f is str.θ.p.I.c. and surjective, $f^{-1}(V_1)$, $f^{-1}(V_2)$ are nonempty disjoint pre-θ-I-open sets and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. It is a contradiction since X is pre-θ-I connected. This shows that Y is connected. ∎

Definition 17. *An ideal topological space X is said to be,*

1. *p-I-closed (resp.p-I-Lindelof) if every pre-I-open cover of X has a finite (resp.countable) subcover whose pre-I-closures cover X.*
2. *countably p-I-closed if every countable pre-I-open cover of X has a finite subcover whose pre-I-closures cover X .*

Theorem 19. *If $f : (X, \tau, I) \to (Y, \vartheta)$ is a str.θ.p.I.c. surjection and X is a p-I-closed space, then Y is compact.*

Proof. Let $(A_i)_{i\in I}$ be an open cover of Y. For each $x \in X$, there exists $i(x) \in I$ such that $f(x) \in A_i(x)$. Since f is str.θ.p.I.c, there exists $U_i(x) \in PIO(X, x)$ such that $f(p\imath Cl(U_i(x))) \subset A_i(x)$. The family $\{U_i(x) : x \in X\}$ is a pre-I-open cover of X. Since X is a p-I-closed space, there exists a finite subset X_0 such that $X = \bigcup\{p\imath Cl(U_i(x) : x \in X_0)$. Therefore, we have $Y = f(X) = \bigcup\limits_{x\in X_0} f(p\imath Cl(U_i(x))) \subset \bigcup\{A_i(x) : x \in X_0\}$. This shows that Y is compact. ∎

Theorem 20. *If $f : (X, \tau, I) \to (Y, \vartheta)$ is a str.θ.p.I.c. surjection and X is a p-I-Lindelof space, then Y is Lindelof space.*

Theorem 21. *If $f : (X, \tau, I) \to (Y, \vartheta)$ is a str.θ.p.I.c. surjection and X is a countably p-I-closed space, then Y is countably compact.*

Let (X, τ, I) be an ideal topological space. A subset K of X is said to be p-I-closed relative to X if every cover of K by pre-I-open sets of X has a finite subcover whose pre-I-closures (with respect to (X, τ, I)) cover K.

Theorem 22. *If a function $f : (X, \tau, I) \to (Y, \vartheta)$ has a strongly pre-I-closed graph, then $f(K)$ is closed in Y for each K which is p-I-closed relative to X.*

Proof. Let $y \in Y - f(K)$. Then, for each $x \in K$, we have $(x, y) \notin G(f)$ and there exist $U_x \in PIO(X, x)$ and an open neighborhood V_x of y such that $f(p\imath Cl(U_x)) \cap V_x = \emptyset$. Since K is p-I-closed relative to X and $K \subset \bigcup\{U\ _x : x \in K\}$, there exist a finite number of $x \in K$, says $x_1, x_2, ..., x_n$ such that $K \subset \bigcup\limits_{i=1}^{n} p\imath Cl(U(x_i))$. Put $V = \bigcap\limits_{i=1}^{n} V(x_i)$. Then V is an open neighbourhood of x and $f(K) \cap V = \bigcup\limits_{i=1}^{n} f(p\imath Cl(U(x_i))) \cap V \subseteq \bigcup\limits_{i=1}^{n} (f(p\imath Cl(U(x_i) \cap V(x_i) = \emptyset$. This shows that $f(K)$ is closed. ∎

Theorem 23. *Let (X, τ, I) be an I-submaximal space. If a function $f : (X, \tau, I) \to (Y, \vartheta)$ has a strongly pre-I-closed graph, then $f^{-1}(K)$ is θ-closed in X for each compact set K of Y.*

Proof. Let $x \notin f^{-1}(K)$. Then, for each $y \in K$ we have $(x, y) \notin G(f)$ and there exist $U(y) \in PIO(X, x)$ and an open neighborhood $V(y)$ of

y such that $f(p\imath Cl(U(y))) \cap V(y) = \emptyset$. Since $\{V(y) : y \in K\}$ is an open cover of K, there exists a finite subset K_0 of K such that $K \subset \bigcup\{V(y) : y \in K_0\}$. Since X is an I-submaximal space, $p\imath Cl(U_i) = Cl(U_i)$. Set $U = \bigcap\{U(y) : y \in K_0\}$. Then, U is an open neighborhood of y and

$$f(Cl(U)) \cap K \subset \bigcup_{y\in K_0} f(Cl(U)) \cap V(y) \subset \bigcup_{y\in K_0} f(Cl(U(y))) \cap V(y) = \emptyset.$$

This shows that $f^{-1}(K)$ is θ-closed in X. ∎

Corollary 3. *Let (X, τ, I) be an I-submaximal space. If a function $f : (X, \tau, I) \to (Y, \vartheta)$ has a strongly pre-I-closed graph, then $f^{-1}(K)$ is θ-I-closed in X for each compact set K of Y.*

Corollary 4. *Let X be an I-submaximal space and Y a compact Hausdorff space. The following properties are equivalent for a function $f : (X, \tau, I) \to (Y, \vartheta)$:*

(1) f is str.θ.p.I.c,
(2) $G(f)$ is strongly pre-I-closed in $X \times Y$,
(3) f is strongly θ-continuous,
(4) f is continuous,
(5) f is pre-I-continuous,
(6) f is weakly I-pre-continuous.

Proof. (1) $\Rightarrow$ (2).This is a consequence of Theorem 17.
(2) $\Rightarrow$ (3).This follows from Theorem 23.
(3) $\Rightarrow$ (4) $\Longrightarrow$ (5) $\Rightarrow$ (6).It can be seen easily by Remark 1.
(6) $\Rightarrow$ (1). Since Y is regular, this follows from Theorem 3. ∎

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