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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 21 (2011), No. 1, 5 - 16

THE MEAN VALUE THEOREMS AND INEQUALITIES OF OSTROWSKI TYPE

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Abstract. The main purpose of this paper is to derive new inequalities of Ostrowski type using mean value theorems. The inequalities for p-norm are also given and the weighted case is considered. New estimations of the remainder term in quadrature formulas are obtained.

1. Introduction

The following result is known in the literature as Ostrowski's inequality ([5]).

Theorem 1. [5] *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with the property that $|f'(t)| \leq M$ for all $t \in (a,b)$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a)M,$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

In the last years the inequalities of Ostrowski type have occupied the attention of many authors ([1], [2], [3], [4], [6], [7], [9], [10]). The mean value theorems were applied to prove this kind of inequalities.

Keywords and phrases: Ostrowski inequality, p-norm, Mean value theorem.
(2010)Mathematics Subject Classification: 26D15, 26D10, 41A55.

In 1946, Pompeiu [8] derive a variant of Lagrange's mean value theorem. In [3], S.S. Dragomir using this theorem evaluates the integral mean of an absolutely continuous function.

Theorem 2. *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ in (x_1, x_2) such that*

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{x_1 - x_2} = f(\xi) - \xi f'(\xi). \tag{1}$$

Theorem 3. [3] *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $[a, b]$ not containing 0. Then for any $x \in [a, b]$, we have the inequality*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{|x|} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f - lf'\|_\infty, \tag{2}$$

where $l(t) = t$, $t \in [a, b]$.

In [9], E.C. Popa using a mean value theorem obtained a generalization of Dragomir's result.

Theorem 4. [9] *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then for any $x \in [a, b]$ we have the inequality*

$$\left| \left[\frac{a+b}{2} - \alpha \right] f(x) + \frac{\alpha - x}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \|f - lf'\|_\infty, \tag{3}$$

where $\alpha \notin [a, b]$ and $l(t) = t - \alpha$, $t \in [a, b]$.

Also, in [6] J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1 \leq p \leq \infty$ which for $p = \infty$ give the Dragomir's result.

Theorem 5. [6] *Let the function $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) with $0 < a < b$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \leq p, q \leq \infty$, and all $x \in [a, b]$, the following inequality holds:*

$$\left| \frac{a+b}{2} \cdot \frac{f(x)}{x} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq PU(x, p) \cdot \|f - lf'\|_p, \tag{4}$$

where $l(t) = t$, $t \in [a, b]$, *and*

$$PU(x,p) = (b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}} + \left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}\right]. \tag{5}$$

Note that in cases $(p, q) = (1, \infty)$, $(\infty, 1)$ and $(2, 2)$ the constant $PU(x, p)$ has to be taken as the limit as $p \to 1, \infty$ and 2, respectively.

The main purpose of this paper is to derive new inequalities of Ostrowski type using mean value theorems, generalizing some results of S.S. Dragomir, J. Pečarić, S. Ungar and E.C. Popa (see [3], [6], [9]). The inequalities for p-norm are also given and the weighted case is considered.

2. The inequalities of Ostrowski type

The following result is a generalization of Pompeiu's mean value theorem.

Theorem 6. *For every real valued function f differentiable on an interval $[a, b]$ not containing 0 and for all pairs $x_1 \neq x_2$ in $[a, b]$, there exists a point ξ in (x_1, x_2) such that*

$$\frac{(x_1-\alpha)f(x_2)-(x_2-\alpha)f(x_1)}{x_1-x_2} = f(\xi)-(\xi-\alpha)f'(\xi), \tag{6}$$

where $\alpha \notin [a, b]$.

Proof. Define $F : \left[\frac{1}{b-\alpha}, \frac{1}{a-\alpha}\right] \to \mathbb{R}$ by

$$F(u) = uf\left(\frac{1}{u}+\alpha\right), \quad \alpha \notin [a, b]. \tag{7}$$

The function F is continuous and differentiable on $\left(\frac{1}{b-\alpha}, \frac{1}{a-\alpha}\right)$ and for all $x, t \in \left[\frac{1}{b-\alpha}, \frac{1}{a-\alpha}\right]$ exists $x < \eta < t$ such that $\frac{F(x)-F(t)}{x-t} = F'(\eta)$, namely

$$\frac{xf\left(\frac{1}{x}+\alpha\right)-tf\left(\frac{1}{t}+\alpha\right)}{x-t} = f\left(\frac{1}{\eta}+\alpha\right)-\frac{1}{\eta}f'\left(\frac{1}{\eta}+\alpha\right).$$

Denote $x = \dfrac{1}{x_2 - \alpha}$, $t = \dfrac{1}{x_1 - \alpha}$, $\xi = \dfrac{1}{\eta} + \alpha$, then $x_1 < \xi < x_2$ and the relation (6) holds.

Remark 7. *If we choose $\alpha = 0$ to obtain the Pompeiu's mean value theorem.*

Remark 8. *From the relation (6) we obtained*

$$|(x_1-\alpha)f(x_2)-(x_2-\alpha)f(x_1)| \le \sup_{\xi\in[a,b]} |f(\xi)-(\xi-\alpha)f'(\xi)|\,|x_1 - x_2|. \tag{8}$$

Integrating (8) over $x_1 \in [a,b]$ we find the the Ostrowski inequality (3) obtained by E.C. Popa in [8].

In the next part of this section we will obtain inequalities of Ostrowski type in p-norm. First of all we will consider the particular cases $p = 2, \infty$, respectively 1.

Theorem 9. *Let the function $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0 < a < b$. Then for all $x \in [a,b]$ the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2} - \alpha\right)\frac{f(x)}{x-\alpha} - \int_a^b f(t)dt\right| \le$$

$$\frac{(b-a)^{\frac{1}{2}}}{3}\|f - lf'\|_2 \cdot \left[\Phi(a,\alpha,x)^{\frac{1}{2}} + \Phi(b,\alpha,x)^{\frac{1}{2}}\right],$$

where $\alpha \notin [a,b]$, $l(t) = t - \alpha$, $t \in [a,b]$ and

$$\Phi(s,\alpha,x) = \ln\left(\frac{x-\alpha}{s-\alpha}\right)^3 + \left(\frac{s-\alpha}{x-\alpha}\right)^3 - 1,\ s \in [a,b].$$

Proof. The function F defined in (7) is continuous and differentiable on $\left(\dfrac{1}{b-\alpha}, \dfrac{1}{a-\alpha}\right)$, and for all $x_1, x_2 \in \left[\dfrac{1}{b-\alpha}, \dfrac{1}{a-\alpha}\right]$ we have

$$\begin{aligned} F(x_1)-F(x_2) &= \int_{x_2}^{x_1} F'(t)dt = \int_{x_2}^{x_1}\left[f\left(\frac{1}{t}+\alpha\right) - \frac{1}{t}f'\left(\frac{1}{t}+\alpha\right)\right]dt \\ &= -\int_{\frac{1}{x_2}+\alpha}^{\frac{1}{x_1}+\alpha} [f(u)-(u-\alpha)f'(u)]\frac{1}{(u-\alpha)^2}\,du. \end{aligned} \tag{9}$$

Denote $x_1 = \dfrac{1}{x-\alpha}$ and $x_2 = \dfrac{1}{t-\alpha}$. Then for all $x, t \in [a,b]$ from (9) we get

$$(t-\alpha)f(x) - (x-\alpha)f(t) = (x-\alpha)(t-\alpha)\int_x^t [f(u)-(u-\alpha)f'(u)]\frac{1}{(u-\alpha)^2}du. \tag{10}$$

Integrating on t and dividing by $x-\alpha$, we obtain

$$(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha} - \int_a^b f(t)dt =$$

$$\int_a^b (t-\alpha)\left(\int_x^t [f(u)-(u-\alpha)f'(u)]\frac{1}{(u-\alpha)^2}du\right)dt$$

and therefore

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha} - \int_a^b f(t)dt\right| \le \tag{11}$$

$$\int_a^b \left|\int_x^t \left|[f(u)-(u-\alpha)f'(u)]\frac{t-\alpha}{(u-\alpha)^2}\right|du\right|dt =$$

$$\int_a^x \left|\int_t^x \left|[f(u)-(u-\alpha)f'(u)]\frac{t-\alpha}{(u-\alpha)^2}\right|du\right|dt +$$

$$\int_x^b \left|\int_x^t \left|[f(u)-(u-\alpha)f'(u)]\frac{t-\alpha}{(u-\alpha)^2}\right|du\right|dt.$$

Applying Hölder's inequality we obtained

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha} - \int_a^b f(t)dt\right| \le$$

$$\left[\int_a^x\left(\int_t^x (f(u)-(u-\alpha)f'(u))^2 du\right)dt\right]^{\frac{1}{2}}\left[\int_a^x\left(\int_t^x \frac{(t-\alpha)^2}{(u-\alpha)^4}du\right)dt\right]^{\frac{1}{2}} +$$

$$\left[\int_x^b\left(\int_x^t (f(u)-(u-\alpha)f'(u))^2 du\right)dt\right]^{\frac{1}{2}}\left[\int_x^b\left(\int_x^t \frac{(t-\alpha)^2}{(u-\alpha)^4}du\right)dt\right]^{\frac{1}{2}} \le$$

$$\left[\int_a^b\left(\int_a^b (f(u)-(u-\alpha)f'(u))^2 du\right)dt\right]^{\frac{1}{2}}$$

$$\cdot\left\{\left[\int_a^x\left(\int_t^x \frac{(t-\alpha)^2}{(u-\alpha)^4}du\right)dt\right]^{\frac{1}{2}} + \left[\int_x^b\left(\int_x^t \frac{(t-\alpha)^2}{(u-\alpha)^4}du\right)dt\right]^{\frac{1}{2}}\right\} =$$

$$(b-a)^{\frac{1}{2}}\|f-lf'\|_2\cdot\left\{\left[\int_a^x\left(\int_t^x\frac{(t-\alpha)^2}{(u-\alpha)^4}du\right)dt\right]^{\frac{1}{2}}+\right.$$

$$\left.\left[\int_x^b\left(\int_x^t\frac{(t-\alpha)^2}{(u-\alpha)^4}du\right)dt\right]^{\frac{1}{2}}\right\}=\frac{(b-a)^{\frac{1}{2}}}{3}\|f-lf'\|_2\cdot\left[\Phi(a)^{\frac{1}{2}}+\Phi(b)^{\frac{1}{2}}\right].$$

Theorem 10. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for all $x\in[a,b]$ the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\leq \tag{12}$$

$$\begin{cases}\|f-lf'\|_\infty\cdot\Psi(a,b,\alpha,x), \ \textit{for } \alpha<a,\\ \\ -\|f-lf'\|_\infty\cdot\Psi(a,b,\alpha,x), \ \textit{for } \alpha>b,\end{cases}$$

where $\alpha\notin[a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ and

$$\Psi(a,b,\alpha,x)=\frac{1}{2(x-\alpha)}\left[(b-x)^2+(x-a)^2\right].$$

Proof. From relation (11) we obtained

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\leq$$

$$\sup_{a\leq u\leq b}|f(u)-(u-\alpha)f'(u)|\left[\int_a^x\left(\int_t^x\frac{|t-\alpha|}{(u-\alpha)^2}du\right)dt+\int_x^b\left(\int_x^t\frac{|t-\alpha|}{(u-\alpha)^2}du\right)dt\right].$$

From the above relation the inequality (12) is proved.

Remark 11. *The inequality (12) coincides with the Ostrowski inequality (3) obtained by E.C. Popa in* [9]*. The proof of the inequality (3) presented in this section was done in a different manner than in* [9].

Theorem 12. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$. Then for all $x\in[a,b]$ the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\leq(b-a)\|f-lf'\|_1\Omega(a,b,\alpha,x), \tag{13}$$

where $\alpha \notin [a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ *and*

$$\Omega(a,b,\alpha,x)=\begin{cases}\dfrac{1}{a-\alpha}+\dfrac{b-\alpha}{(x-\alpha)^2}, & \text{for } \alpha<a,\\[2ex] \dfrac{\alpha-a}{(\alpha-x)^2}+\dfrac{1}{\alpha-b}, & \text{for } \alpha>b.\end{cases}$$

Proof. From relation (11) we obtained

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\le$$

$$\int_a^x\left(\int_t^x|f(u)-(u-\alpha)f'(u)|\max_{t\le u\le x,\, a\le t\le x}\frac{|t-\alpha|}{(u-\alpha)^2}du\right)dt$$

$$+\int_x^b\left(\int_x^t|f(u)-(u-\alpha)f'(u)|\max_{x\le u\le t,\, x\le t\le b}\frac{|t-\alpha|}{(u-\alpha)^2}du\right)dt.$$

From the above relation the inequality (13) is proved.

Theorem 13. *Let the function* $f:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$, *with* $1<p,q<\infty$, $p,q\neq 2$ *and all* $x\in[a,b]$ *the following inequality holds*

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\le \tag{14}$$

$$\begin{cases}(b-a)^{\frac{1}{p}}\|f-lf'\|_p\left[\Theta(a,x,\alpha)^{\frac{1}{q}}+\Theta(b,x,\alpha)^{\frac{1}{q}}\right], & \text{for } \alpha<a,\\[2ex] -(b-a)^{\frac{1}{p}}\|f-lf'\|_p\left[\Theta(a,x,\alpha)^{\frac{1}{q}}+\Theta(b,x,\alpha)^{\frac{1}{q}}\right], & \text{for } \alpha>b,\end{cases}$$

where $\alpha \notin [a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ *and*

$$\Theta(s,x,\alpha)=\frac{1}{1-2q}\left\{\frac{(x-\alpha)^{2-q}}{q+1}+\frac{(x-\alpha)^{2-q}}{q-2}\right.$$
$$\left.-\frac{(x-\alpha)^{1-2q}(s-\alpha)^{q+1}}{q+1}-\frac{(s-\alpha)^{2-q}}{q-2}\right\},\ s\in[a,b].$$

Proof.

Applying Hölder's inequality in (11), we obtained

$$\left|(b-a)\left(\frac{a+b}{2}-\alpha\right)\frac{f(x)}{x-\alpha}-\int_a^b f(t)dt\right|\le$$

$$\left[\int_a^x\left(\int_t^x|f(u)-(u-\alpha)f'(u)|^p\,du\right)dt\right]^{\frac{1}{p}}\left[\int_a^x\left(\int_t^x\frac{|t-\alpha|^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}+$$

$$\left[\int_x^b\left(\int_x^t|f(u)-(u-\alpha)f'(u)|^p\,du\right)dt\right]^{\frac{1}{p}}\left[\int_x^b\left(\int_x^t\frac{|t-\alpha|^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}\leq$$

$$\left[\int_a^b\left(\int_a^b|f(u)-(u-\alpha)f'(u)|^p\,du\right)dt\right]^{\frac{1}{p}}\eta(a,b,\alpha,x)=$$

$$(b-a)^{\frac{1}{p}}\|f-lf'\|_p\eta(a,b,\alpha,x),$$

where

$$\eta(a,b,\alpha,x)=\left[\int_a^x\left(\int_t^x\frac{|t-\alpha|^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}+\left[\int_x^b\left(\int_x^t\frac{|t-\alpha|^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}.$$

Calculating the integrals from the above relation, the inequality (14) is proved.

3. The weighted case

In this section we consider the weighted case of Ostrowski inequality.

Theorem 14. *Let the function $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $0<a<b$, and let $w:[a,b]\to\mathbb{R}$ be a nonnegative integrable function. Then for $\frac{1}{p}+\frac{1}{q}=1$, with $1\leq p,q\leq\infty$, and all $x\in[a,b]$ the following inequality holds*

(15)
$$\left|\frac{f(x)}{x-\alpha}\int_a^b(t-\alpha)w(t)dt-\int_a^b f(t)w(t)dt\right|\leq(b-a)^{\frac{1}{p}}\|f-lf'\|_p\Lambda(a,b,\alpha,x),$$

where $\alpha\notin[a,b]$, $l(t)=t-\alpha$, $t\in[a,b]$ and

$$\Lambda(a,b,\alpha,x)=\left[\int_a^x\left(\int_t^x\frac{|t-\alpha|^q w(t)^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}+\left[\int_x^b\left(\int_x^t\frac{|t-\alpha|^q w(t)^q}{(u-\alpha)^{2q}}du\right)dt\right]^{\frac{1}{q}}.$$

Proof. Multiplying (10) by $\dfrac{w(t)}{x-\alpha}$ and integrating on t, we get

$$\frac{f(x)}{x-\alpha}\int_a^b(t-\alpha)w(t)dt-\int_a^b w(t)f(t)dt=$$

$$\int_a^b w(t)(t-\alpha)\left(\int_x^t[f(u)-(u-\alpha)f'(u)]\cdot\frac{1}{(u-\alpha)^2}du\right)dt,$$

and as in the proof of Theorem 13 we have

$$\left|\frac{f(x)}{x-\alpha}\int_a^b (t-\alpha)w(t)dt - \int_a^b f(t)w(t)dt\right| \leq$$

$$\left[\int_a^x\left(\int_t^x |f(u)-(u-\alpha)f'(u)|^p\, du\right) dt\right]^{\frac{1}{p}} \left[\int_a^x\left(\int_t^x \frac{|t-\alpha|^q w(t)^q}{(u-\alpha)^{2q}} du\right) dt\right]^{\frac{1}{q}} +$$

$$\left[\int_x^b\left(\int_x^t |f(u)-(u-\alpha)f'(u)|^p\, du\right) dt\right]^{\frac{1}{p}} \left[\int_x^b\left(\int_x^t \frac{|t-\alpha|^q w(t)^q}{(u-\alpha)^{2q}} du\right) dt\right]^{\frac{1}{q}}$$

which gives (15).

4. The estimate of the remainder in quadrature formula

In this section, in the same way with the reasoning used in [3] and [9] we will give new estimations of the remainder term in the quadrature formula constructed by E.C. Popa in [9].

We assume in the following that $0 < a < b$ and $\alpha \notin [a,b]$. Consider the division of the interval $[a,b]$ given by

$$\Delta : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

Let ξ_i be a sequence of intermediate points $\xi_i \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$ and denote $h_i = x_{i+1} - x_i$. In [9] was defined the following quadrature formula

$$\int_a^b f(t)dt = S_\Delta(f, \xi_i) + \mathcal{R}_\Delta(f, \xi_i), \text{ where}$$

$$S_\Delta(f,\xi_i) = \sum_{i=0}^{n-1} \frac{f(\xi_i)}{\xi_i - \alpha}\left(\frac{x_{i+1}+x_i}{2} - \alpha\right) h_i.$$

The following estimation for remainder term $\mathcal{R}_\Delta(f,\xi_i)$ was given in [9].

Theorem 15. [9] *Assume that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then we have*

$$|\mathcal{R}_\Delta(f,\xi_i)| \leq \frac{1}{2h}\|f - lf'\|_\infty \cdot \sum_{i=0}^{n-1} h_i^2,$$

where $h = \min\{|a-\alpha|, |b-\alpha|\}$ and $l(t) = t - \alpha$, $t \in [a,b]$.

In the next part of this section new estimations of the remainder term $\mathcal{R}_\Delta(f,\xi_i)$ will be given.

Theorem 16. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then we have*

$$|\mathcal{R}_\Delta(f,\xi_i)|\le \frac{2}{3}\mathbb{K}(a,b,\alpha)\cdot\|f-lf'\|_2\sum_{i=0}^{n-1}h_i^{\frac{1}{2}}, \tag{16}$$

where $\mathbb{K}(a,b,\alpha)=\left[\ln\left(\frac{b-\alpha}{a-\alpha}\right)^3+\left(\frac{b-\alpha}{a-\alpha}\right)^3-1\right]^{\frac{1}{2}}$ *and* $l(t)=t-\alpha$, $t\in[a,b]$.

Proof. Applying the Theorem 9 on the interval $[x_i,x_{i+1}]$ for the intermediate points ξ_i, to obtain

$$\left|\frac{f(\xi_i)}{\xi_i-\alpha}\left(\frac{x_{i+1}+x_i}{2}-\alpha\right)h_i-\int_{x_i}^{x_{i+1}}f(t)dt\right|\le \tag{17}$$
$$\frac{h_i^{\frac{1}{2}}}{3}\|f-lf'\|_2\left[\Phi(x_i,\alpha,\xi_i)^{\frac{1}{2}}+\Phi(x_{i+1},\alpha,\xi_i)^{\frac{1}{2}}\right],$$

where

$$\Phi(x_i,\alpha,\xi_i)=\ln\left(\frac{\xi_i-\alpha}{x_i-\alpha}\right)^3+\left(\frac{x_i-\alpha}{\xi_i-\alpha}\right)^3-1\le\ln\left(\frac{b-\alpha}{a-\alpha}\right)^3+\left(\frac{b-\alpha}{a-\alpha}\right)^3-1,$$

$$\Phi(x_{i+1},\alpha,\xi_i)=\ln\left(\frac{\xi_i-\alpha}{x_{i+1}-\alpha}\right)^3+\left(\frac{x_{i+1}-\alpha}{\xi_i-\alpha}\right)^3-1\le\ln\left(\frac{b-\alpha}{a-\alpha}\right)^3+\left(\frac{b-\alpha}{a-\alpha}\right)^3-1.$$

The relation (17) can be written

$$\left|\frac{f(\xi_i)}{\xi_i-\alpha}\left(\frac{x_{i+1}+x_i}{2}-\alpha\right)h_i-\int_{x_i}^{x_{i+1}}f(t)dt\right|\le\frac{2}{3}h_i^{\frac{1}{2}}\|f-lf'\|_2\mathbb{K}(a,b,\alpha).$$

Summing over i from 0 to $n-1$ we deduce the desired estimate (16).

Theorem 17. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then we have*

$$|\mathcal{R}_\Delta(f,\xi_i)|\le\tilde{\Omega}(a,b,\alpha)\|f-lf'\|_1\cdot\sum_{i=0}^{n-1}h_i, \tag{18}$$

where $l(t)=t-\alpha$, $t\in[a,b]$, *and* $\tilde{\Omega}(a,b,\alpha)=\begin{cases}\frac{a+b-2\alpha}{(a-\alpha)^2},\ \text{for}\ \alpha<a,\\[2mm] \frac{2\alpha-a-b}{(\alpha-b)^2},\ \text{for}\ \alpha>b.\end{cases}$

Proof. Applying the Theorem 12 on the interval $[x_i, x_{i+1}]$ for the intermediate points ξ_i, to obtain
(19)
$$\left|\frac{f(\xi_i)}{\xi_i-\alpha}\left(\frac{x_{i+1}+x_i}{2}-\alpha\right)h_i-\int_{x_i}^{x_{i+1}} f(t)dt\right| \le h_i\|f-lf'\|_1\Omega(x_i,x_{i+1},\alpha,\xi_i),$$

where

$$\Omega(x_i,x_{i+1},\alpha,\xi_i)=\begin{cases}\dfrac{1}{x_i-\alpha}+\dfrac{x_{i+1}-\alpha}{(\xi_i-\alpha)^2}, & \text{for } \alpha<a,\\[2ex] \dfrac{\alpha-x_i}{(\alpha-\xi_i)^2}+\dfrac{1}{\alpha-x_{i+1}}, & \text{for } \alpha>b.\end{cases}$$

Since $\Omega(x_i,x_{i+1},\alpha,\xi_i)\le\tilde{\Omega}(a,b,\alpha)$ the relation (19) can be written

$$\left|\frac{f(\xi_i)}{\xi_i-\alpha}\left(\frac{x_{i+1}+x_i}{2}-\alpha\right)h_i-\int_{x_i}^{x_{i+1}} f(t)dt\right| \le h_i\|f-lf'\|_1\cdot\tilde{\Omega}(a,b,\alpha).$$

Summing over i from 0 to $n-1$ we deduce the desired estimate (18).

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