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NORMAL APPROXIMATIONS OF GEODESICS ON SMOOTH TRIANGULATED SURFACES

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Abstract. In this paper we study relations between normal curves and geodesic curves on triangulated smooth surfaces. Based on a curvature measure for normal curves, we define normal geodesics and build a semi discrete curvature flow under which normal geodesics converge to classical geodesic curves and vice versa, each geodesic in the classic differential geometric sense can be approximated by a sequence of normal geodesics under the defined flow. We give experimental results for the approximation of geodesics on both synthetic as well as on meshes generated from point clouds obtained by sampling of real data.

1. Introduction

Geodesics and their approximations represent not only a fundamental subject of research in Differential Geometry, see, e.g. [2], [33], [32], [26], [1]

and its various applications in other fields of mathematics [10], they also are of great importance in various aspects of Computer Graphics [27], [19], Imaging [30] Computational Geometry [12], Pattern Analysis [21], Learning [3], Computation Theory [10], Networks [9], etc.

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In particular, geodesics and quasi-geodesics (see below) on PL (piecewise linear) and piecewise flat surfaces are of great interest both in their theoretic as well as computer driven implementations (see, amongst others, [32], [26] and [27], [30], [19] respectively).

It is in this context that our study is undertaken, hence the ambient space is a triangulated surface, and we study relations between geodesics on that surface, and *shortest normal curves* (to be defined in Section 2), with respect to the given triangulation and its subdivisions. Informally, normal curves can be regarded as a specific type of quasi-geodesics in the sense of [2], representing a subset of the set of all quasi-geodesics, formed of those quasi-geodesics that will be termed as "straight".

In [5], two flavors of results where proved.

- *Convergence results*: There is a semi discrete variational method under which a sequence of locally shortest normal curve will converge to a geodesic curve, not necessarily shortest. If, in addition, the triangulation is assumed to be *fat* (see Section 4), then we can guarantee convergence to shortest geodesics.
- *Approximation results*: Every geodesic on a surface is the limit curve, under the variational method mentioned above, of a sequence of short normal curves with respect to some triangulation and its subdivisions.

In order to achieve the above goals we will define *length* and *curvature* measures on normal curves according to which *least-weight, minimal, shortest* and *straight* curves will be defined. This will be done in Section 3. Note that straight curves are of interest also in the paper [27], however, since the curvature measure we will define herein is different from the one in [27], (see also [32]), so does the meaning of being straight.

In Section 4 the variational process is described, basically a *curvature flow*, given in [5] for normal curves, and show that we obtain a family of *straight* normal curves which converges to a limit curve that is a geodesic. Two types of flows will be considered: One is merely an extension of the standard curvature flow as studied in [11], [14], [13] and others, to piecewise smooth curves; the second one is a semi-discrete version of the former, where we restrict the flow only to the points at which the curve is non-smooth. In this section we will review convergence results for both flows, as proved in [5]. In Section 5, we will change the direction of interest and briefly discuss the approximation theorems proven in [5] for geodesic curves. A special type of triangulations will be taken into account, namely the so called *fat triangulations*. We will rely on works such as [31] for the ability to build a fat triangulation desirable for our needs. *PL-approximations* of the surface will also be considered, based on the constructed fat triangulation and its subdivisions, and we will show that for every geodesic curve on a smooth surface there is an *approximating sequence* of *PL* straight normal curves. In addition, we will also deduce an algorithmic way to find such a sequence in the case of *shortest geodesics*. In Section 6 some very preliminary experimental results are presented, regarding the algorithm mentioned above, for approximating shortest geodesics by straight normal curves. Finally, in Section 7 we briefly discuss study in progress, its goal being to prove some similar convergence and approximation results for *least area normal surface*, with respect to minimal surfaces, see also [4]. However, we begin in the following Section 2, with a brief introduction to normal surfaces and normal curves. As normal surfaces where introduced first, we shall also start with surfaces.

2. Preliminaries - Normal curves

2.1. **Normal Curves.**

Definition 2.1. *A curve C on a triangulated orientable surface Σ is called* normal curve *iff all intersections of C with 2-simplecies of the triangulation $\mathcal{T}$ are made of normal pieces, i.e. made of pathes which run from one edge to a different edge of the same triangle. We assume C is smooth in the interior of the 2-cells it passes through.*

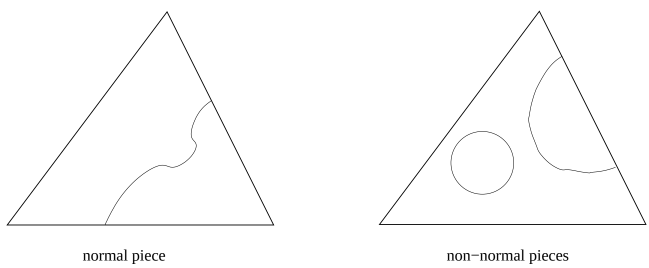

Figure 1. Normal and not normal pieces.

2.1.1. *Matching equations.* Consider a normal curve γ and let τ, μ be two 2-cells, adjacent along an edge e. We may code each normal piece of γ in τ say, according to the single vertex of τ it separates from the other two vertices. Since each normal piece in τ which pathes through e into μ will separate either of the vertices of e, we get an equation of the form,

$$x_\tau + y_\tau = x_\mu + y_\mu,$$

where x_τ, x_μ are the number of normal pieces separating the vertex x in τ, μ respectively.

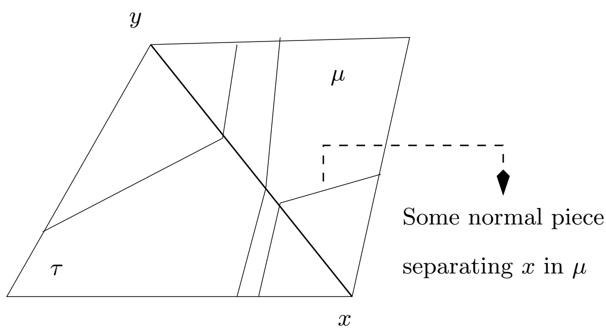

Figure 2. Illustration of matching equations.

Theorem 2.2. *Every normal curve on a triangulated surface induces a set of matching equations and vice versa every* integral solution *all values of which are* non negative*, of a system of matching equations can be realized as a normal curve, not necessarily connected.*

2.1.2. *Historical remark.* Normal curves can be regarded as a reminiscent of Normal Surfaces theory first introduced during 1930's by H.Kneser, [22], in his studies of the topology of 3-manifolds. In 1960's the theory of normal surfaces was revised and put in fully rigorous manner by W.Haken, [15], and where shown to give some first algorithmic solutions to problems related to 3-manifolds topology and geometry. In [17] a version of Theorem 2.2 for surfaces is given. We will refer to normal surfaces again in Section 7.

3. Length and curvature measures for normal curves

In this section we will define length and curvature measures for normal curves with respect to which we will have definitions of minimal (shortest) curves. These definitions will lead in turn to curvature flows minimizing the length of normal curves. In this process we will go from fine to coarse in the sense that, first we will have the usual length function of a curve that will result in the curvature flow very well known in the literature, [14], [13], [11]. The second will be a discrete version of length measure that will induce a discrete curvature flow that yields a sequence of minimal normal curves which converges to a smooth geodesic curve.

3.1. **Length and weight.** Let $(\Sigma, \mathcal{T})$ be a triangulated surface where $\mathcal{T} = (V, E, F)$ is a geodesic triangulation with respect to a given Riemannian structure on Σ.

Definition 3.1. (1) *The weight ω of an embedded curve on a triangulated surface is the total number of intersection points of the curve with $\mathcal{T}^1$.*
A normal curve is called least-weight *iff its weight is minimal in its isotopy class. For curves with* non empty fixed boundary*, we can also have curves with* non-zero *least-weight amongst all isotopy classes rel-boundary points. We will term such a curve as* global least-weight *curve.*
(2) *The length L, of a normal curve on a surface is the sum of lengths of all its normal pieces.*
   (a) *A curve will be called* minimal *iff its length is stationary with respect to a small variation.*
   (b) *A normal curve is* shortest *iff its length is non zero and minimal with respect to its isotopy classes. The same distinguish as above between closed curves and curves with boundary, holds also for shortest curves. For curves with fixed boundary we can have* non-zero *length, minimal with respect to all isotopy classes. Such a curve can be termed* global shortest *curve.*

Unless mentioned otherwise, when dealing with curves with boundary, least-weight/shortest *will always mean in the global sense.*

Remark 3.2. *Defining weight as above for general embedded curves on a surface, one can easily show that any curve on a triangulated surface can always be isotoped to a curve which is either normal or completely contained in a single triangle. Hence, a* least-weight *curve is always normal or has zero weight. If the curve is closed and essential, i.e. not null-homotopic, it is isotopic to a normal curve. Yet,* *all normally isotopic curves (i.e. isotopies that do not deform a normal curve through any vertex) have the same weight so, there is no uniqueness of a least-weight normal curve.*

Definition 3.3. *Similar to vertex linking sphere* [18]*, we define a linking circle to be a circle in the link of a small disc neighborhood of a vertex. A multiple of linking circle is a finite brunched cover of a linking circle.*

Using elementary hyperbolic geometry one can show,

Lemma 3.4. [5] *If* Σ *is a hyperbolic surface and* $\mathcal{T}$ *is a* geodesic ideal triangulation*, of* Σ*, and* Γ *is a normal curve which is not a multiple of a linking circle, then there exists a minimal length normal curve, in the normal homotopy class of* Γ*. every.*

3.2. **Dual graph reflections of normal curves.** Let γ be a normal curve on the triangulated surface $(\Sigma, \mathcal{T})$. Suppose $\partial\gamma = \{p, q\}$ are two fixed points on Σ interior in some two 2-cells, τ_p, τ_q, respectively. It is possible that $p = q$.

γ naturally defines a weighted path γ^* in the dual graph $\mathcal{T}^*$ of $\mathcal{T}$, in the following way.

- Each vertex of the dual path γ^* corresponds to a triangle intersected by γ.
- Each edge of γ^* is assigned to an edge of $\mathcal{T}$ at which γ crosses from one 2-cell to an adjacent one.
- We weight each edge of γ^* according to the number of times γ crosses through the corresponding edge of $\mathcal{T}$.
- We define the *length* of γ^* to be the sum of weights of all its edges.

Evidently each normal isotopy class uniquely determines such a dual path. It is our intension to find efficient characteristics for *least-weight* normal curves. In this course we would like to view such a normal curve as a realization of a shortest weighted path in the dual graph. Yet, some caution must be taken since, generally, we cannot expect a general weighted path in the dual graph to be realizable as a normal curve on the surface. Restrictions on this are due to the matching equations. We will show in the following that for a least-weight normal curve the situation is simpler and that, in fact, it can indeed be viewed as a realization of a shortest path in the dual graph.

Decoding of the matching equations in the dual setting is doable and should give a characterization of the weighted dual paths that can be realized as normal curves yet, to this point, it is left out of the scope of this current paper.

Remark 3.5. *It is essential in this context to consider curves that are* least-weight *in the global sense, for while proving some of the preceding lemmas we may not be able to keep isotopy class fixed. If the curve is closed, we will assume it is essential with some fixed initial point* p*, and consider* least weight *curves with respect to all isotopy classes of closed essential curves initialized at* p*. In the case of a non-essential closed curve where the* least-weight *is zero, results below are evident.*

Lemma 3.6. [5] *Let* γ *be a normal curve from* p *to* q*, that has* least-weight *amongst all normal curves connecting* p *to* q*. Then,* γ *has at most one normal piece in every triangle of* $\mathcal{T}$*.*

Lemma 3.7. [5] *A path in the dual graph for which all weights along its edges equal to* 1 *is realizable as a normal curve with respect to* $\mathcal{T}$*.*

Theorem 3.8. [5] *A normal curve is a* global*(w.r.t isotopy classes)*least-weight *curve if and only if its corresponding dual path is a shortest path according to the length function defined in 3.2.*

Corollary 3.9. *There is an algorithm which finds in finite time least-weight normal curve on a triangulated surface.*

Definition 3.10. *Let* Γ *be a normal curve with respect to a triangulation* $\mathcal{T}$ *on a surface* Σ*. We will define the curvature of* Γ *at a point* x *as follows.*

(1) *If* x *is an internal point on a segment of* $\Gamma \cap \mathcal{T}^2$*, we will take the curvature to be,*

$$\mathfrak{K}(x) = k_g(\Gamma, x),$$

where $k_g(\Gamma, x)$ *denotes the geodesic curvature of* Γ *at* x*.*

(2) *If* x *is a vertex formed by the intersection of* Γ *with* $\mathcal{T}^1$*, and let* t_1, t_2*, be the two vectors tangent to* Γ *at* x*. The curvature at* x *is defined to be*

$$\mathfrak{K}(x) = \cos(\theta_1) + \cos(\theta_2),$$

where θ_1, θ_2 *are the angles between* t_1, t_2 *and* t_e*, the vector tangent to the edge* e *at* x*.*

Remark 3.11. *There exists a vast amount of literature concerning curvature measures for piecewise smooth curves. Usually, to some angle defect is considered at points where the curve is not smooth. The definition given herein is meant to be compatible with a variation*

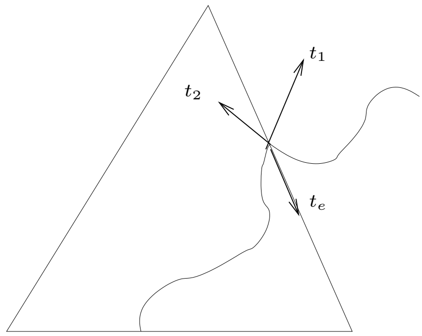

FIGURE 3. Three tangent vectors defining curvature along an edge.

of length formula which will be referred to soon, and also for being compatible with a definition of curvature for normal surfaces given in [18] *(to be referred to in Section* 6*).*

4. Curve shortening flows

Following Cheeger and Ebin, [7], we use first variation of arc length to find out conditions for which a normal curve is minimal. suppose $\Gamma_s(t)$ is a small variation of Γ with arc length parameter t. Let S be the variation vector field and T be the tangent vector field ($S = \frac{\partial}{\partial s}$, $T = \frac{\partial}{\partial t}$). Note that at each point of $\Gamma \cap \mathcal{T}^1$, S coincides with the vector field tangent to $\mathcal{T}^1$ at that point. Then the length of $\Gamma_s(t)$ is given by,

$$L_s = \sum_i L_i = \sum_i \int_0^{L_i} |\Gamma'_{s,\,i}(t)|dt = \sum_i \int_0^{L_i} < T, T >^{1/2} dt \ ,$$

where the indices i stands for the i^{th} normal piece of Γ.
For a curve to be minimal we demand

$$\frac{\partial L_s}{ds}|_{(s=0)} = 0.$$

$$\frac{\partial L_s}{ds} = \sum_i \int_0^{L_i} \frac{d}{ds} < T, T >^{1/2} dt = \sum_i \int_0^{L_i} S < T, T > dt =$$

$$= \sum_i \int_0^{L_i} (T < S, T > - < S, \nabla_T T >)dt =$$

$$= \sum_i < S, T > |_0^{L_i} - \sum_i \int_0^{L_i} < S, \nabla_T T >)dt =$$

$$\sum_i \int_0^{L_i} \nabla_T T dt + \sum_i (\cos(\theta_1^i) + \cos(\theta_2^i)).$$

This gives therefore, that in order for a curve to be a minimal normal curve it should both be a geodesic arc along the interior of each 2-cell it intersects, as well being “straight” along the intersections with the 1-skeleton. Straight means that along each edge, $(\cos(\theta_1) + \cos(\theta_2)) = 0$ which in turn generalizes the condition $(\theta_1 + \theta_2 = \pi)$, traditionally obtained from other common curvature measures for polygonal curves, [27].

Definition 4.1. *From the above we define the semi smooth curvature flow to be*

$$\frac{\partial \Gamma}{\partial t} = \mathfrak{K}(\Gamma) \tag{1}$$

Where Γ *is a normal curve and* $\mathfrak{K}$ *is its curvature as defined previously.*

Lemma 4.2. [5] *Let* Γ *be a piecewise smooth normal curve on a smooth triangulated surface* $(\Sigma, \mathcal{T})$*. Suppose* $x \in \Gamma \cap \mathcal{T}^1$ *is an intersection point of the curve on the interior of some edge* e*, so that* Γ *is* straight *at* x *in the sense defined above. Then the geodesic curvature of* Γ *at* x *exists and equals zero.i.e.*

$$\kappa_g(\Gamma, x) = 0$$

Corollary 4.3. *The curvature flow defined above transforms a normal curve embedded in a smooth surface to a geodesic curve.*

As already mentioned, the formulation above is a rephrasing of the curvature flow, extensively studied in the past, for piecewise smooth curves. In the following, we will further modify the flow by restricting it to $\Gamma \cap \mathcal{T}^1$.

Definition 4.4. *Let the weight of a curve be as in Definition 3.1.*

(1) *The* normalized weight *of a normal curve is given by*

$$n\omega(\Gamma) = L(\Gamma) \cdot \lambda^2$$

where λ *is the parameter of* $\mathcal{T}$*, i.e. the maximal edge length, where we range over all edges of* $\mathcal{T}$*.*

4.0.1. *Minimizing through straightening.* Let C be a rectifiable (i.e of finite total length) curve on a triangulated surface $(\Sigma, \mathcal{T})$. In this section we alter the minimization process while restricting it merely to points on $C \cap \mathcal{T}^1$. More precisely, consider the following procedure.

(1) Normalize C with respect to $\mathcal{T}$.
(2) Take a least-weight normal curve $\widehat{C}$, isotopic to C.
(3) Straighten $\widehat{C}$ at all intersections with the edges of $\mathcal{T}$.
(4) Take a subdivision of $\mathcal{T}$ to obtain a new triangulation $\mathcal{T}_1$.
(5) Go to (1) while C is replaced by $\widehat{C}$.

Remark 4.5. *Step* 3 *is done by moving each point* x *of* $C \cap \mathcal{T}^1$ *according to,*

$$\frac{\partial x}{\partial t} = \cos(\theta_1) + \cos(\theta_2) \ , \qquad (2)$$

where θ_1, θ_2 *are as before. Using say, partition of unity we can smoothly extend this process inside some small collar neighborhood of* $\mathcal{T}^1$*.*

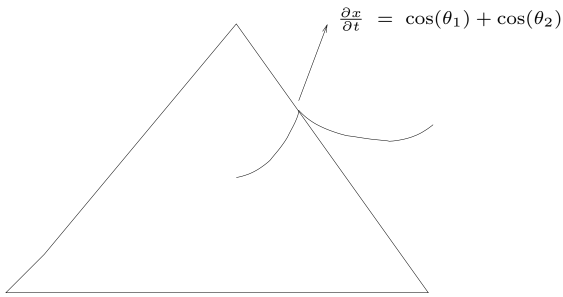

FIGURE 4. Combinatorial flow.

Remark 4.6. *There exists a variety of subdivisions that can be considered in the above procedure. The two that will be used herein are the barycentric subdivision and the "median" subdivision which is obtained by connecting the middle point of the edges of each triangle to each other, see Figure 4.2. If not necessary, we will not specify the actual subdivision taken.*

Theorem 4.7. [5] *The obtained sequence of straight curves converges to a smooth geodesic on* Σ

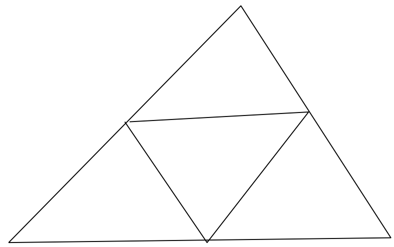

Figure 5. Median Subdivision.

Remark 4.8. *Normality of curves is crucial and best illustrated by the lemma above, for if the curve was not normal then we could not assume a bound on the arc length of its normal pieces, relative to the distance between y_1 and y_2 above, even when the division parameter λ gets very small.*

A natural question that arises is, if one restricts the above straightening procedure to *global* least-weight curves, do one gets convergence to shortest geodesics?

For having a chance of affirmative question to that question it is essential that the triangulation we work with, capture somehow the geometry of a collar neighborhood of a shortest geodesic. To be more precise, suppose Γ is a geodesic curve between two points P and Q on Σ. Then restricting to all triangles intersected by Γ, we get some triangular strip section of Σ. The length of Γ is naturally a monotonic increasing function of the (average) curvature of Σ along this section. If we can have a triangulation that admits similar behavior between the number of triangles in a neighborhood and (some averaging of) the curvature in that neighborhood, we can hope for answering the question posed above. It turns out that such triangulations do exist and in fact can be specifically built.

4.1. **Fat triangulations.** In this subsection we will give basic definition notations and quote relevant results. All details can be found in the relevant cited literature.

Definition 4.9.
(1) *A triangle in $\mathbb{R}^2$ is called φ-*fat *iff all its angles are larger than a prescribed value $\varphi > 0$.*
(2) *A k-simplex $\tau \subset \mathbb{R}^n$, $2 \leq k \leq n$, is φ-*fat *if there exists $\varphi > 0$ such that the ratio $\frac{r}{R} \geq \varphi$, where r and R, are resp. the radii of the inscribed and circumscribed (k-1)-spheres of τ.*

(3) *A triangulation* $\mathcal{T} = \{\sigma_i\}_{i \in \mathbf{I}}$ *is fat if all its simplices are* φ-fat *for some* $\varphi > 0$.

Proposition 4.10 ([8]). *There exists a constant* $c(k)$ *that depends solely upon the dimension* k *of* τ *such that*

$$\frac{1}{c(k)} \cdot \varphi(\tau) \leq \min_{\sigma < \tau} \measuredangle(\tau, \sigma) \leq c(k) \cdot \varphi(\tau)\,, \tag{3}$$

and

$$\varphi(\tau) \leq \frac{Vol_j(\sigma)}{diam^j\, \sigma} \leq c(k) \cdot \varphi(\tau)\,, \tag{4}$$

where φ *denotes the fatness of the simplex* τ, $\measuredangle(\tau, \sigma)$ *denotes the (*internal*)* dihedral angle *of the face* $\sigma < \tau$ *and* $Vol_j(\sigma)$*;* $diam\, \sigma$ *stand for the Euclidian* j*-volume and the diameter of* σ *respectively. (If* $dim\, \sigma = 0$*, then* $Vol_j(\sigma) = 1$*, by convention.)*

Condition 3 is just the expression of fatness as a function of dihedral angles in all dimensions, while Condition 4 expresses fatness as given by "large area/diameter". Diameter is important since fatness is independent of scale. Existence of fat triangulations of Riemannian manifolds is guaranteed by the studies given in [6], [25], [28] and [29]. These are summarized below.

Theorem 4.11 ([6]). *Every compact* $\mathcal{C}^2$ *Riemannian manifold admits a fat triangulation.*

Theorem 4.12 ([25]). *Every open (unbounded)* $\mathcal{C}^\infty$ *Riemannian manifold admits a fat triangulation.*

Theorem 4.13 ([29]). *Let* M^n *be an* n*-dimensional* $\mathcal{C}^1$ *Riemannian manifold with boundary, having a finite number of compact boundary components. Then, any fat triangulation of* ∂M^n *can be extended to a fat triangulation of* M^n.

Following the above, a scheme for achieving fat triangulations for smooth n-Riemannain manifolds, $n \geq 2$ is given in

Theorem 4.14 ([31]). *Let* Σ *be a connected, non-necessarily compact, smooth of class* $\mathcal{C}^r, r \geq 2$*,* n*-Riemannian manifold,* $n \geq 2$*, with finitely many boundary components. Then, there exists a sampling scheme (i.e a way of building fat triangulation) of* Σ*, with a proper density* $\mathcal{D} = \mathcal{D}(p) = \mathcal{D}\left(\frac{1}{k(p)}\right)$*, where* $k(p) = \max\{|k_1|, |k_2|, ..., |k_n|\}$*, and* $k_1, ..., k_n$ *are the principal curvatures of* Σ*, at the point* $p \in \Sigma$*.*

- According to the above Theorem 4.14 the density should be understood in the sense that the length of each edge of the triangulation is given locally as $\frac{1}{\sqrt{|K|}}$ where $|K|$ is the appropriate maximal curvature at a neighborhood of an edge. The fatness condition of the triangulation ensures that the "radial" density of the triangulation in the sense of the number of triangles around every vertex is almost uniform. As a result we have a greater number of triangles at curved areas of the surface and fewer and larger triangles at flat regions. Hence a shortest geodesic curve between two points will intersect fewer triangles than non-shortest geodesics.

Resulting from the remark above we get that at each iteration, a shortest geodesic and a global least-weight normal curve are contained in the same triangular section of the surface Σ. This in turn shows,

Theorem 4.15. *If in the shortening procedure described previously, we take in step (2) global least-weight normal curves then the obtained sequence of curves converge to a shortest geodesic between the end points.*

5. PL approximations of surfaces and geodesics

In this section we alter our attention in the opposite direction from the previous one. While in previous section we showed that we can have sequences of normal curves that converge to geodesics, which can be titled as *convergence results*, in this section we are interested in *approximation results*. We wish to show that every geodesic curve on a smooth surface is a limit of a sequence of normal curves with respect to some triangulation and its subdivisions. Opposed to previous section, we will have to consider a specific triangulation for our purpose or rather, a triangulation satisfying specific geometric constrains, and as a result we will also have to specify the subdivision scheme that can be used in order to fulfill these constrains.

Input: A surface Σ, two points p and q on it and a geodesic curve Γ from p to q.

Output: A sequence of curves Γ_n, each of which is a shortest/least-weight normal curve on a PL-surface Σ_n, so that

$$(i)\,\Sigma_n \to \Sigma \text{ and } (ii)\,\Gamma_n \to \Gamma.$$

5.1. **Fat triangulations and PL-approximations.** Before stating the main theorem of this section we will review fat triangulations some more while at this point we focus on their usage for PL-approximating of surfaces.

Definition 5.1. (1) *Let $f : K \to \mathbb{R}^n$ be a $\mathcal{C}^r$ map, and let $\delta : K \to \mathbb{R}_+^*$ be a continuous function. Then $g : |K| \to \mathbb{R}^n$ is called a* δ-approximation to f *iff:*
(i) There exists a subdivision K' of K such that $g \in \mathcal{C}^r(K', \mathbb{R}^n)$;
(ii) $d_{eucl}\big(f(x), g(x)\big) < \delta(x)$, for any $x \in |K|$;
(iii) $d_{eucl}\big(df_a(x), dg_a(x)\big) \leq \delta(a) \cdot d_{eucl}(x, a)$, for any $a \in |K|$ and for all $x \in \overline{St}(a, K')$.

(2) *Let K' be a subdivision of K, $U = \overset{\circ}{U}$, and let $f \in \mathcal{C}^r(K, \mathbb{R}^n)$, $g \in \mathcal{C}^r(K', \mathbb{R}^n)$. g is called a* δ-approximation *of f (on U) iff conditions (ii) and (iii) above hold for any $a \in U$.*

Definition 5.2 (*Secant map*)**.** *Let $f \in \mathcal{C}^r(K)$ and let s be a simplex, $s < \sigma \in K$. Then the linear map: $L_s : s \to \mathbb{R}^n$, defined by $L_s(v) = f(v)$ where v is a vertex of s, is called the* secant map induced by f.

The motivation for having fat triangulations for manifolds in terms of PL-approximations is stressed by the following theorem.

Theorem 5.3 ([23])**.** *Let $f : \sigma \to \mathbb{R}^n$ be of class $\mathcal{C}^k$. Then, for $\delta > 0$, there exists $\varepsilon, \varphi_0 > 0$, such that, for any $\tau < \sigma$, fulfilling the following conditions:*
(i) $diam(\tau) < \varepsilon$ and,
(ii) τ is φ_0-fat,
then, the secant map L_τ is a δ-approximation of $f|\tau$.

Following the theorem above combined with Theorem 4.14 we use the secant map as defined in order to reproduce a PL-surface as a δ-approximation for the sampled surface, [31].

We are now in a position to state the following approximation theorem.

Theorem 5.4. [5] *Let Γ be a geodesic curve between two points p and q on a triangulated surface $(\Sigma, \mathcal{T})$ where $\mathcal{T}$ is a fat triangulation of Σ, for which $\Gamma \cap \mathcal{T}^{(0)} = \emptyset$. Let $\mathcal{T}_{(n)}$ be the n^{th} median subdivision of $\mathcal{T}$. Let Σ_n be the PL-approximation of Σ built as the secant map on $\mathcal{T}_n$. Then, there exists a sequence of curves Γ_n embedded in Σ_n, each of which is a shortest/least-weight normal curve, where Σ_n is given as a triangulated polyhedral surface with a triangulation induced from $\mathcal{T}_n$, such that*

$$\Gamma_n \to \Gamma.$$

Corollary 5.5. [5] *The sequence of shortest normal curves* Γ_n *converges to* Γ.

While in the above theorem we showed that every *pre-given* geodesic curve can be *PL approximated* by *PL normal* curves, we will show that if only the end points are given, we can still find a sequences of *PL normal* curves that approximates the *shortest* geodesics between these points.

Theorem 5.6. [5] *Let* p *and* q *be two given points on a smooth surface* Σ. *Then there is an algorithm to find a sequence of* PL*-normal curves, each of which embedded inside some* PL-approximation Σ_n *of* Σ *such that this sequence converges to a shortest geodesic curve from* p *to* q.

6. Experimental Results

In this section we give some preliminary experimental results obtained from testing the algorithm described in Section 3 on both analytical surfaces as well as a surface which is part of a CT-Colonoscopy.

6.1. **Analytic surfaces.** We bring here results obtained on a sphere with different resolutions. Similar results where obtained also for the torus and in the final version of this paper will be referred to in the authors web site.

The tests where run on a sphere of radius 1. On such a sphere, the angle between two radii ending at the start and end points is equal to the length of the shortest path between these 2 points on the face of the sphere. The sphere was approximated 5 times, using 80, 320, 1280, 5120 and 20480 triangles, respectively. The following Figure 6 depicts the algorithm results. For each mesh, several pairs of start and end points were picked, according to the angle between them, and the algorithm was applied.

It is important to notice that a mesh of a sphere is bounded by the sphere, and a path on its surface can be shorter than the minimum path on the sphere itself. The higher the accuracy of the mesh, the smaller this possible difference is.

6.2. **Colon Surface.** In Figure 7 there are shown results of computing shortest normal curve with respect a triangulated mesh of some resolution obtained from a CT-Colonoscopy. Even in this fairly low resolution one can see a reasonable approximation of the shortest geodesic between the start and end points by the computed normal geodesic.

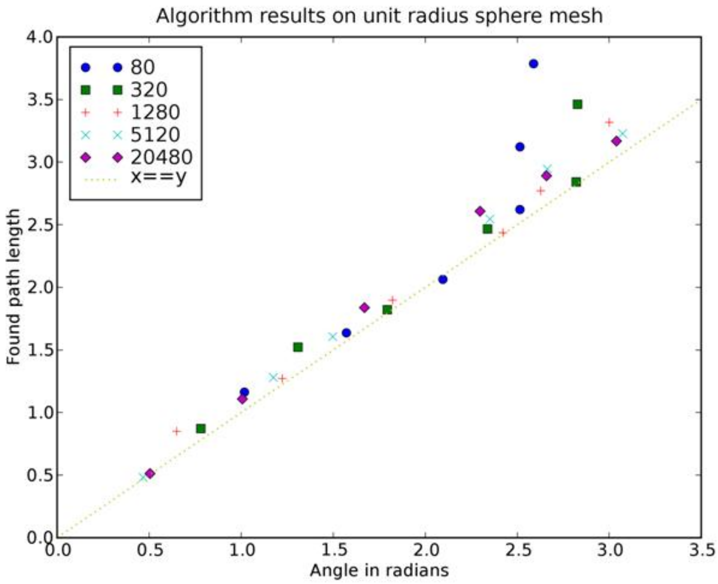

FIGURE 6. The dotted line is describes the length of the true shortest path on the face of the sphere.

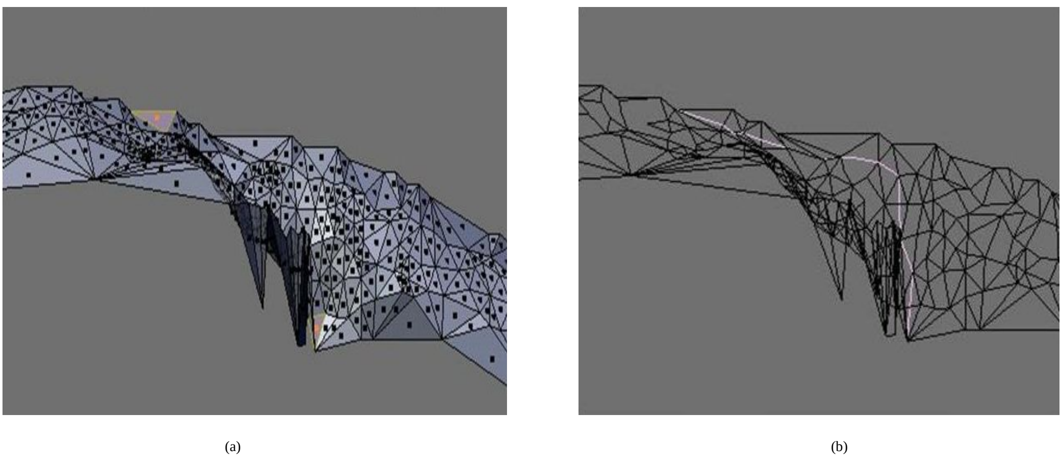

FIGURE 7. Shortest Normal curve on a Colon mesh: (a) The mesh with indicating start and end points; (b) Computed shortest normal curve.

6.3. **Convergence.** The following examples show an initial normal approximation of a geodesic on the sphere and its convergence to the actual geodesic while subdivisions of the triangulation is taken, see Figure 8 and Figure 9 to the case of a non simply connected surface. It can be seen that subdivision are actually taken only inside the initial triangle path.

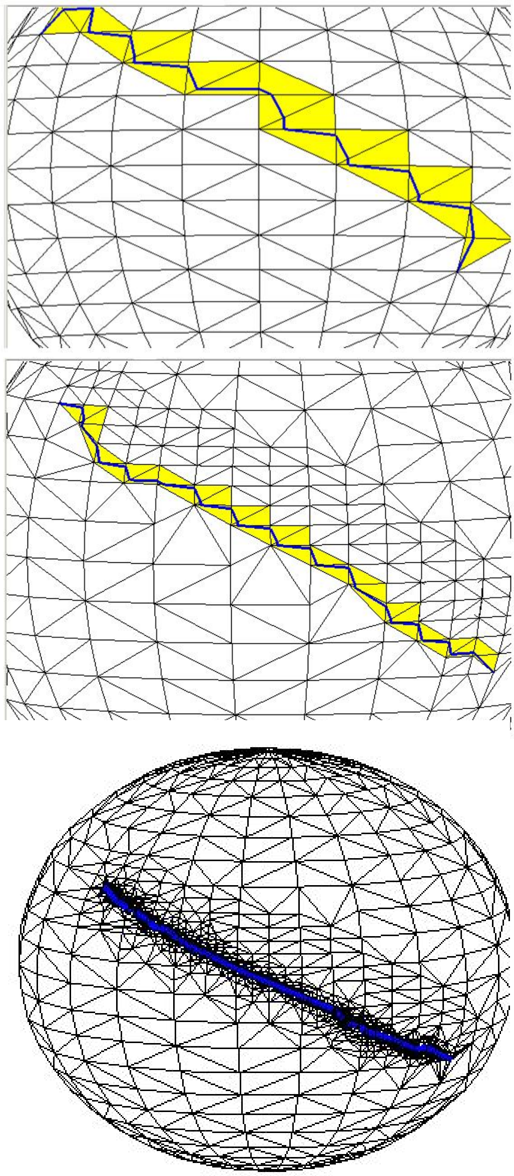

FIGURE 8. Initial normal curve (top) and the curve after 1 (middle) and 9 (bottom) iterations.

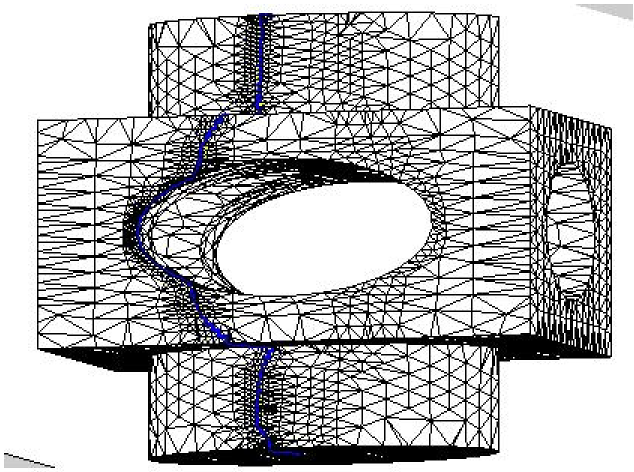

FIGURE 9. Normal approximation of a geodesic curve on a surface of high genus. Results shown is obtained after 12 iterations.

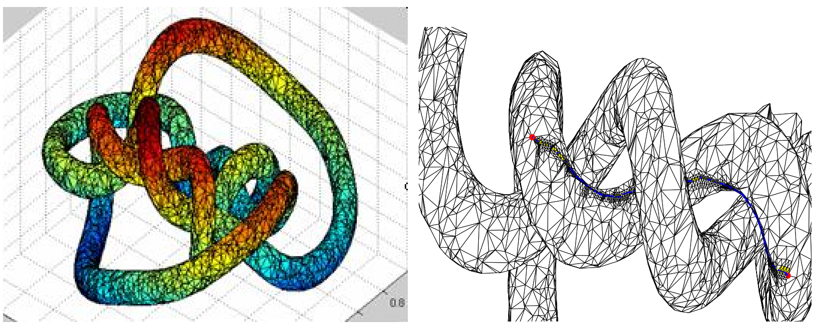

FIGURE 10. Normal geodesic curve on a knotted torus. *Top* image shows the surface and *bottom* shows results obtained after 12 iterations.

7. Further study

7.1. **Geometric Measures on Normal Surfaces.** A natural direction for further study is the possibility to extend the study presented herein to surfaces embedded in three dimensional manifolds. As noted before in Section 2, normal surface theory do exists and used in the context of three dimensional manifolds topology and geometry. In the seminal paper [18], length and area of a normal surface and the definitions of pl-minimal and of least-area normal surfaces are defined. Existence and uniqueness of *PL minimal* surfaces is shown in [18] and also in [24]. In a succeeding work we intend to prove results similar to

Theorems 4.7 and 5.4 for least-weight normal surfaces with respect to minimal surfaces in the classical differential geometric sense. Doing so, enables one to give an algorithmic way to define semi-discrete version of the Laplace-Beltrami operator yielding with a constructive ability of finding minimal surfaces. Applications will be in the direction of using this semi-discrete flow for images in the spirit of [16], [20].

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