"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 21 (2011), No. 1, 37 - 50

ON A ONE-DIMENSIONAL MATHEMATICAL MODEL RELATED TO SOIL BIOREMEDIATION

ELENA- ROXANA ARDELEANU

Abstract. In this paper we present a mathematical model associated to a bioremediation process. We consider a one-dimensional soil composed of a single layer. In this bioremediation process the bacteria migrates by diffusion and chemotaxis, where the diffusion coefficient is supposed to be constant.

The mathematical model is given by a system of nonlinear partial differential equations. In order to study this system of equations, we use the perturbation method for small parameters. The existence and uniqueness of the solution is studied within the framework of the equations' evolution theory based on m-accretive operators.

1. Model formulation

1.1. **Description of the physical process.** In this paper we consider the bioremediation model (F-G model) given by A. Fasano and D. Giorni in [2]. The F-G model is a special case of the Keller and Segel model [7]. We suppose a one-dimensional polluted soil, homogeneous, occupying the layer $0 < x < L$. We consider in this soil the propagation of a single pollutant and one type of bacteria able to distroy it. During bioremediation, bacteria migrate by diffusion and chemotaxis. Diffusion is the movement of molecules within in terms of decreasing concentration of bacterial population (from an area with a higher concentration to an area with a lower concentration).

Keywords and phrases: m-Accretive operators; Evolution equations; Diffusion and Chemotaxis.
(2010)Mathematics Subject Classification: 47B44; 93C20; 35R20; 30E25.

Chemotaxis phenomenon is particular sensitivity of some microorganisms (bacteria in our case) to the attraction of a chemical (in our case the pollutant). Consider also that the soil contains adequate nutrients for maintenance of the bacteria, at a temperature that is convenient and does not contain other chemicals that could inhibit the bioremediation process.

We denote by t the time, by x the spatial coordinate, by $c(t,x)$ the concentration of pollutant agent and by $b(t,x)$ the concentration of bacteria able to distroy it. Bioremediation process starts at $t=0$. We denote by $c_0(x)$ and $b_0(x)$ the concentration of the pollutant, respectively the concentration of bacteria, at $t=0$.

1.2. **F-G one-dimensional mathematical model.** Let $\Omega=(0,L)$ be an open, bounded subset of $\mathbb{R}$ with the boundary $\Gamma=\partial\Omega=\{x\in\mathbb{R}: x=0 \text{ şi } x=L\}$. Let $(0,T)$ be a finite time interval in which takes place the bioremediation process. We denote $Q=(0,T)\times\Omega$ and $\Sigma=(0,T)\times\Gamma$.

In this model, the bacteria also migrate by diffusion and chemoattraction. In order to write the equation for the flux of bacteria $j(t,x)$ we denote with $D>0$ the diffusion cofficient, supposed constant, and with $K(b,c)\geq 0$ the function measuring the chemotactic response:

$$j(t,x)=-D\frac{\partial b}{\partial x}+bK(b,c)\frac{\partial c}{\partial x}.$$

Substituting the flux in the continuity equation we obtain the mass conservation equation:

$$\frac{\partial b}{\partial t}-D\frac{\partial^2 b}{\partial x^2}+\frac{\partial}{\partial x}\left[bK(b,c)\frac{\partial c}{\partial x}\right]=f(b,c) \text{ for } (t,x)\in Q. \tag{1.1}$$

The function $f(b,c)$ denoting the proliferation rate of the bacteria per unit volume depends on the concentration of bacteria, as well as on the concentration of pollutant.

The kinetic law for pollutant absorption is as follows:

$$\frac{\partial c}{\partial t}=-\frac{\beta_1 c}{1+\beta_2 c}b \text{ for } (t,x)\in Q, \tag{1.2}$$

where $\beta_1>0$ and $\beta_2\geq 0$ are constants. The initial conditions for concentrations of pollutant and bacteria are:

$$c(0,x) = c_0(x) \text{ for } x\in\Omega, \tag{1.3}$$

$$b(0,x) = b_0(x) \text{ for } x\in\Omega, \tag{1.4}$$

with $c_0(x)$, $b_0(x)$ pozitive. To obtain the boundary condition we consider the flux of bacteria null on the both sides of the layer:

$$-D\frac{\partial b}{\partial x}+bK\left(b,c\right)\frac{\partial c}{\partial x}=0 \text{ for } (t,x)\in\Sigma. \tag{1.5}$$

Thus, the model to be studied is a nonlinear parabolic problem with Neumann boundary conditions and initial conditions, represented by system (1.1)- (1.5).

1.3. **Dimensionless F-G model.** We consider the characteristic sizes, denoted by index "a": L_a for length, T_a for time, b_a, c_a for concentrations of bacteria and of pollutant, respectively, D_a for the diffusion coefficient, K_a for the chemotactic response, f_a for the rate of multiplication of bacteria. With their help, the system of equations (1.1)- (1.5) will be written in dimensionless form.

We introduce the following notations:

$$x=x^*L_a,\ \ t=t^*T_a,\ \ b=b^*b_a,\ \ c=c^*c_a,$$
$$D=D^*D_a,\ \ K=K^*K_a,\ \ f=f^*f_a,$$

where by superior index * we denote the dimensionless variables. Next we replace into the dimensional system.

The equation (1.1) becomes:

$$\frac{b_a}{T_a}\frac{\partial b^*}{\partial t^*}-\frac{D_ab_a}{L_a}D^*\frac{\partial^2 b^*}{\partial x^{*2}}+\frac{b_aK_ac_a}{L_a}$$
$$\times\frac{\partial}{\partial x^*}\left[b^*K^*\left(b^*b_a,c^*c_a\right)\frac{\partial c^*}{\partial x^*}\right]=f_af^*\left(b^*b_a,c^*c_a\right) \text{ for } (t,x)\in Q.$$

We multiply the equation obtained by T_a/b_a. We make the following notations for the dimensionless expressions of f and K:

$$\widetilde{f}^*\left(b^*,c^*\right)=f^*\left(b^*b_a,c^*c_a\right),$$
$$\widetilde{K}^*\left(b^*,c^*\right)=K^*\left(b^*b_a,c^*c_a\right),$$

and we obtain:

$$\frac{\partial b^*}{\partial t^*}-D^*\frac{T_aD_a}{L_a}\frac{\partial^2 b^*}{\partial x^{*2}}+\frac{T_aK_ac_a}{L_a}$$
$$\times\frac{\partial}{\partial x^*}\left[b^*\widetilde{K}^*\left(b^*,c^*\right)\frac{\partial c^*}{\partial x^*}\right]=\frac{f_aT_a}{b_a}\widetilde{f}^*\left(b^*,c^*\right) \text{ for } (t,x)\in Q.$$

We denote:

$$\overline{D}=\frac{T_aD_a}{L_a},\ \overline{K}=\frac{T_aK_ac_a}{L_a} \text{ and } \overline{f}=\frac{f_aT_a}{b_a} \tag{1.6}$$

and we obtain the following dimensionless form for the equation (1.1):

(1.7)
$$\frac{\partial b^*}{\partial t^*} - \overline{D} D^* \frac{\partial^2 b^*}{\partial x^{*2}} + \overline{K} \frac{\partial}{\partial x^*} \left[b^* \widetilde{K}^* (b^*, c^*) \frac{\partial c^*}{\partial x^*} \right] = \overline{f} \widetilde{f}^* (b^*, c^*), \ (t, x) \in Q.$$

The dimensionless form of kinetic law (1.2) has the next form:

$$\frac{\partial c^*}{\partial t^*} = -\frac{\overline{\beta_1} c^* b^*}{1 + \overline{\beta_2} c^*}, \ (t, x) \in Q, \tag{1.8}$$

where

$$\overline{\beta_1} = \beta_1 T_a b_a \text{ and } \overline{\beta_2} = \beta_2 c_a. \tag{1.9}$$

The initial conditions (1.3)- (1.4) are written in dimensionless form as:

$$c^*(0, x) = c_0^*(x) \text{ for } x \in \Omega, \tag{1.10}$$

$$b^*(0, x) = b_0^*(x) \text{ for } x \in \Omega, \tag{1.11}$$

where $c_0^*(x) = \dfrac{c_0(x)}{c_a}$ and $b_0^*(x) = \dfrac{b_0(x)}{b_a}$. The boundary condition (1.5) is:

$$-\overline{D} D^* \frac{\partial b^*}{\partial x^*} + \overline{K} b^* \widetilde{K}^* (b^*, c^*) \frac{\partial c^*}{\partial x^*} = 0, \ (t, x) \in \Sigma. \tag{1.12}$$

Since the diffusion coefficient is constant, we can assume $D_a = D$ and $L_a = L$. In this case $D^* = 1$. Finally, we get the following dimensionless system, where we agree to not denote index *:

$$\frac{\partial b}{\partial t} - \overline{D} \frac{\partial^2 b}{\partial x^2} + \overline{K} \frac{\partial}{\partial x} \left[b K (b, c) \frac{\partial c}{\partial x} \right] = \overline{f} f (b, c), \ (t, x) \in Q, \tag{1.13}$$

$$\frac{\partial c}{\partial t} = -\frac{\overline{\beta_1} b c}{1 + \overline{\beta_2} c}, \ (t, x) \in Q, \tag{1.14}$$

$$c(0, x) = c_0(x), \ x \in \Omega, \tag{1.15}$$

$$b(0, x) = b_0(x), \ x \in \Omega, \tag{1.16}$$

$$-\overline{D} \frac{\partial b}{\partial x} + \overline{K} b K (b, c) \frac{\partial c}{\partial x} = 0, \ (t, x) \in \Sigma. \tag{1.17}$$

To study the equations system obtained we use the small parameter perturbation method. The case where $\overline{\beta_2}$ is the small parameter is treated in the paper [2]. In this paper we consider the model described by equations (1.13)- (1.17) with the small parameter $\overline{\beta_1}$. We make series expansion with respect to the small parameter $\overline{\beta_1}$ denoted with ε and we consider here only the approximation of order "0" system.

Let

$$
\begin{aligned}
b(t,x) &= b^0(t,x) + \varepsilon b^1(t,x) + ..., \\
c(t,x) &= c^0(t,x) + \varepsilon c^1(t,x) + ..., \\
K(b,c) &= K\left(b^0,c^0\right) + \varepsilon K_b\left(b^0,c^0\right) b^1(t,x) + \varepsilon K_c\left(b^0,c^0\right) c^1(t,x) + ..., \\
f(b,c) &= f\left(b^0,c^0\right) + \varepsilon f_b\left(b^0,c^0\right) b^1(t,x) + \varepsilon f_c\left(b^0,c^0\right) c^1(t,x) + ...,
\end{aligned}
$$

where K_b, K_c, f_b, f_c represent the derivatives of K and f with respect to b and c. Replacing in the system of equations (1.13)-(1.17) and identifying coefficients of ε^0 we get the system for approximation of order "0":

$$
\frac{\partial b^0}{\partial t} - \overline{D}\frac{\partial^2 b^0}{\partial x^2} + \overline{K}\frac{\partial}{\partial x}\left[b^0 K\left(b^0,c^0\right)\frac{\partial c^0}{\partial x}\right] = \overline{f} f\left(b^0,c^0\right),\ (t,x)\in Q, \tag{1.18}
$$

$$
\frac{\partial c^0}{\partial t}\left(1+\overline{\beta_2}c^0\right) = 0,\ (t,x)\in Q, \tag{1.19}
$$

$$
c^0(0,x) = c_0(x),\ x\in\Omega, \tag{1.20}
$$

$$
b^0(0,x) = b_0(x),\ x\in\Omega, \tag{1.21}
$$

$$
-\overline{D}\frac{\partial b^0}{\partial x} + \overline{K} b^0 K\left(b^0,c^0\right)\frac{\partial c^0}{\partial x} = 0,\ (t,x)\in\Sigma. \tag{1.22}
$$

Since (1.19) implies $\frac{\partial c^0}{\partial t} = 0$ for all $(t,x)\in Q$, it follows by (1.19) and (1.20) that for all $(t,x)\in Q$

$$
c^0(t,x) = c_0(x). \tag{1.23}
$$

In the following we assume that $c_0(x) = c_0$ is constant. In this case, in the equation (1.18) we will denote by $f(b^0) = \overline{f} f\left(b^0, c_0\right)$. We thus get the following system of equations for approximation of order "0":

$$
\frac{\partial b^0}{\partial t} - \overline{D}\frac{\partial^2 b^0}{\partial x^2} = f(b^0),\ (t,x)\in Q, \tag{1.24}
$$

$$
b^0(0,x) = b_0(x),\ x\in\Omega, \tag{1.25}
$$

$$
\frac{\partial b^0}{\partial x} = 0,\ (t,x)\in\ \Sigma. \tag{1.26}
$$

The condition (1.26) amount to assume that the bacteria does not go through the boundaries.

It is known that in general, a nonlinear equation with a nonlinear term of the form $f(b,x)$ does not admit a global solution in time [4], [5]. We consider a condition for the function $f(b,x)$ for which the problem admits a global solution. We suppose that the function $f(b^0)$ is negative, which means that the mortality rate of bacteria is grater that its rate of multiplication. We denote by $\mu(r)r = -f(r)$. In the context of population dynamics, an important case is the one where the nonnegative function $\mu(r)$ is locally Lipschitz on $\mathbb{R}$, i.e., there exist $L_\mu(R) > 0$, such that as $|r| \le R$ and $|\overline{r}| \le R$ we have:

$$|\mu(r) - \mu(\overline{r})| \le L_\mu(R)\,|r - \overline{r}|\,. \tag{1.27}$$

We also consider that the diffusion coefficient

$$\overline{D} \ge \rho > 0,\ b_0 \ge 0. \tag{1.28}$$

$$0 \le \mu(r) \text{ and } \mu(0) = 0. \tag{1.29}$$

For the simplicity in writing we shall no longer indicate the ”^{0}” symbol.

2. The existence, uniqueness and properties of the solution

We study the existence of solutions for the approximation of order ”0” system. The mathematical formulation of the problem falls within the equations‘ evolution theory based on m-accretive operators.

2.1. **Functional framework.**

We set $\Omega = (0, L)$ and we consider the spaces $V = H^1(\Omega)$; $H = L^2(\Omega)$ with $V \subset H \subset V'$ (V' the dual of V). V is endowed with the norm $\|\psi\|_V^2 = \left\|\frac{\partial \psi}{\partial x}\right\|^2 + \|\psi\|^2$. We mention that by $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the scalar product and respectively the norm in $L^2(\Omega)$. We denote the value of $g \in V'$ in $\psi \in V$ with $g(\psi) = \langle g, \psi\rangle_{V',V}$, representing duality between V' and V.

We define the linear operator $A_0 : V \to V'$ by

$$\langle A_0 b, \psi\rangle_{V',V} = \int_\Omega \left[\overline{D}\frac{\partial b}{\partial x}\frac{\partial \psi}{\partial x} - f(b)\,\psi\right] dx \tag{2.1}$$

$$= \int_\Omega \left[\overline{D}\frac{\partial b}{\partial x}\frac{\partial \psi}{\partial x} + b\mu(b)\,\psi\right] dx \tag{2.2}$$

for all $\psi \in V$ and the operator $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$Ab = A_0 b, \tag{2.3}$$

for all $b \in D(A)$ where

$$D(A) = \{b \in V,\ A_0 b \in V'\}.$$

Everywhere in the following we shall use the standard notation for the Sobolev spaces on Ω. Moreover $W^{1,2}(0,T;H) = \left\{ b \in L^2(0,T;H);\ \frac{db}{dt} \in L^2(0,T;H) \right\}$, where $\frac{db}{dt}$ is in sense of distributions. Recall that any $b \in W^{1,2}(0,T;H)$ is absolutely continuous on $[0,T]$ and $\frac{db}{dt}$ exists a.e. on $(0.T)$. We may notice that $b_0 \in H$ and $b \in C([0,T];H)$.

So, we get the next Cauchy problem:

$$\frac{db}{dt}(t) + Ab(t) = 0, \text{ a.e. } t \in (0,T), \tag{2.4}$$

$$b(0) = b_0. \tag{2.5}$$

Note that if b is a strong solution [1] for the problem (2.4)-(2.5), then $b^0(t,x) := b(t)(x)$ satisfies (1.24) in the sense of distributions. Indeed, if $f \in L^2(\Omega)$, then (2.4) can be written as:

$$\int_\Omega \left[\frac{\partial b}{\partial t} + \overline{D} \frac{\partial b}{\partial x} \frac{\partial \psi}{\partial x} - f(b)\,\psi \right] dx = 0, \tag{2.6}$$

a.e. $t \in (0,T)$ and for all $\psi \in L^2(\Omega)$. After some calculations we get

$$\int_\Omega \left[\frac{\partial b}{\partial t}\psi - \overline{D} \frac{\partial^2 b}{\partial x^2} - f(b) \right] \psi dx = 0,$$

which implies that (1.24) is satisfied in sense of distributions on $(0,T) \times (0,L)$.

Conversely, starting from (1.24), with $b^0(t,x) := b(t)(x)$, multiplying by ψ and integrating on Ω we get

$$\int_\Omega \frac{\partial b}{\partial t}\psi dx - \overline{D}\frac{\partial b}{\partial x}\psi \,\Big|_0^L + \int_\Omega \overline{D} \frac{\partial b}{\partial x}\frac{\partial \psi}{\partial x} dx - \int_\Omega f(b)\,\psi dx = 0. \tag{2.7}$$

Using (1.26) we get $\overline{D}\frac{\partial b}{\partial x}\psi \,\Big|_0^L = 0$, hence the above equality implies (2.6), therefore (2.4) is satisfied.

On the other hand, the initial condition (2.5) corresponds to (1.24).

We define the function $b \longrightarrow E(b) \equiv \mu(b)b$ from $L^2(\Omega)$ to $L^2(\Omega)$ and we prove the existence of the solution in two step. In the first one we suppose that the function $b \longrightarrow E(b)$ is globally Lipschitz on

$L^2(\Omega)$ and we prove that in this case the problem (2.4) has a unique solution. In the second step we show the existence of the solution of problem (2.4) while the function $b \longrightarrow E(b)$ is locally Lipschitz [6].

In order to prove the existence of a solution to problem (2.4), we are going to show the quasi m-accretiveness of the operator A.

Assume that for every $R > 0$, there exist $C(R) > 0$ such that

$$\left|\mu(b) - \mu(\overline{b})\right| \leq C\left(R\right)\left\|b - \overline{b}\right\| \tag{2.8}$$

a.e. on Ω, whenever b, $\overline{b} \in H$ with $\|b\| \leq R$ and $\left\|\overline{b}\right\| \leq R$. The next lemma will be used in the second part of the demonstration.

Lemma 1. *Assume that (2.8) is satisfies. Then for any $R > 0$, there exist $M(R) > 0$ such that if $\|b\| \leq R$ and $\left\|\overline{b}\right\| \leq R$, then*

$$\left\|E(b) - E(\overline{b})\right\| \leq M(R)\left\|b - \overline{b}\right\|. \tag{2.9}$$

Proof. Let b, $\overline{b} \in L^2(\Omega)$ such that $\|b\| \leq R$ and $\left\|\overline{b}\right\| \leq R$. Then

$$\begin{aligned}
\left|E(b) - E(\overline{b})\right| &= \left|\mu(b,\cdot)b - \mu(\overline{b},\cdot)\overline{b}\right| \\
&= \left|\mu(b,\cdot)b + \mu(b,\cdot)\overline{b} - \mu(b,\cdot)\overline{b} - \mu(\overline{b},\cdot)\overline{b}\right| \\
&\leq |\mu(b,\cdot)|\left|b - \overline{b}\right| + \left|\overline{b}\right|\left|\mu(b,\cdot) - \mu(\overline{b},\cdot)\right| \qquad (2.10)\\
&\leq C\left(R\right)\|b\|\left|b - \overline{b}\right| + \left|\overline{b}\right| C\left(R\right)\left\|b - \overline{b}\right\|. \qquad (2.11)
\end{aligned}$$

Then:

$$\left\|E(b) - E(\overline{b})\right\| \leq M(R)\left\|b - \overline{b}\right\|,$$

where $M(R) = 2RC(R)$. ∎

In this section we will assume that $b \to E\left(b\right)$ is Lipschitz continuous on $L^2(\Omega)$, i.e., there exist $M > 0$, such that for b, $\overline{b} \in L^2(\Omega)$ we have:

$$\left\|E(b) - E(\overline{b})\right\| \leq M\left\|b - \overline{b}\right\|. \tag{2.12}$$

We first prove the quasi-accretiveness of the operator A.

Lemma 2. *Assume (1.28)- (1.29) and the additional condition (2.12). Then the operator A is quasi-accretive on $L^2(\Omega)$.*

Proof. We have to prove that for $\lambda > 0$ sufficiently large the operator $\lambda I + A$ is accretive, namely:

$$\left(\left(\lambda I + A\right)b\left(t\right) - \left(\lambda I + A\right)\overline{b}\left(t\right), b\left(t\right) - \overline{b}\left(t\right)\right) \geq 0,$$

for b, $\overline{b} \in D(A)$. We have

$$\begin{aligned}
&\left((\lambda I + A)\, b\,(t) - (\lambda I + A)\, \overline{b}\,(t)\,, b\,(t) - \overline{b}\,(t)\right) = \\
&= \left(\lambda b\,(t) + Ab\,(t) - \lambda \overline{b}\,(t) - A\overline{b}\,(t)\,, b\,(t) - \overline{b}\,(t)\right) \\
&= \lambda \left(b\,(t) - \overline{b}\,(t)\,, b\,(t) - \overline{b}\,(t)\right) + \left(Ab\,(t) - A\overline{b}\,(t)\,, b\,(t) - \overline{b}\,(t)\right) \\
&= \lambda \left\|b\,(t) - \overline{b}\,(t)\right\|^2 + \int_\Omega \overline{D} \left(\frac{\partial b}{\partial x} - \frac{\partial \overline{b}}{\partial x}\right)\left(\frac{\partial b}{\partial x} - \frac{\partial \overline{b}}{\partial x}\right) dx \\
&+ \int_\Omega \left[E\,(b) - E\left(\overline{b}\right)\right]\left(b - \overline{b}\right) dx \\
&\geq \lambda \left\|b\,(t) - \overline{b}\,(t)\right\|^2 - M \left\|b\,(t) - \overline{b}\,(t)\right\|^2 \\
&= (\lambda - M) \left\|b\,(t) - \overline{b}\,(t)\right\|^2 \geq 0,
\end{aligned}$$

for λ large enough, $\lambda \geq \lambda_0 \geq M$. In conclusion, the operator $(\lambda I + A)$ is accretive, so the operator A is quasi- accretive. ∎

Lemma 3. *Assume the same conditions as in Lemma 2. Then A is quasi m-accretive on $L^2(\Omega)$.*

Proof. We know that the operator A is quasi-accretive. It remains to show that $R(\lambda I + A) = L^2(\Omega)$, for λ large enough. This is equivalent to show that for all $g \in L^2(\Omega)$, there exists $b \in D(A)$ so that $(\lambda I + A)b = g$. We show that this equation has a unique solution through a fixed point theorem. We fix $\omega \in L^2(\Omega)$ and we study the associated Cauchy problem:

$$(\lambda I + A_V)\, v = g - E(\omega) \tag{2.13}$$

where $A_V : V \to V'$ is defined by:

$$\langle A_V v, \psi\rangle_{V',V} = \int_\Omega \overline{D} \frac{\partial v}{\partial x}\frac{\partial \psi}{\partial x} dx,$$

for all $\psi \in V$.

In order to prove that the problem (2.13) has a solution we use Lax-Milgram lemma. To come to this end we show that $(\lambda I + A_V)$ is coercive for $\lambda > \lambda_0$ and bounded.

$$\begin{aligned}
\langle(\lambda I + A_V)\, v, v\rangle_{V',V} &= \lambda \langle v, v\rangle_{V',V} + \langle A_V v, v\rangle_{V',V} \\
&= \lambda \|v\|^2 + \int_\Omega \overline{D} \frac{\partial v}{\partial x}\frac{\partial v}{\partial x} dx \\
&\geq \lambda \|v\|^2 + \rho \left\| \frac{\partial v}{\partial x} \right\|^2 \\
&= (\lambda - \rho) \|v\|^2 + \rho \|v\|_V^2 \,,\ \forall v \in V,
\end{aligned}$$

and so the operator $(\lambda I + A_V)$ is coercive and bounded since

$$\|A_V v\|_{V'} = \sup_{\|\psi\|_V \leq 1} \left| \langle A_V v, \psi\rangle_{V',V} \right| \leq \|v\|_V \,.$$

So, the operator $A_V \in V'$ is surjective. Furthermore, $E(\omega) \in L^2(\Omega) \subset V'$ and there exists $v \in V$ so $(\lambda I + A_V)\, v = g - E(\omega)$, for all $\omega \in L_2(\Omega)$. Let us set two solutions v and $\overline{v}$ corresponding to ω and $\overline{\omega}$. We have the equations:

$$\begin{aligned}
(\lambda I + A_V)\, v &= g - E(\omega), \\
(\lambda I + A_V)\, \overline{v} &= g - E(\overline{\omega}).
\end{aligned}$$

Let us multiply scalarly by $(v - \overline{v})$ their difference

$$\lambda (v - \overline{v}, v - \overline{v}) + (A_V (v - \overline{v}), v - \overline{v}) + (E(\omega) - E(\overline{\omega}), v - \overline{v}) = 0$$

and we have:

$$\begin{aligned}
&\lambda \|v - \overline{v}\|^2 + \int_\Omega \overline{D} \left(\frac{\partial v}{\partial x} - \frac{\partial \overline{v}}{\partial x} \right)^2 dx \\
&+ \int_\Omega (E(\omega) - E(\overline{\omega}))\, (v - \overline{v})\, dx = 0.
\end{aligned} \tag{2.14}$$

But $A_V v = g - E(\omega) - \lambda v \in L^2(\Omega)$.

$$\|E(\omega)\|^2 \leq M^2 \|\omega\|^2 \leq C,$$

and from (2.14) we get:

$$\|v - \overline{v}\|^2 \leq \frac{M}{\lambda} \|\omega - \overline{\omega}\|^2 ,$$

for λ large enough, $\lambda > \max [M, 1]$.

Denoting by $\phi : L^2(\Omega) \to L^2(\Omega)\,,\ \phi(\omega) = v \in L^2(\Omega)$ a solution of equation (2.13). Since $\|v - \overline{v}\|_{L^2(\Omega)} \leq \sqrt{\frac{M}{\lambda}} \|\omega - \overline{\omega}\|_{L^2(\Omega)}$ and $\sqrt{\frac{M}{\lambda}} < 1$

for λ large enought, it yields that ϕ is a contraction on $L^2(\Omega)$. By Banach fixed point theorem, ϕ has a fixed point $\phi(\omega) = \omega = v$, and the equation (2.13) has a solution $v \in L^2(\Omega)$ and the operator A is quasi m-accretive on $L^2(\Omega)$. ∎

Now we can formulate the existence theorem for the solutions of problem (2.4)-(2.5).

Theorem 4. *Assume (1.28)- (1.29), (2.12) and let $b_0 \in D(A)$. Then problem (2.4)-(2.5) has a unique strong solution*

$$b \in C([0,T], L^2(\Omega)) \cap L^2(0,T; D(A)), \tag{2.15}$$

which satisfies the estimates:

$$\left\| b(t) - \overline{b}(t) \right\| \leq \left\| b_0 - \overline{b_0} \right\| e^{Mt}, \tag{2.16}$$

$$\|b(t)\|^2 + 2\overline{D} \int_0^t \left\| \frac{\partial b}{\partial x} \right\|^2 d\tau \leq \|b_0\|^2, \tag{2.17}$$

for any $t \in [0,T]$, where $\overline{b}$ is another solution of (2.4), with $\overline{b}(0) = \overline{b_0}$.

Proof. As the operator A is quasi m-accretive in $L^2(\Omega)$, by Lemma 2 and Lemma 3, the existence of the solution and (2.15) follows immediately from the fundamental theory of the existence of solution for evolution equations with m-accretive operators in Hilbert spaces [1].

The first inequality is an immediate consequence of the quasi-accretiveness of operator A. Suppose two solutions b and $\overline{b}$ corresponding to the initial data b_0 and $\overline{b_0}$ and multiplying the equation:

$$\frac{d}{dt}(b - \overline{b}) + Ab - A\overline{b} = 0$$

with $(b - \overline{b})$, we get:

$$\frac{1}{2}\frac{d}{dt}\left\| b(t) - \overline{b}(t) \right\|^2 + \left(Ab(t) - A\overline{b}(t), b(t) - \overline{b}(t) \right) = 0.$$

We integrate on $(0,T)$:

$$\frac{1}{2}\int_0^t \frac{d}{d\tau}\left\| b(t) - \overline{b}(t) \right\|^2 d\tau +$$

$$\int_0^t \left(Ab(t) - A\overline{b}(t), b(t) - \overline{b}(t) \right) d\tau = 0.$$

But $\left(Ab\left(t\right)-A\overline{b}\left(t\right),b\left(t\right)-\overline{b}\left(t\right)\right)\geq -M\left\|b\left(t\right)-\overline{b}\left(t\right)\right\|^{2}$ and so

$$\frac{1}{2}\left\|b\left(t\right)-\overline{b}\left(t\right)\right\|^{2}-\frac{1}{2}\left\|b_{0}-\overline{b_{0}}\right\|^{2}-M\int\limits_{0}^{t}\left\|b\left(\tau\right)-\overline{b}\left(\tau\right)\right\|^{2}d\tau\leq 0$$

$$\left\|b\left(t\right)-\overline{b}\left(t\right)\right\|^{2}\leq\left\|b_{0}-\overline{b_{0}}\right\|^{2}+2M\int\limits_{0}^{t}\left\|b\left(\tau\right)-\overline{b}\left(\tau\right)\right\|^{2}d\tau. \tag{2.18}$$

Next we apply Gronwall's lemma and we get (2.16)

$$\left\|b\left(t\right)-\overline{b}\left(t\right)\right\|\leq\left\|b_{0}-\overline{b_{0}}\right\|e^{Mt},\forall t\in[0,T].$$

To obtain (2.17), we multiply (2.4) by b and integrate over $(0,T)$. We get:

$$\left\|b(t)\right\|^{2}-\left\|b_{0}\right\|^{2}+2\int\limits_{0}^{t}\left(Ab\left(\tau\right),b\left(\tau\right)\right)d\tau=0 \tag{2.19}$$

and from here we have (2.17)

$$\left\|b(t)\right\|^{2}+2\overline{D}\int\limits_{0}^{t}\left\|\frac{\partial b}{\partial x}\right\|^{2}d\tau\leq\left\|b_{0}\right\|^{2}.$$

∎

Once completed the first stage, we proceed to demonstrate the existence of solution of the problem (2.4) while the function $b\longrightarrow E(b)$ is locally Lipschitz. In fact we use Theorem 4.

Theorem 5. *Assume conditions (1.28)- (1.29) and let $b_0\in D(A)$. Then the problem (2.4)-(2.5) has a unique solution*

$$b\in W^{1,2}\left(0,T;L^{2}\left(\Omega\right)\right)\cap C([0,T],L^{2}(\Omega))\cap L^{2}\left(0,T;V\right), \tag{2.20}$$

such that

$$\left\|b(t)\right\|^{2}+2\rho\int\limits_{0}^{t}\left\|\frac{\partial b}{\partial x}\right\|^{2}d\tau\leq\left\|b_{0}\right\|^{2}, \tag{2.21}$$

for any $t\in[0,T]$.

Proof. From Lemma 1 we know that $b\longrightarrow E(b)\equiv\mu\left(b\right)b$ is locally Lipschitz on $L^{2}\left(\Omega\right)$. We reduce the problem to the previous case in which this function is Lipschitz continuous. For this we approximate the function $E(\cdot)$ for each $N \geq 1$ setting:

$$E_N(b) = \begin{cases} E(b) & , \ \|b(t)\| \leq N \\ E\left(\dfrac{Nb}{\|b\|_{L^2(\Omega)}}\right), & \|b(t)\| > N \end{cases}. \tag{2.22}$$

Actually this truncated functions is Lipschitz continous on $L^2(\Omega)$ for each N fixed.

We consider the approximating problem:

$$\frac{\partial b_N}{\partial t}(t) + A_N b_N(t) = 0, \text{ a.e } t \in (0, T), \tag{2.23}$$

$$b_N(0) = b_0, \tag{2.24}$$

where A_N is defined by (2.3)-(2.12) in which $E(b)$ is replaced by $E_N(b)$. For each N, the assumptions of theorem 4 are fulfilled. So we find that for $b_0 \in D(A)$, the problem (2.23)-(2.24) has a unique solution $b_N \in C([0,T], L^2(\Omega)) \cap L^2(0,T;V)$ which satisfies (2.17):

$$\|b_N(t)\|^2 + \int_0^t \left\|\frac{\partial b_N(\tau)}{\partial x}\right\|^2 d\tau \leq R = \|b_0(\tau)\|^2 < \infty.$$

For N large enough, $N > R$ we get $A_N b_N(t) = A b_N(t)$, so that $b_N(t)$ is a solution to problem (2.4)-(2.5).

To prove the uniqueness we consider two solutions b and $\bar{b}$ corresponding to b_0. By the previous proof we have that if

$$N > \sup_{t\in[0,T]} \|b\|_{L^2(\Omega)} + \sup_{t\in[0,T]} \|\bar{b}\|_{L^2(\Omega)}$$

then $b(t) = b_N(t)$ and $\bar{b}(t) = b_N(t)$, where $b_N(t)$ is the solution to (2.23)-(2.24). This proves the uniqueness of solution. ∎

References

[1] Barbu V., Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff International Publishing, Leyden, 1976

[2] Fasano A. , Giorni D., On a one- dimensional problem related to bioremediation of polluted soils, Advances in Mathematical Sciences and Applications, vol. 14, no. 2 (2004), pp. 443-455, Tokyo, Japan

[3] Fasano A., Petracco M., Talamucci F. , The espresso-coffe problem in complex flows in industrial processes, A. Fasano ed., MSSET Birkhäuser, Chapt. 8 (2000) 241-280

[4] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \triangle u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sect. I, Part 2, 13 (1966), 109-124

[5] H. Fujita, On some non existence and non uniqueness theorems for nonlinear parabolic equations, Proceding of Amer Math. Symposionum on Nonlinear Functional Analysis, Chicago, April 1968

[6] Caterina Cusulin, Mimmo Iannelli, Gabriela Marinoschi, Age-strucutured diffusion in a multi-layer environment, Nonlinear Analysis Real World Applications, 6, 1, 207-223, 2005

[7] Tindall M., Porter S., Maini P., Armitage J., Overview of mathematical approaches used to model bacterial chemotaxis II: Bacterial populations, Bull. Math. Biol., 2008, pp 1570-1607

Vasile Alecsandri" University of Bacau,
Faculty of Sciences,
Department of Mathematics and Informatics,
Bacau, Romania,
e-mail: rardeleanu@ub.ro