

SLIGHTLY CONTINUOUS FUZZY MULTIFUNCTIONS IN MINIMAL STRUCTURES

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Abstract. In this paper we introduce the class of slightly m -continuous fuzzy multifunctions, which is a generalization of the concept called a slightly m -continuous multifunction introduced by Valeriu Popa and Takashi Noiri in the general topology ([7]). At the same time we introduce the concept of fuzzy generalized multifunction and we investigate the properties.

1. PRELIMINARIES

Let Y be an arbitrary non-empty set and the unit interval $J = [0, 1] \subset \mathbf{R}$. A fuzzy set in Y is an application $\lambda : Y \rightarrow [0, 1]$. Denote by $F(Y)$ the class of fuzzy sets in Y . The set Y , known as the Y space, will be identified with the constant function $\mathbf{1}$ and the empty set \emptyset will be identified with the constant function $\mathbf{0}$. Let I an index set and $\{\lambda_i\}_{i \in I}$ a class of fuzzy sets in Y . The union and the intersection of this class, denoted $\bigcup_{i \in I} \lambda_i$ and as $\bigcap_{i \in I} \lambda_i$ are defined by $\left(\bigcup_{i \in I} \lambda_i\right)(y) = \sup_{i \in I} \lambda_i(y)$ respectively $\left(\bigcap_{i \in I} \lambda_i\right)(y) = \inf_{i \in I} \lambda_i(y)$, $(\forall) y \in Y$. If $\lambda, \mu \in F(Y)$, the inclusion denoted as $\lambda \leq \mu$ (or $\lambda \geq \mu$) is defined by $\lambda(y) \leq \mu(y)$ and the equality denoted as $\lambda = \mu$ is defined by $\lambda(y) = \mu(y)$, $(\forall) y \in Y$.

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The complement of $\lambda \in F(Y)$, denoted as λ^c , is defined by $\lambda^c = I - \lambda$, $\lambda^c(y) = 1 - \lambda(y)$, $(\forall)y \in Y$.

In this paper a very important role will be played by the notion called quasi-coincidence (q -coincidence) introduced by Pu Pao-Ming and Liu Ying-Ming ([9]): the sets $\lambda, \mu \in F(Y)$ are said to be quasi-coincident (or q -coincident) if there exists $y \in Y$ so that $\lambda(y) + \mu(y) > 1$ and this is denoted as $\lambda q \mu$; otherwise, we obtain $\lambda(y) + \mu(y) \leq 1$, which is denoted as $\lambda \bar{q} \mu$ ([9]). If λ and μ are q -coincident in $y \in Y$, then $\lambda(y) \neq 0$, $\mu(y) \neq 0$ and consequently $(\lambda \cap \mu)(y) \neq 0$ ([9]).

It is known that $\lambda \leq \mu$ if and only if λ and $I - \mu$ are not q -coincident, denoted by $\lambda \bar{q}(I - \mu)$ ([9]). A fuzzy topology on Y (according to Chang, [3]) is a class $\tau \subseteq F(Y)$ which satisfies the following conditions (or axioms):

(T_1) $\mathbf{0}, \mathbf{1} \in \tau$;

(T_2) if $\delta_i \in \tau, i = \overline{1, n}$, then $\bigcap_{i=1}^n \delta_i \in \tau$, where $\bigcap_{i=1}^n \delta_i = \min_{1 \leq i \leq n} \delta_i$;

(T_3) $\delta_i \in \tau, i \in I$, then $\bigcup_{i \in I} \delta_i \in \tau$.

The couple (Y, τ) is defined as a fuzzy topological space (according to Chang, [3]) abbreviated *f. t. s.* Each element of the τ class is a fuzzy set τ -open and the complement of a τ -open fuzzy set is called a τ -closed fuzzy set. A fuzzy set which is simultaneously τ -open and τ -closed is called a τ -clopen fuzzy set (or clopen set).

The interior and the closure of set $\lambda \in F(Y)$ are defined by (see [3]):

$$Int \lambda = \overset{\circ}{\lambda} = \bigcup \{ \delta \mid \delta \leq \lambda, \delta \in \tau \} \text{ resp. } Cl \lambda = \bar{\lambda} = \bigcap \{ \sigma \mid \lambda \leq \sigma, \sigma^c \in \tau \}.$$

Let (Y, τ) be a *f. t. s.* and $\lambda \in F(Y)$. The set λ is called:

a) F semi-open if $\lambda \leq \bar{\overset{\circ}{\lambda}}$;

b) F regular closed (resp. F regular open) if $\lambda = \bar{\overset{\circ}{\lambda}}$ (resp. $\lambda = \overset{\circ}{\bar{\lambda}}$) ([1]).

The complement of a F semi-open set is said to be F -semi-closed set. The intersection of all F -semi-closed sets of Y containing λ is called the F semi-closure of λ and is denoted by $FsCl \lambda$ or $Fs \bar{\lambda}$.

The union of all F semi-open sets of Y contained in λ is called the F semi-interior of λ and is denoted by $FsInt \lambda$ or $Fs \overset{\circ}{\lambda}$.

2. MINIMAL STRUCTURES

In this section we remind the concept by minimal structure introduced in the general topology by Valeriu Popa and Takashi Noiri, as well as any definitions and lemmas established by these authors (see [5], [6], [7]).

Definition 1. Let X be a nonempty set and $P(X)$ the power set of X . The subfamily $m_X \subseteq P(X)$ is called a minimal structure (briefly m -structure) on X if $\emptyset \in m_X$ and $X \in m_X$. The couple (X, m_X) is said to be a minimal space. Each member of m_X is said to be m_X -open and the complement of a m_X -open is said to be m_X -closed.

Definition 2. Let the minimal space (X, m_X) and A a subset of X . The m_X -closure of A and the m_X -interior of A are defined as follows:

- (1) $m_X - ClA = \cap \{G \mid A \subseteq G, X \setminus G \in m_X\}$,
- (2) $m_X - IntA = \cup \{V \mid V \subseteq A, V \in m_X\}$.

Lemma 1. Let the minimal space (X, m_X) and A, B the subsets of X . The following properties hold:

- (1) $m_X - Cl(X \setminus A) = X \setminus (m_X - IntA)$, $m_X - Int(X \setminus A) = X \setminus (m_X - ClA)$;
- (2) If $(X \setminus A) \in m_X$, then $m_X - ClA = A$ and if $A \in m_X$, then $m_X - IntA = A$;
- (3) $m_X - Cl\emptyset = \emptyset$, $m_X - ClX = X$, $m_X - Int\Phi = \Phi$ and $m_X - IntX = X$;
- (4) If $A \subseteq B$, then $m_X - ClA \subseteq m_X - ClB$ and $m_X - IntA \subseteq m_X - IntB$;
- (5) $A \subseteq m_X - ClA$ and $m_X - IntA \subseteq A$;
- (6) $m_X - Cl(m_X - ClA) = m_X - ClA$ and $m_X - Int(m_X - IntA) = m_X - IntA$.

Lemma 2. Let (X, m_X) be a minimal space and A a subset of X . Then $x \in m_X \setminus ClA$ if and only if $U \cap A = \emptyset$ for every $U \in m_X$ with $x \in U$.

Definition 3. A minimal structure m_X on a nonempty set X is said to have the property **(B)** if the union of any family of subsets belonging to m_X belongs to m_X .

Lemma 3. For a minimal structure m_X on a nonempty set X , the following are equivalent:

- (1) m_X has the property **(B)**;
- (2) If $m_X - IntV = V$, then $V \in m_X$;
- (3) If $m_X - ClF = F$, then $X \setminus F \in m_X$.

Lemma 4. Let X be a nonempty set, m_X a minimal structure on X satisfying **(B)** and A a subset of X . Then the following properties hold:

- (1) $A \in m_X$ if and only if $m_X - IntA = A$;
- (2) A is m_X -closed if and only if $m_X - ClA = A$;
- (3) $m_X - IntA \in m_X$ and $m_X - ClA$ is m_X -closed.

3. SLIGHTLY M -CONTINUOUS FUZZY MULTIFUNCTIONS

In 1985 Papageorgiu ([8]) introduced the notion of a fuzzy multifunction and defined the fuzzy upper inverse and fuzzy lower inverse. In 1991 N. N. Mukherjee and S. Malakar ([4]) have redefined the fuzzy lower inverse of a fuzzy multifunction in terms of the notion of quasi-coincidence of Pu and Liu ([9]).

Definition 4 ([8]). Let (X, t) be a topological space in the classical sense and (Y, τ) be a fuzzy topological space. An application $F : X \rightarrow F(Y)$ is called a fuzzy multifunction.

Remark 1. Obviously, for each $x \in X$, $F(x)$ is a fuzzy set in Y , that $F(x) \in F(Y)$.

Definition 5 ([4]). For a fuzzy multifunction $F : X \rightarrow F(Y)$, the upper inverse $F^+(\lambda)$ and the lower inverse $F^-(\lambda)$ of a fuzzy set $\lambda \in F(Y)$ are defined as follows:

$$F^+(\lambda) = \{x \in X \mid F(x) \leq \lambda\} \subset X \text{ and } F^-(\lambda) = \{x \in X \mid F(x) q \lambda\} \subset X.$$

Lemma 5 ([4]). For a fuzzy multifunction $F : X \rightarrow F(Y)$, we have $F^+(1 - \lambda) = X \setminus F^-(\lambda)$ and $F^-(1 - \lambda) = X \setminus F^+(\lambda)$, for any fuzzy set $\lambda \in F(Y)$.

In this section we introduce the class of slightly m -continuous fuzzy multifunction through

Definition 6. Let the minimal space (X, m_X) and (Y, τ) a *f. t. s.* A fuzzy multifunction $F : X \rightarrow F(Y)$ is said to be:

- a) fuzzy upper slightly m -continuous at a point $x \in X$ if for each fuzzy clopen set $\lambda \in F(Y)$ containing $F(x)$ (i.e. $F(x) \leq \lambda$), there exists $U \in m_X$ containing x such that $F(U) \leq \lambda$;
- b) fuzzy lower slightly m -continuous at a point $x \in X$ if for each fuzzy clopen set $\mu \in F(Y)$ with $F(x) q \mu$, there exists $U \in m_X$ containing x such that $F(u) q \mu$ for each $u \in U$;
- c0 fuzzy upper (resp. fuzzy lower) slightly m -continuous if it has the property at each point $x \in X$.

We introduce now

Definition 7. Let (X, m_X) a minimal space and (Y, τ) a fuzzy topological space. A fuzzy multifunction $F : (X, m_X) \rightarrow F(Y)$ is said to be a fuzzy generalized multifunction.

We have the following theorems of characterization: Theorem 1, Theorem 2 and Corollaries 1 and 2.

Theorem 1. For a generalized multifunction $F : (X, m_X) \rightarrow F(Y)$ the following are equivalent:

- (1) F is fuzzy upper slightly m -continuous;
- (2) $F^+(\mu) = m_X - \text{Int}(F^+(\mu))$ for each μ fuzzy clopen set in (Y, τ) ;
- (3) $F^-(\mu) = m_X - \text{Cl}(F^-(\mu))$ for each μ fuzzy clopen in (Y, τ) .

Proof. (1) \Rightarrow (2); If μ is any fuzzy clopen set in (Y, τ) and $x \in F^+(\mu)$, then $F(x) \leq \mu$ and by the hypothesis, there exists $U \in m_X$ with $x \in U$ such that $F(U) \leq \mu$. Thus $x \in U \leq F^+(\mu)$, next $x \in m_X - \text{Int}(F^+(\mu))$ and we have $F^+(\mu) \leq m_X - \text{Int}(F^+(\mu))$. By Lemma 1, we obtain $F^+(\mu) = m_X - \text{Int}(F^+(\mu))$.
 (2) \Rightarrow (3): If μ is any fuzzy clopen set in (Y, τ) then $1 - \mu$ is fuzzy clopen set in (Y, τ) . By (2), lemma 5 and Lemma 1, we have

$X \setminus F^-(\mu) = F^+(1 - \mu) = m_X - \text{Int}(F^+(1 - \mu)) = X \setminus (m_X \setminus \text{Cl}(F^-(\mu)))$. Then we obtain $F^-(\mu) = m_X - \text{Cl}(F^-(\mu))$.

(3) \Rightarrow (2): This follows from Lemma 5.

(2) \Rightarrow (1). Let $x \in X$ and μ be any fuzzy clopen set in (Y, τ) with $F(x) \leq \mu$. Then $x \in F^+(\mu) = m_X - \text{Int}(F^+(\mu))$ and there exists $U \in m_X$ with $x \in U$ such that $x \in U \leq F^+(\mu)$, next $f(U) \leq \mu$ and hence F is fuzzy upper slightly m -continuous.

Theorem 2. For a fuzzy generalized multifunction $F : (X, m_X) \rightarrow F(Y)$ the following are equivalent:

- (1) F is fuzzy lower slightly m -continuous;
- (2) $F^-(\mu) = m_X - \text{Int}(F^-(\mu))$ for each μ fuzzy clopen in (Y, τ) ;
- (3) $F^+(\mu) = m_X - \text{Cl}(F^+(\mu))$ for each μ fuzzy clopen in (Y, τ) .

Proof. (1) \Rightarrow (2): Let μ a fuzzy clopen set in (Y, τ) and $x \in F^-(\mu)$. Then $F(x) q \mu$ and by the hypothesis there exists $U \in m_X$ with $x \in U$ such that $F(u) q \mu$ for each $u \in U$. Then we have $U \leq F^-(\mu)$ and by Lemma 1, $F^-(\mu) = m_X - \text{Int}(F^-(\mu))$.

(2) \Rightarrow (3): If μ is fuzzy clopen in (Y, τ) , then $1 - \mu$ is fuzzy clopen in (Y, τ) . It is known that $X \setminus F^+(\mu) = F^-(1 - \mu)$ (see Lemma 5) and by (2) we have $X \setminus F^+(\mu) = F^-(1 - \mu) = m_X - \text{Int}(F^-(1 - \mu)) = X \setminus (m_X - \text{Cl}(F^+(\mu)))$ and hence we obtain $F^+(\mu) = m_X - \text{Cl}(F^+(\mu))$.

(3) \Rightarrow (1): Let $x \in X$ and μ any fuzzy clopen set in (Y, τ) such that $F(x) q \mu$. Then $x \in F^-(\mu)$ and hence $x \notin X \setminus F^-(\mu) = F^+(1 - \mu)$. By the hypothesis (3), we have $x \notin m_X - \text{Cl}(F^+(1 - \mu))$ and by Lemma 2, there exists $U \in m_X$ with $x \in U$ such that $U \cap F^+(1 - \mu) = \emptyset$ and $U \subseteq F^-(\mu)$. Therefore $F(u) q \mu$ for each $u \in U$ and hence F is fuzzy lower slightly m -continuous.

Let (X, m_X) a minimal space, where m_X has property **(B)**, (Y, τ) is a fuzzy topological space and $F : (X, m_X) \rightarrow F(Y)$ is any fuzzy generalized multifunction. Then the following corollary holds:

Corollary 1. For a fuzzy generalized multifunction the following are equivalent:

- (1) F is fuzzy upper slightly m -continuous;

- (2) $F^-(\mu)$ is m_X -open in X for each μ fuzzy clopen in (Y, τ) ;
- (3) $F^+(\mu)$ is m_X -closed in X for each μ fuzzy clopen in (Y, τ) .

We introduce here the notion called a fuzzy extremely disconnected space (briefly *F. E. D.*, after [7]).

Definition 8. A fuzzy topological space (Y, τ) is said to be a fuzzy extremely disconnected if the closure (in the sense of Chang) of each τ -open set is τ -open set.

If (Y, τ) is a extremely disconnected, we obtain the following theorems of characterization for a fuzzy generalized multifunction, Theorems 3 and 4.

Theorem 3. For a fuzzy generalized multifunction $F : (X, m_X) \rightarrow F(Y)$, where (Y, τ) is *F. E. D.* the following are equivalent:

- (1) F is fuzzy upper slightly m -continuous;
- (2) $m_X - Cl(F^-(\mu)) \subseteq F^-(Cl\mu)$ for every set $\mu \in \tau$;
- (3) $F^+(Int\sigma) \subseteq m_X - Int(F^+(\sigma))$ for every set $\sigma \in F(Y)$, where $\sigma^c \in \tau$.

Proof. (1) \Rightarrow (2): Let $\mu \in F(Y)$, $\mu \in \tau$, hence $Cl\mu$ is clopen set. Obviously $F^-(\mu) \subseteq F^-(Cl\mu)$ and by Theorem 1 (3), $F^-(Cl\mu) = m_X - Cl(F^-(Cl\mu))$. By Lemma 1 we have $m_X - Cl(F^-(\mu)) \subseteq m_X - Cl(F^-(Cl\mu)) = F^-(Cl\mu)$, hence $m_X - Cl(F^-(\mu)) \subseteq F^-(Cl\mu)$.

(2) \Rightarrow (3): Let $\sigma \in F(Y)$ with $\sigma^c \in \tau$ and $\mu = 1 - \sigma$. Obviously $\mu \in \tau$ and by lemma 1, we have $X \setminus (m_X \setminus Int(F^+(\sigma))) = m_X - Cl(X \setminus F^+(\sigma)) = m_X - Cl(F^-(1 - \sigma)) \subseteq F^-(Cl(1 - \sigma)) = F^-(1 - Int\sigma) = X \setminus F^+(Int\sigma)$ (see and Lemma 5). Therefore $F^+(Int\sigma) \subseteq m_X - Int(F^+(\sigma))$.

(3) \Rightarrow (1): Let $x \in X$ and $\mu \in F(Y)$, clopen fuzzy set with $F(x) \leq \mu$. By hypothesis (3) we have $x \in F^+(\mu) = F^+(Int\mu) \subseteq m_X - Int(F^+(\mu))$. Then, there exists $U \in m_X$ such that $x \in U \subseteq F^+(\mu)$, hence $F(U) \leq \mu$, that is F is fuzzy upper slightly m -continuous.

Theorem 4. For a fuzzy generalized multifunction $F : (X, m_X) \rightarrow F(Y)$, where (Y, τ) is *F. E. D.* the following are equivalent:

- (1) F is fuzzy lower slightly m -continuous;
- (2) $m_X - Cl(F^+(\mu)) \subseteq F^+(Cl\mu)$ for every set $\mu \in \tau$;
- (3) $F^-(Int\sigma) \subseteq m_X - Int(F^-(\sigma))$ for every set $\sigma \in F(Y)$, with $\sigma^c \in \tau$.

Proof. The proof is similar to that of Theorem 3 and Theorem 2.

Definition 9. Let (X, m_X) a minimal space and (Y, τ) a *f. t. s.* A fuzzy generalized multifunction $F : (X, m_X) \rightarrow F(Y)$ is said to be:

- a) upper weakly m -continuous if for each point $x \in X$ and each fuzzy set $\delta \in \tau$ with $F(x) \leq \delta$, there exists $U \in m_X$ such that $F(U) \leq \bar{\delta}$;
- b) lower weakly m -continuous if for each point $x \in X$ and each fuzzy set $\delta \in \tau$ such $F(x) q \delta$, there exists $U \in m_X$ with $x \in U$ such that $F(u) q \bar{\delta}$ for each $u \in U$.

Now we investigate the relationships between upper/lower slightly m -continuous fuzzy multifunctions and the notions of Definition 9.

Theorem 5. If a fuzzy generalized multifunction $F : (X, m_X) \rightarrow F(Y)$ is upper weakly m -continuous, then is upper slightly m -continuous.

Proof. Let $x \in X$ and $\delta \in F(Y)$, clopen set with $F(x) \leq \delta$. By hypothesis, there exists $U \in m_X$ with $x \in U$ such that $F(U) \leq \bar{\delta} = \delta$, therefore F is upper slightly m -continuous.

Theorem 6. If a fuzzy generalized multifunction $F : (X, m_X) \rightarrow F(Y)$ is lower weakly m -continuous, then F is lower slightly m -continuous.

Proof. Let $x \in X$ and $\delta \in F(Y)$, clopen set such $F(x) q \delta$. By hypothesis, there exists $U \in m_X$ with $x \in U$ such that $F(u) q \bar{\delta}$ for each $u \in U$, therefore F is lower slightly m -continuous.

Let (X, t) a topological space in the classical sense, (Y, τ) a fuzzy topological space and $F : (X, t) \rightarrow F(Y)$ a fuzzy multifunction.

Definition 10 ([4]). The fuzzy multifunction F is said to be:

- a) upper weakly continuous if for each point $x \in X$ and each fuzzy set $\delta \in \tau$ with $F(x) \leq \delta$, there exists $U \in t$ with $x \in U$ such that $F(x) \leq \bar{\delta}$;
- b) lower weakly continuous if for each point $x \in X$ and each fuzzy set $\delta \in \tau$ such that $F(x) \not\leq \delta$, there exists $U \in t$ with $x \in U$ such that $F(u) \not\leq \delta$ for each $u \in U$.

Let (X, t) be a topological space and (Y, τ) a fuzzy topological space.

Theorem 7. If the fuzzy multifunction $F : (X, m_x) \rightarrow F(Y)$ is upper weakly continuous, then $F^+(\mu) \in t$ for every fuzzy clopen set $\mu \in F(Y)$.

Proof. It follows easy from Corollary 1 and Theorem 5.

Theorem 8. If the fuzzy multifunction $F : (X, t) \rightarrow F(Y)$ is lower weakly continuous, then $F^-(\mu) \in t$ for every fuzzy clopen set $\mu \in F(Y)$.

Proof. It follows easy from Corollary 2 and Theorem 6.

Through analogy with Definition 5.5 ([7]) we introduce here

Definition 11. Let (X, m_x) a minimal space and $A \subset X$. The m_x -frontier of A , denoted by $m_x - FrA$, is defined by $m_x - FrA = m_x \setminus ClA \cap m_x \setminus Cl(X \setminus A) = m_x \setminus Cl(m_x \setminus IntA)$.

Then, is true

Theorem 9. Let (X, m_x) a minimal space, (Y, τ) a fuzzy topological space and $F : (X, m_x) \rightarrow F(Y)$ a fuzzy generalized multifunction. The set of all points $x \in X$ at which the fuzzy generalized multifunction F is not upper/lower slightly m -continuous is identical with the union of m_x -frontiers of the upper/lower inverse images of clopen sets containing (resp. meeting) $F(x)$.

REFERENCES

- [1]. AZZAD, K K., **On a fuzzy Semicontinuity, Fuzzy Almost Continuity and Fuzzy Weakly Continuity**, Journal of Mathematical Analysis and Applications, No. 82, 1981, pp. 14- 32.

- [2]. BRESCAN, M., **Sur quelques formes faibles de continuité pour les multifunctions floues**, Studii și cercetări științifice, Seria: Matematică, Nr. 12, 2002, pp. 13-30, Universitatea din Bacău.
- [3]. CHANG, C. L., **Fuzzy Topological Spaces**, Journal of Mathematical Analysis and Applications, 43 (1973), pp. 734-742.
- [4]. MUKHERJEE, M. N., MALAKAR, S., **On Almost Continuous and Weakly Continuous Multifunctions**, Fuzzy Sets and Systems, Elsevier Science Publishers B. V. North Holland, pp. 114-125, 1991.
- [5]. NOIRI, T., POPA, V., **On M-Continuous Functions**, Analele Universității “Dunărea de Jos” Galați, Fascicola II, XVIII (2000), pp. 31-41.
- [6]. NOIRI, T., POPA, V., **On the Definitions of Some generalized Forms of Continuity under Minimal Conditions**, Memoirs Faculty of Sciences Kochi University, Series A, Mathematics, 22, 2001.
- [7]. NOIRI, T., POPA, V., **Slightly m-Continuous Multifunctions**, Bulletin of the Institute of Mathematics, Academica Sinica (new Series), vol. 1, No. 4, pp. 485-505, December, 2006.
- [8]. PAPAGEORGIOU, N. S., **Fuzzy Topology and Fuzzy Multifunctions**, Journal Math. Anal. Appl. 109 (1985), pp. 397-425.
- [9]. PAO-MING PU, YING-MING LIU, **Fuzzy Topology I**, Journal Math. Anal. Appl., 76, pp. 571-599, 1980.
- [10]. TUNA, H., YALVAS, S., **Fuzzy Sets and Functions of Fuzzy Spaces**, Journal Math. Anal. Appl. 126, 1987, pp. 409-423.

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