"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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MULTIPLE SOLUTIONS FOR A CLASS OF NONLINEAR EQUATIONS VIA THE MOUNTAIN PASS THEOREM

JENICĂ CRÎNGANU

Abstract. The aim of this paper is to study the existence of the multiple solutions for the abstract equation

$$J_p u = N_f u,$$

where J_p is the duality mapping on a real reflexive and smooth Banach space X, corresponding to the gauge function $\varphi(t) = t^{p-1}$, $1 < p < \infty$.

It is assumed that X is compactly imbedded in $L^q(\Omega)$, where Ω is a bounded domain in $R^N, N \geq 2$, $1 < q < p^*$, p^* being the Sobolev conjugate exponent, $N_f : L^q(\Omega) \to L^{q'}(\Omega), \frac{1}{q} + \frac{1}{q'} = 1$, being the Nemytskii operator generated by a Caratheodory function $f : \Omega \times R \to R$ which satisfies some appropriate conditions.

In order to prove the existence of the multiple solutions we use a multiple variant of the the Mountain Pass theorem.

1. Introduction

In this paper we study the existence of multiple solutions for the abstract equation

$$J_p u = N_f u \tag{1.1}$$

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in the following functional framework:

(H_1) $1 < p < \infty$;

(H_2) X is a real reflexive and smooth Banach space, compactly imbedded in $L^q(\Omega)$, where $\Omega \subset R^N, N \geq 2$, is a bounded domain with smooth boundary and

$$1 \ < q < p^* = \begin{cases} \frac{Np}{N-p} & \text{if} \quad N > p \\ +\infty & \text{if} \quad N \leq p \end{cases} \quad ;$$

(H_3) $J_p : X \to X^*$ is the duality mapping corresponding to the gauge function
$\varphi(t) = t^{p-1}, t \geq 0$;
We assume that J_p is continuous and satisfies the (S_+) condition: if $u_n \rightharpoonup u$ (weakly) in X and $\limsup_{n\to\infty} \langle J_p u_n, u_n - u \rangle \leq 0$ then $u_n \to u$ (strongly) in X;

(H_4) $N_f : L^q(\Omega) \to L^{q'}(\Omega)$, where $\frac{1}{q} + \frac{1}{q'} = 1$, defined by $(N_f u)(x) = f(x, u(x))$, for $u \in L^q(\Omega)$, $x \in \Omega$, is the Nemytskii operator generated by the Caratheodory function $f : \Omega \times R \to R$, which satisfies the growth condition

$$|f(x,s)| \leq c(|s|^{q-1} + 1), \text{ for } x \in \Omega, s \in R, \tag{1.2}$$

whith $c > 0$ a constant.

The following two problems are the departure point in considering the abstract equation (1.1). The first is a Dirichlet problem :

$$-\Delta_p u = f(x,u) \quad \text{in} \quad \Omega, \tag{1.3}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{1.4}$$

where $1 < p < +\infty$, $\Omega \subset R^N, N \geq 2$, is a bounded domain with smooth boundary,
$\Delta_p u = \mathrm{div}(|\nabla u|^{p-2} \nabla u) = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right)$, is the so-called p-Laplacian and
$f : \Omega \times R \to R$ is a Carathéodory function which satisfies the growth condition (1.2).

By solution of the problem (1.3),(1.4) we mean an element $u \in W_0^{1,p}(\Omega)$ which satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx = \int_\Omega f(x,u) v \, dx \quad \text{for all} \quad v \in W_0^{1,p}(\Omega). \tag{1.5}$$

On the other hand, if $W_0^{1,p}(\Omega)$ is endowed with the norm $\| u \|_{1,p} = \| \,|\nabla u|\, \|_{0,p}$, the duality mapping corresponding to the gauge function $\varphi(t) = t^{p-1}$ is exactly $-\Delta_p$:

$$J_p = -\Delta_p : W_0^{1,p}(\Omega) \to \left(W_0^{1,p}(\Omega)\right)^* = W^{-1,p'}(\Omega),\ \frac{1}{p} + \frac{1}{p'} = 1,$$

$$\langle -\Delta_p u, v\rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \, dx \quad \text{for all} \quad u, v \in W_0^{1,p}(\Omega)$$

(see e.g.[7] or [11]).
It is easy to see that $u \in W_0^{1,p}(\Omega)$ is a solution of the problem (1.3), (1.4) in the sense of (1.5) if and only if u is a solution of the operator equation

$$J_p u = N_f u.$$

So the problem (1.3), (1.4) is a particular case of the abstract equation (1.1), corresponding to $X = \left(W_0^{1,p}(\Omega), \|\cdot\|_{1,p}\right)$.

The second is a Neumann problem (see e.g. [4]):

$$-\Delta_p u + |u|^{p-2} u = f(x,u) \quad \text{in} \quad \Omega, \tag{1.6}$$

$$|\nabla u|^{p-2} \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.7}$$

The hypothesis about f are the same as for the preceding Dirichlet problem. By solution of the problem (1.6), (1.7) we mean an element $u \in W^{1,p}(\Omega)$ which satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx + \int_\Omega |u|^{p-2} uv dx = \int_\Omega f(x,u) v dx \quad \text{for all} \quad v \in W^{1,p}(\Omega). \tag{1.8}$$

Now we define on $W^{1,p}(\Omega)$ a new equivalent norm

$$\||u\||_{1,p}^p = \| u \|_{0,p}^p + \| \,|\nabla u|\, \|_{0,p}^p \text{ for all } u \in W^{1,p}(\Omega)$$

(see [4]).

Then $W^{1,p}(\Omega)$ is a real smooth and reflexive Banach space, compactly imbedded in $L^q(\Omega)$. Moreover, the duality mapping corresponding to the gauge function $\varphi(t) = t^{p-1}$ is single valued and it is defined by

$$J_\varphi : \left(W^{1,p}(\Omega), \||\cdot\||_{1,p}\right) \to \left(W^{1,p}(\Omega), \||\cdot\||_{1,p}\right)^*$$
$$J_\varphi u = -\Delta_p u + |u|^{p-2} u \qquad \text{for all} \quad u \in W^{1,p}(\Omega) \quad \text{(see [4]).}$$

It is easy to see that $u \in W^{1,p}(\Omega)$ is a solution of problem (1.6), (1.7) in the sense of (1.8) if and only if

$$J_\varphi u = N_f u. \tag{1.9}$$

So the problem (1.6), (1.7) is a particular case of the abstract equation (1.1), corresponding to $X = \left(W^{1,p}(\Omega), \|\cdot\|_{1,p}\right)$. Thus the Dirichlet and the Neumann problems have the same structure: in the left member we have the duality mapping corresponding to $\varphi(t) = t^{p-1}$ and in the right member a Nemytskii operator, only the spaces over which those operators act are different. Thus many of the methods used for the Dirichlet problem can be adapted to the Neumann problem. Returning to the equation (1.1) by its solution we mean an element $u \in X$ which satisfies

$$J_p u = N_f u, \text{ in } X^*, \text{ so that}$$

$$\langle J_\varphi u, v\rangle = \int_\Omega f(x,u)v dx, \qquad \text{for all} \quad v \in X. \tag{1.10}$$

By N_f we mean $i' N_f i$, where i is the compact imbedding of X into $L^q(\Omega)$ and $i' : L^{q'}(\Omega) \to X^*$ be its adjoint. By the properties of the duality mapping

$$J_p u = \partial\Phi(u) \text{ for all } u \in X, \text{ where},$$

$$\Phi(u) = \int_0^{\| u \|} \varphi(t) dt = \frac{1}{p} \| u \|^p \quad \text{for all } u \in X,$$

and $\partial\Phi : X \to \mathcal{P}(X^*)$ is the subdifferential of Φ in the sense of convex analysis, i.e.

$$\partial\Phi(u) = \{x^* \in X^* : \quad \Phi(v) - \Phi(u) \geq \langle x^*, v-u\rangle, \qquad \text{for all} \quad v \in X\}$$

(see e.g. [3], [7] or [11]).

Since Φ is convex and X is smooth, it results that Φ is Gâteaux differentiable on X and

$$\partial\Phi(u) = \{\Phi'(u)\} \qquad \text{for all} \quad u \in X.$$

So, $J_\varphi u = \Phi'(u)$ for all $u \in X$ and, by continuity of J_p , it results that $\Phi \in C^1(X, R)$.

By the properties of the Nemytskii operator, the functional
$\Psi : L^q(\Omega) \to R, \Psi(u) = \int_\Omega F(x,u)dx$, where $F(x,s) = \int_0^s F(x,t)dt$ is C^1 on $L^q(\Omega)$
and then on X and $\quad \Psi'(u) = N_f u \qquad$ for all $\quad u \in X$.

Consequently, the functional $\mathcal{F} : X \to R$ defined by

$$\mathcal{F}(u) = \Phi(u) - \Psi(u) = \int_0^{\| u \|} \varphi(t)\mathrm{d}t - \int_\Omega F(x,u)dx \qquad \text{for all} \quad u \in X,$$

is C^1 on X and

$$\mathcal{F}'(u) = \Phi'(u) - \Psi'(u) = J_\varphi u - N_f u \qquad \text{for all} \quad u \in X.$$

Then, $u \in X$ is a solution of equation (1.1) if and only if u is a critical point for $\mathcal{F}$, i.e.

$$\mathcal{F}'(u) = 0.$$

For to prove that the equation (1.1) has an infinity of solutions in X, we prove that $\mathcal{F}$ possesses an unbounded sequence of critical values. For this we use a multiple variant of the Mountain Pass theorem:

Theorem 1.1. *Let X be an infinite dimensional real Banach space and let $I \in C^1(X,R)$ be even, satisfying (PS) condition, $I(0) = 0$, and:*

(i) there are constants $\rho > 0, \alpha > 0$ such that $I\big|_{\| x \| = \rho} \geq \alpha$;

(ii) for each finite dimensional subspace X_1 of X, the set $\{x \in X_1 : I(x) \geq 0\}$ is bounded.

Then I possesses an unbounded sequence of critical values.

For the proof and the details see e.g. Kavian [10] or Mawhin and Willem [12].

Let us recall that the functional $I \in C^1(X,R)$ satisfies the (PS) condition if any sequence $(u_n) \subset X$ for which $(I(u_n))$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence.

2. The main result

We need the following result

Proposition 2.1. *Suppose that the Caratheodory function $f : \Omega \times R \to R$ satisfies*

(i) there is $q \in (1, p^*)$ such that

$$| f(x,s) | \leq c(| s |^{q-1} + 1) \ \ \textit{for}\, x \in \Omega, s \in R,$$

with $c > 0$ constant.

(ii) there are numbers $\theta > p$ and $s_0 > 0$ such that

$$0 < \theta F(x,s) \leq s f(x,s) \ \ \textit{for}\ x \in \Omega, | s | \geq s_0.$$

Then, if X_1 is a finite dimensional subspace of X, the set $S = \{v \in X_1 : \mathcal{F}(v) \geq 0\}$ is bounded in X.

Proof. From (i) F satisfies

$$|\,F(x,s)\,| \leq c_1(|\,s\,|^q + 1) \ \text{ for } \ x \in \Omega,\ s \in R, \tag{2.1}$$

with $c_1 \geq 0$ constant.

As in the proof of proposition 7 in [7] there is $\gamma \in L^\infty(\Omega)$, $\gamma > 0$ in Ω such that

$$F(x,s) \geq \gamma(x)\,|\,s\,|^\theta \ \text{ for } x \in \Omega,\ |\,s\,| \geq s_0. \tag{2.2}$$

We shall prove that $\mathcal{F}$ satisfies

$$\mathcal{F}(v) \leq \frac{1}{p}\,\|\,v\,\|^p - \int_\Omega \gamma(x)\,|\,v\,|^\theta\,dx + K, \text{ for all } v \in X, \tag{2.3}$$

with K constant.

For $v \in X$ we denote $\Omega_v = \{x \in \Omega : |\,v(x)\,| < s_0\}$.

By (2.1) we have
$\int_{\Omega_v} F(x,v)dx \geq -c_1 \int_{\Omega_v}(|\,v\,|^p + 1)dx \geq -c_1 \int_\Omega (s_0^q + 1)dx =$
$= -c_1(s_0^q + 1)meas(\Omega) = k_1,$
and by (2.2) it holds

$$\int_{\Omega\setminus\Omega_v} F(x,v)dx \geq \int_{\Omega\setminus\Omega_v} \gamma(x)\,|\,v\,|^\theta\,dx.$$

Then

$$\begin{aligned}
\mathcal{F}(v) &= \frac{1}{p}\,\|\,v\,\|^p - \left(\int_{\Omega_v} F(x,v)dx + \int_{\Omega\setminus\Omega_v} F(x,v)dx\right) \leq \\
&\leq \frac{1}{p}\,\|\,v\,\|^p - \int_{\Omega\setminus\Omega_v} \gamma(x)\,|\,v\,|^\theta\,dx - k_1 = \\
&= \frac{1}{p}\,\|\,v\,\|^p - \int_\Omega \gamma(x)\,|\,v\,|^\theta\,dx + \int_{\Omega_v} \gamma(x)\,|\,v\,|^\theta\,dx - k_1 \leq \\
&\leq \frac{1}{p}\,\|\,v\,\|^p - \int_\Omega \gamma(x)\,|\,v\,|^\theta\,dx + K,
\end{aligned}$$

where $K = \|\,\gamma\,\|_{0,\infty}\, s_0^q meas(\Omega) - k_1$, and (2.3) is proved.

The functional $\|\cdot\|_\gamma : X \to R$ defined by

$$\|\,v\,\|_\gamma = \left(\int_\Omega \gamma(x)\,|\,v\,|^\theta\,dx\right)^{\frac{1}{\theta}}$$

is a norm on X. On the finite dimensional subspace X_1 the norms $\| \cdot \|_X$ and $\| \cdot \|_\gamma$ being equivalent, there is a constant $\widetilde{K}=\widetilde{K}\,(X_1) > 0$ such that

$$\| v \|_X \leq \widetilde{K} \left(\int_\Omega \gamma(x) \,|\, v \,|^\theta \, dx \right)^{\frac{1}{\theta}} \text{ for all } v \in X_1.$$

Consequently, by (2.3) on X_1 it holds

$$\begin{aligned} \mathcal{F}(v) &\leq \frac{1}{p} \widetilde{K}^p \left(\int_\Omega \gamma(x) \,|\, v \,|^\theta \, dx \right)^{\frac{p}{\theta}} - \int_\Omega \gamma(x) \,|\, v \,|^\theta \, dx + K = \\ &= \frac{1}{p} \widetilde{K}^p \,\| v \|_\gamma^p - \| v \|_\gamma^\theta + K. \end{aligned}$$

Therefore

$$\frac{1}{p} \widetilde{K}^p \,\| v \|_\gamma^p - \| v \|_\gamma^\theta + K \geq 0 \text{ for all } v \in S$$

and since $\theta > p$ it results that S is bounded. □

Now we can state

Theorem 2.1. *Assume that the hypothesis $(H_1), (H_2), (H_3)$ and (H_4) hold. Moreover, assume that the Caratheodory function f is odd in the second argument :*
$f(x, -s) = -f(x, s)$ and are satisfied the conditions
(i) there is $q \in (1, p^*)$ such that

$$|\, f(x, s) \,| \leq c(|\, s \,|^{q-1} + 1) \textit{ for } x \in \Omega, s \in R,$$

with $c > 0$ constant;
(ii) $\limsup_{s \to 0} \dfrac{f(x, s)}{|\, s \,|^{p-2}\, s} < \lambda_1$ uniformly with $x \in \Omega$,
where λ_1 is the smallest eigenvalue for J_p;
(iii) there are constants $\theta > p$ and $s_0 > 0$ such that

$$0 < \theta F(x, s) \leq s f(x, s) \ for\ x \in \Omega, |\, s \,| \geq s_0.$$

Then the problem (1.1) has an unbounded sequence of solutions.

Proof. Since f is odd it results that $\mathcal{F}$ is even. It is obvious that $\mathcal{F}(0) = 0$.

From (i) and (iii), reasoning as in the theorem 15 in [7] we get that $\mathcal{F}$ satisfies the (PS) condition. Furthemore, from $(i), (ii)$ and theorem 17 in [7], there are constants $\rho, \alpha > 0$ such that $\mathcal{F}_{\big|\| u \|=\rho} \geq \alpha$.

Proposition 2.1 and $(i), (iii)$ show that the set $\{v \in X_1 : \mathcal{F}(v) \geq 0\}$ is bounded in X, whenever X_1 is a finite dimensional subspace of X.

By the theorem 1.1 it results that $\mathcal{F}$ possesses an unbounded sequence of critical values, so that the problem (1.1) has an unbounded sequence of solutions. $\square$

3. Examples

Example 3.1. If $X = W_0^{1,p}(\Omega)$ then $J_p = -\Delta_p$ and the solutions set of equation $J_p u = N_f u$ coincides with the solutions set of the Dirichlet problem (1.3) (see the Introduction). Consequently, the existence result given in section 2 becomes the existence results obtained in [7] for the Dirichlet problem (1.3), (1.4).

Example 3.2. We consider $X = W^{1,p}(\Omega)$, endowed with the norm

$$\| u \|_{1,p}^p = \| u \|_{0,p}^p + \| \, | \nabla u | \, \|_{0,p}^p \text{ for all } u \in W^{1,p}(\Omega).$$

In this case $J_p u = -\Delta_p u + | u |^{p-2} u$ for all $u \in W^{1,p}(\Omega)$ (see the Introduction).

The space $\left(W^{1,p}(\Omega), \| \cdot \|_{1,p}\right)$ is a smooth reflexive Banach space, compactly imbedding in the space $L^q(\Omega)$, $J_p : \left(W^{1,p}(\Omega), \| \cdot \|_{1,p}\right) \to \left(W^{1,p}(\Omega), \| \cdot \|_{1,p}\right)^*$ is single valued, continuous and satisfies the (S_+) condition, so we are in the functional framework (H_2), (H_3) about X (see [4]). Consequently, the existence result given in section 2 becomes the existence result obtained in [4] for the Neumann problem (1.8), (1.9).

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"Dunărea de Jos" University of Galaţi,
Department of Mathematics, Galaţi - ROMANIA
E-mail: jcringanu@ugal.ro