"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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A DEFORMATION OF A QUARTIC CARTAN METRIC

OTILIA LUNGU

Abstract. In this paper we consider a Cartan space $C^n = (M, K(x,p))$ with the metric $K(x,p) = \sqrt[4]{a^{ijkh}(x) p_i p_j p_k p_h}$ and a deformation using $\omega(x,p) = b_i(x) p_i$ .We call it a Randers- Quartic Cartan space and we are going to study some of its properties.

1. PRELIMINARIES

Let M be a real n-dimensional differential manifold and (T^*M, π^*, M) its cotangent bundle. If (U, x^i) is a local chart on M, then (x^i, p_i) is the induced system of coordinates on $\pi^{*-1}(U)$.

A Cartan metric on M is a real positive function $K : T^*M \to R_+$ satisfying the following properties:
i) K is differentiable on $T^*M - \{0\}$ and continuous on the null section of the projection π^*;
ii) K is positively homogeneous of order one with respect to p_i :

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(1.1) $$K(x,\lambda p)=\lambda K(x,p),\ \lambda>0;$$

iii) $\forall(x,p)\in T^*M-\{0\}$, $g^{ij}(x,p)$ is positive and nondegenerate, where

(1.2) $$g^{ij}(x,p)=\frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j}.$$

The pair $C^n=(M,K)$ is called a Cartan space and $g^{ij}(x,p)$ is called the fundamental tensor of the Cartan space.

Let g_{ij} be the covariant tensor of g^{ij}: $g_{ij}g^{jh}=\delta_i^h$ .The Christoffel symbols of $g^{ij}(x,p)$ are given by

(1.3) $$\gamma_{jk}^{i}=\frac{1}{2}g^{ih}\left(\frac{\partial g_{hj}}{\partial x^k}+\frac{\partial g_{hk}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^h}\right).$$

We denote

(1.4) $$\gamma_{jk}^{0}=\gamma_{jk}^{i}p_i$$

and

(1.5) $$\gamma_{j0}^{0}=\gamma_{jk}^{i}p_i p^k.$$

The canonical nonlinear connection of the Cartan space $C^n=(M,K)$ is given by the coefficients

(1.6) $$N_{ij}=\gamma_{ij}^{0}-\frac{1}{2}\gamma_{h0}^{0}\frac{\partial g_{ij}}{\partial p_h}.$$

If we consider the canonical metrical connection $C\Gamma(N)=\left(H_{jk}^{i},C_{jk}^{i}\right)$ of the Cartan space, the local components of $C\Gamma(N)$ have the expressions:

(1.7) $$H_{jk}^{i}=\frac{1}{2}g^{is}\left(\frac{\delta g_{sj}}{\delta x^k}+\frac{\delta g_{sk}}{\delta x^j}-\frac{\delta g_{jk}}{\delta x^s}\right)$$
$$C_i^{jk}=g_{is}C^{sjk}=-\frac{1}{2}g_{is}\left(\frac{\partial g^{sk}}{\partial p_j}+\frac{\partial g^{js}}{\partial p_k}-\frac{\partial g^{jk}}{\partial p_s}\right),$$

Where

(1.8) $$\frac{\delta}{\delta x^k}=\frac{\partial}{\partial x^k}+N_{ks}\frac{\partial}{\partial p_s}$$

2. QUARTIC CARTAN SPACE

Let $C^n = (M, K)$ be a Cartan space with the metric

(2.1) $$K(x, p) = \sqrt[4]{a^{ijkh}(x) p_i p_j p_k p_h},$$

where $a^{ijkh}(x)$ are the components of a symetric tensor field of (4,0)-type.
We call this space a *quartic Cartan space* and we denote it by QC^n.
For an easier calculation we also denote:

(2.2) $$\begin{aligned} a^{ijkh}(x) p_h &= K a^{ijk}(x, p); \\ a^{ijkh}(x) p_k p_h &= K^2 a^{ij}(x, p); \\ a^{ijkh}(x) p_j p_k p_h &= K^3 a^i(x, p). \end{aligned}$$

The angular metrical tensor of QC^n space is

(2.3) $$h^{ij} = 3\left(a^{ij} - a^i a^j\right).$$

and the fundamental tensor is

(2.4) $$g^{ij} = 3a^{ij} - 2a^i a^j.$$

Supposing that $\left(a^{ij}\right)$ is regular, there exists the inverse matrix $\left(a_{ij}\right) = \left(a^{ij}\right)^{-1}$. So we get

(2.5) $$a_i = a_{ij} a^j, a_s a^{sjk} = a^{jk}, a_{is} a^{sjk} = a_i^{jk}$$

and the components $g_{ij} = \left(g^{ij}\right)^{-1}$:

(2.6) $$g_{ij} = \frac{1}{3} a_{ij} + \frac{2}{3} a_i a_j$$

The components of the v-torsion tensor $C^{ijk} = -\frac{1}{2} \frac{\partial g^{ij}}{\partial p_k}$ are

(2.7) $$C^{ijk} = -\frac{3}{K}\left(a^{ijk} - a^{ij} a^k - a^{jk} a^i - a^{ki} a^j + 2a^i a^j a^k\right).$$

3. RANDERS QUARTIC CARTAN SPACE

It is well known that a Randers metric is a deformation of a Riemannian metric $\alpha(x,y)=\sqrt{a_{ij}(x)y^i y^j}$ with a 1-form $\beta(x,y)=b_i(x)y^i$. In the same way now we are considering a deformation of a quartic Cartan metric $K(x,p)=\sqrt[4]{a^{ijkh}(x)p_i p_j p_k p_h}$ with $\omega(x,p)=b^i(x)p_i$.

Definition 3.1. Let $QC^n=(M,K(x,p))$ be a quartic Cartan space with the metric $K(x,p)=\sqrt[4]{a^{ijkh}(x)p_i p_j p_k p_h}$. The space $RQC^n=(M,F(x,p))$, where $F(x,p)=K(x,p)+\omega(x,p)$ is called *Randers Quartic Cartan space.*

Proposition 3.1. The fundamental tensor of RQC^n is

(3.1) $$\hat{g}^{ij}=3\varepsilon a^{ij}-(3\varepsilon-1)a^i a^j+a^i b^j+a^j b^i+b^i b^j$$

where $\varepsilon=\dfrac{F}{K}$

Theorem 3.1. Let $QC^n=(M,K)$ a Quartic Cartan space and g^{ij} his fundamental tensor. The fundamental tensor of the space $RQC^n=(M,F=K+\omega)$ is

(3.2) $$\hat{g}^{ij}=\frac{F}{K}\left(g^{ij}-a^i a^j\right)+\left(a^i+b^i\right)\left(a^j+b^j\right).$$

Theorem 3.2 The components of the v-torsion tensor $\hat{C}^{ijk}$ of RQC^n are given by

(3.3) $$\begin{aligned}\hat{C}^{ijk}&=-2\frac{F}{K}C^{ijk}-\frac{1}{2}h^{ij}\frac{b^k K-\omega a^k}{K^2}\\&+3\frac{a^{ik}-a^i a^k}{K}\left(a^j+b^j-\frac{F}{K}a^j\right)+3\frac{a^{jk}-a^j a^k}{K}\left(a^i+b^i-\frac{F}{K}a^i\right)\end{aligned},$$

where C^{ijk} are the components of the v-torsion tensor of QC^n space .

Proof: We have $\hat{C}^{ijk} = -\frac{1}{2}\frac{\partial \hat{g}^{ij}}{\partial p_k}$ and from (3.2) we get

(3.4)
$$\hat{C}^{ijk} = -\frac{1}{2}\left[\frac{\partial \varepsilon}{\partial p_k}\left(g^{ij} - a^i a^j\right) + \varepsilon\left(\frac{\partial g^{ij}}{\partial p_k} - \frac{\partial a^i}{\partial p_k}a^j - \frac{\partial a^j}{\partial p_k}a^i\right)\right.$$
$$\left. + \frac{\partial a^i}{\partial p_k}\left(a^j + b^j\right) + \frac{\partial a^j}{\partial p_k}\left(a^i + b^i\right)\right]$$

By a direct calculation we obtain

(3.5)
$$\frac{\partial a^i}{\partial p_k} = 3\frac{a^{ik} - a^i a^k}{K},$$

(3.6)
$$\frac{\partial a^{ij}}{\partial p_k} = 2\frac{a^{ijk} - a^{ij} a^k}{K}$$

and

(3.7)
$$\frac{\partial \varepsilon}{\partial p_k} = \frac{\partial}{\partial p_k}\left(\frac{\omega}{K}\right) = \frac{b^k K - \omega a^k}{K^2}.$$

Replacing (3.5), (3.6) and (3.7) in (3.4) we get immediately the conclusion.

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"Vasile Alecsandri" University of Bacău
Department of Mathematics and Informatics,
Bacău, Romania
otilas@yahoo.com