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MYLLER CONFIGURATIONS AND VRĂNCEANU NONHOLONOMIC MANIFOLDS

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Abstract. One studies the relations between Myller Configurations and Vrănceanu Nonholonomic Manifolds.

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1. Introduction

Sixty years ago, Academician Octav Mayer remarked that the notion of nonholonomic manifold, discovered by Academician Gheorghe Vrănceanu in 1926 was suggested by the notion of Myller configuration, introduced by Academician Alexandru Myller in 1922. In this note we point out the arguments which justify this assertion.

2. Myller Configurations

Alexandru Myller introduced and studied the configurations determined by: a curve C (having s as canonical parameter), a field of planes $\pi(s)$, along C and a field of directions given by a versor field $\xi(s)$ on C with the property that $\xi(s)$ belongs to $\pi(s)$. These geometric objects are considered in the 3-dimensional Euclidean space. Thus, Alexandru Myller considered in 1922 the Geometry of versor field $\xi(s)$ in the plane field $\pi(s)$ and discovered a first important invariant $G(s)$, called *the deviation of parallelism.*

In the case of $G(s) = 0$, the versor field $\xi(s)$ is parallel *in Myller sense* in the plane field $\pi(s)$. Five years before, the great Italian mathematician Levi-Civita discovered the notion of parallelism in curved manifolds, in particular, in surfaces of Euclidean space. But, it is important to remark that the Myller parallelism is an essential generalization of Levi-Civita parallelism. It is well- known that the Levi-Civita parallelism is a fundamental notion in Einstein theory of relativity.

The invariant G is a new intrinsic invariant on surfaces and allows to introduce a new concept, *the Myller concurence.*

Alexandru Myller published a first paper entitled "Quelques propriétés des surfaces réglées en liaison avec la theory du parallelism de Levi-Civita" in C.R. de l'Academie des Sciences, Paris, tome 174, 1922, p.997. Three years later, Levi-Civita published in the Appendix of his book "Lezioni di Calcolo Differenziale Assoluto" (Zanichelli, 1925) the paper of Alexandru Myller. This was a great success of the School of Geometry founded by Myller at the University of Iaşi.
In 1960, when the "Alexandru Myller" Seminar of Mathematics in Iaşi celebrated 50 years of existence, the author of the present Note defined the notion of Myller configuration and delivered a first lecture, "The Geometry of Myller Configurations", published in the Journal Analele Universităţii "Al.I.Cuza" din Iaşi. This is a triple of geometric objects $M = (C, \xi(s), \pi(s))$, formed by a smooth curve C, a versor field $\xi(s)$ defined along C and a plane field $\pi(s)$ on C with the property $\xi(s) \subset \pi(s)$. To a Myller configuration $M = (C, \xi(s), \pi(s))$ a Darboux frame $\Re_D = (P(s), \xi(s), \mu(s), n(s))$ can be uniquely associated , where $n(s)$ is a versor normal to the plane $\pi(s)$ and $\mu(s)$ is a versor normal to $n(s)$ and to $\xi(s)$, i.e. $\mu(s) = n(s) \times \xi(s)$.

The motion equations of the Darboux frame give the *Fundamental equations of Myller Configuration.* These equations allow to determine the main invariants: geodesic curvature $G(s)$, normal curvature $K(s)$ and geodesic torsion $T(s)$. At the same time we prove a *Fundamental Theorem.* The invariant $G(s)$ is the invariant discovered by Alexandru Myller, the second invariant $K(s)$ has been established by O. Mayer and the last one, $T(s)$, has been introduced by E.Bertolotti, [1].

The parallelism of the versor field $\xi(s)$, in the Myller configuration $M = (C, \xi(s), \pi(s))$ is characterized by property $G(s) = 0$.

In fact, the whole Differential Geometry of Myller configuration is governed by the fundamental equations, as one can see in the recent English version of the book " The Geometry of Myller Configurations" [1].

3. **Vrănceanu Nonholonomic Manifolds** E_3^2

The notion of Nonholonomic manifold was discovered by Academician Gheorghe Vrănceanu in 1926. At that time he was the Ph.D. student of the great Italian mathematician Tulio Levi-Civita, (1873,1941) and at the same time Vrănceanu was awarded a scholarship by Academician Alexandru Myller.

Vrănceanu published his original results in the famous journal *Comptes Rendus de l'Academie des Sciences Paris* in 1926 and he submitted a very important paper [3] in Memorial des Sciences Mathematiques, in 1936.

Academician Octav Mayer, remarked at the Fourth International Congress of Romanian Mathematicians, in 1956, that the notion of Myller Configuration introduced by Myller in 1922 allowed to Vrănceanu in 1926 to discover the concept of Nonholonomic Manifold. This is almost true, since Vrănceanu discovered the idea of Nonholonomic Manifold starting from Theory of Nonholonomic Mechanical Systems.

A nonholonomic manifold E_3^2, in the 3-dimensional Euclidean space E_3 is defined as a 2-dimensional distribution D in E_3. But D can be characterized as a plane field $\pi(P)$ in E_3 for any point $P \in E_3$.

If one considers a tangent field of plans $\pi(P)$ along to a curve C from a nonholonomic manifold E_3^2 and a versor field $\xi(s)$ tangent to the same nonholonomic manifold along the curve C , we obtain a Myller configuration $M = (C, \xi(s), \pi(s))$, geometrically associated to E_3^2. The invariants $G(s)$, $K(s)$ and $T(s)$ are the fundamental invariants of the versor field $\xi(s)$ in the nonholonomic manifold E_3^2. One proves that the versor field $\xi(s)$ is parallel in Levi-Civita sense along the curve C in the nonholonomic manifold E_3^2, it is characterized by the equations $G(s) = 0$ and $K(s) = 0$, determines a conjugation relation for $\xi(s)$ and that $T(s) = 0$ defines an orthogonal conjugation.

In particular, if the versor $\xi(s)$ coincides with the tangent versor field $\alpha(s)$ to the curve C, then the invariant $G(s)$ is the geodesic curvature $k_g(s)$ of curve C in the manifold E_3^2, $K(s)$ is the normal curvature $k_n(s)$ of C in E_3^2 and $T(s)$ is the geodesic torsion $\tau_g(s)$ on C in the nonholonomic manifold E_3^2. Clearly, C is geodesic if $k_g = 0$; C is an asymptotic line of E_3^2 if $k_n(s) = 0$. Finally, C is a curvature line of E_3^2 if $\tau_g(s) = 0$.

By using the extremal values of the normal curvature $k_n(s)$ one can determine an Euler formula and a Dupin indicatrix for E_3^2. The notions of mean curvature and total curvature can be introduced. But we can study in the same manner the variation of the geodesic torsion in a point P of E_3^2. The extremal values of $\tau_g(s)$ allow to determine a Bonnet formula and a Bonnet indicatrix , as well as a mean torsion T_m and a total torsion T_t. The manifold E_3^2 is holonomic (integrable) if and only if the mean curvature T_m vanishes. But in this case the total curvature T_t is exactly the famous Bacaloglu curvature from the theory of surfaces in Euclidean space E_3 (Romanian physicist Emanoil Bacaloglu discovered his notion of curvature in the year 1859, [1]).

As applications we can study the nonholonomic plane discovered by Academician Grigore Moisil, [1]. He wrote for the first time the explicite Pfaff equation for nonholonomic sphere, [1].

In general theory of nonholonomic quadrics has been elaborated by Gheorghe Vrănceanu, [2] and the Gauss-Codazzi equations have been established by A. Dobrescu [2]. Many Romanian geometers, as M.Haimovici, R.Miron, I.Vaisman et.al. , have important original contributions to this topic, [1].

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