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THREE EXTENSIONS OF ORLICZ-SOBOLEV SPACES TO METRIC MEASURE SPACES AND THEIR MUTUAL EMBEDDINGS

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Abstract. We study the mutual embeddings between three extensions of Orlicz-Sobolev spaces to a metric measure space, the Orlicz-Sobolev spaces of Newtonian type, of Hajłasz type and of Cheeger type.

1. Introduction and preliminaries

Sobolev spaces play a fundamental role in the theory of partial differential equations and calculus of variations. There are several extensions of first order Sobolev spaces to metric measure spaces, namely Hajłasz spaces [8], Newtonian spaces [18], Cheeger spaces [4]. These Sobolev-type spaces, each of which extending some features of Sobolev spaces on Euclidean spaces, are an indispensable tool in the metric generalizations of quasiconformal theory and nonlinear potential theory. Generalizations of Hajłasz spaces and of Newtonian spaces that extend Orlicz-Sobolev spaces to the metric setting have been introduced and studied in [1] and [19], respectively. Recently, Orlicz-Sobolev spaces on metric measure spaces have been also generalized by replacing the Orlicz space by a Banach function space [15].

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It is natural to compare various extensions of Sobolev, respectively of Orlicz-Sobolev, to the metric setting. In this paper we study the mutual embeddings between three extensions of Orlicz-Sobolev spaces to a metric measure space, namely the spaces of Newtonian type $N^{1,\Psi}(X)$, of Hajłasz type $M^{1,\Psi}(X)$ and of Cheeger type $H_{1,\Psi}(X)$.

It is known that $M^{1,\Psi}(X)$ continuously embeds into $N^{1,\Psi}(X)$, whenever Ψ is an $N-$function satisfying the Δ_2-condition, by [19, Theorem 6.22]. We provide sufficient conditions for the existence of a continuous embedding of $N^{1,\Psi}(X)$ into $M^{1,\Psi}(X)$, using as a main tool the Hardy-Littlewood maximal operator. Let X be a doubling metric measure space supporting a weak $(1,\Phi)-$Poincaré inequality, where Φ is a Young function satisfying the Δ_2-condition . Assuming that Ψ is an $N-$function satisfying the $\Delta'-$condition, together with its inverse, and that the Hardy-Littlewood maximal operator is bounded in $L^{\Psi\circ\Phi^{-1}}(X)$, it follows that there exists a continuous embedding $N^{1,\Psi}(X) \subset M^{1,\Psi}(X)$.

We define the Cheeger type Orlicz-Sobolev space $H_{1,\Psi}(X)$ as a natural generalization of the Cheeger space $H_{1,p}(X)$ introduced in [4]. We show that a continuous embedding $H_{1,\Psi}(X) \subset N^{1,\Psi}(X)$ holds whenever Ψ is a Young function, while $N^{1,\Psi}(X)$ embeds continuously into $H_{1,\Psi}(X)$ provided that the Young function Ψ satisfies the $\Delta_2-$and ∇_2-conditions for large values of the variable (equivalently, the Banach space $L^{\Psi}(X)$ is reflexive).

Our results generalize Theorems 4.8 and 4.10 of [18], where $\Psi(t) = t^p$, with $1 < p < \infty$.

First we recall some important notions from the theory of Orlicz spaces [17]. The notions of Young function and $N-$function are well-known.

We deal with the growth rates of Young functions given by Δ_2-, ∇_2- and $\Delta'-$conditions. Let $\Phi : [0,\infty] \to [0,\infty]$ be a Young function. Φ is said to satisfy a Δ_2-condition if there is a constant $C_\Phi > 0$ such that $\Phi(2t) \leq C_\Phi \Phi(t)$ for every $t \in [0,\infty)$. A Young function satisfying a Δ_2-condition is called *doubling* (globally). Every doubling Young function is real-valued, strictly increasing and continuous. The inverse Φ^{-1} of every strictly increasing Young function $\Phi : [0,\infty) \to [0,\infty)$ is subadditive, hence it is doubling with $C_{\Phi^{-1}} = 2$. The Δ_2-condition for an increasing Young function Φ implies the power growth estimate: $\Phi(\lambda t) \leq C_\Phi \lambda^{\log_2 C_\Phi} \Phi(t)$, for all $\lambda \geq 1, t \geq 0$. A Young function $\Phi : [0,\infty) \to [0,\infty)$ is said to satisfy a ∇_2-condition if there is a constant $C > 1$ such that $\Phi(t) \leq \frac{1}{2C}\Phi(Ct)$ for all $t \in [0, \infty)$. It is well-known that Ψ satisfies a ∇_2-condition if and only if its complementary Young function is doubling. It is said that Φ satisfies the Δ_2-condition (respectively, the ∇_2-condition) if there exist the constants $t_0 > 0$ and $C > 0$ such that for every $t \geq t_0$ we have $\Phi(2t) \leq C\Phi(t)$ (respectively, $\Phi(t) \leq \frac{1}{2C}\Phi(Ct)$). Φ is said to satisfy a $\Delta'-$condition if there is a constant $C > 0$ such that $\Phi(ts) \leq C\Phi(t)\Phi(s)$ for all $t, s \geq 0$.

Let $(X, \mathcal{A}, \mu)$ be a measure space with a complete and $\sigma-$finite measure μ and let $\Phi : [0, \infty] \to [0, \infty]$ be a Young function.

The Orlicz space $L^\Phi(X)$ is the set of all measurable functions $u : X \to [-\infty, \infty]$ for which there exists $\lambda > 0$ such that

$$\int_X \Phi\left(\frac{|u(x)|}{\lambda}\right) dx < \infty.$$

The Orlicz space $L^\Phi(X)$ is a Banach space with the Luxemburg norm defined by

$$\|u\|_{L^\Phi(X)} = \inf\left\{k > 0 : \int_X \Phi(\frac{|u|}{k})d\mu \leq 1\right\}.$$

For every measurable function $u : X \to [-\infty, +\infty]$, denote $I_\Phi(u) = \int_X \Phi(|u|)d\mu$. If $I_\Phi(u) < \infty$, then $u \in L^\Phi(X)$ and the converse is true provided that Φ is doubling.

Remark 1. *By* [20, Lemma 4]*, for every doubling Young function Φ and all $u \in L^\Phi(X)$, the following inequalities hold:*

$$\|u\|_{L^\Phi(X)} \leq f_\Phi(I_\Phi(u)) \text{ and } I_\Phi(u) \leq h_\Phi(\|u\|_{L^\Phi(X)}),$$

where we denoted $f_\Phi(t) = \max\{t, 2t^{1/\log_2 C_\Phi}\}$ and $h_\Phi(t) = \max\{t, C_\Phi t^{\log_2 C_\Phi}\}$.

Throughout this paper we deal with a metric measure space (X, d, μ), which is a metric space (X, d) equipped with a Borel regular outer measure μ. Assume that μ is finite and positive on balls. Recall that a metric space is called *proper* if every closed ball is compact.

Remark 2. *Since μ is finite on balls, for every doubling $N-$function $\Phi : [0, \infty) \to [0, \infty)$ we have $L^\Phi(X) \subset L^1_{loc}(X)$, by* [17, Proposition 3.1.7].

Definition 1. *The measure μ on the metric space (X, d, μ) is said to be doubling if there is a constant $C_d \geq 1$ such that*

$$\mu(B(x, 2r)) \leq C_d \mu(B(x, r)) \tag{1.1}$$

for every ball $B(x, r) \subset X$.

For every doubling measure μ there are some positive constants C_b and Q so that $\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq C_b \left(\frac{r}{r_0}\right)^Q$, for all $0 < r \leq r_0$ and $x \in B(x_0, r_0)$. Here Q is called a *homogeneous dimension* of the metric measure space X.

Recall the notion of Hardy-Littlewood maximal function of $f \in L^1_{loc}(X)$, which is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu.$$

In the presence of the doubling condition, some classical results, such as Vitali covering theorem, Lebesgue's differentiation theorem and the maximal function theorem have natural extensions to the setting of metric measure spaces [12]. In harmonic analysis, doubling metric measure spaces are known as homogeneous spaces [5].

In the following it is assumed that the measure μ is doubling.

The Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded in $L^p(X)$ provided that X is a doubling metric measure space and $p > 1$ [12]. If Φ is an $N-$function satisfying the ∇_2-condition, then $\mathcal{M}$ is bounded as an operator from $L^\Phi(X)$ into itself (see [16] for a more detailed discussion). Under this assumptions on Φ it follows by [7, Theorem 2.2] that there exist some positive constants A and b such that

$$I_\Phi(\mathcal{M}f) \leq A I_\Phi(bf) \tag{1.2}$$

for every $f \in L^\Phi(X)$, a property stronger in general than the boundedness of $\mathcal{M}$ in $L^\Phi(X)$.

A substitute for the norm of the gradient in analysis on metric measure spaces is the concept of upper gradient.Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0, +\infty]$ is said to be an *upper gradient* of u in X if

$$|u(\gamma(1)) - u(\gamma(0))| \leq \int_\gamma g \, ds, \tag{1.3}$$

for every compact rectifiable path $\gamma : [0, 1] \to X$.

Since upper gradients are unstable under changes $\mu-$a.e. and under limits, the more general notion of weak upper gradient has been introduced [11]. This modification of the notion of upper gradient is essential in defining and studying some Sobolev-type spaces on metric measure spaces.

The notion of $\Phi-$modulus of a path family is indispensable in the definition of the notion of $\Phi-$weak upper gradient, on which the definition of Orlicz- Sobolev spaces of Newtonian type is based.

Definition 2. [19] *Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function. The $\Phi-$modulus of a family Γ of paths in X is $M_\Phi(\Gamma) = \inf \|\rho\|_{L^\Phi(X)}$, where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ satisfying $\int\limits_\gamma \rho ds \geq 1$ for all locally rectifiable paths $\gamma \in \Gamma$.*

Definition 3. [19] *Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0, +\infty]$ is called a $\Phi-$weak upper gradient of u if (1.3) holds for all compact rectifiable paths $\gamma : [0, 1] \to X$ except for a path family Γ_0 with $M_\Phi (\Gamma_0) = 0$ in X.*

The collection $\widetilde{N}^{1,\Phi}(X)$ of all functions $u \in L^\Phi (X)$ having a $\Phi-$weak upper gradient $g \in L^\Phi (X)$ is a vector space. For $u \in \widetilde{N}^{1,\Phi}(X)$ define $\|u\|_{1,\Phi} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)}$, where the infimum is taken over all $\Phi-$weak upper gradients $g \in_{L^\Phi(X)}$ of u. Consider the equivalence relation $u \sim v \Leftrightarrow \|u - v\|_{1,\Phi} = 0$. Then $N^{1,\Phi}(X) = \widetilde{N}^{1,\Phi}(X)/\sim$ is a Banach space with the norm $\|u\|_{N^{1,\Phi}} := \|u\|_{1,\Phi}$[19].

If $X = \Omega \subset \mathbb{R}^n$ is a domain and Φ is a doubling Young function, then $N^{1,\Phi}(X) = W^{1,\Phi}(\Omega)$ as Banach spaces and the norms are equivalent [19, Theorem 6.19].

The notion of $\Phi-$modulus of a path family, a generalization of the notion of $p-$modulus, is indispensable in order to define Orlicz-Sobolev spaces of Newtonian type.

Definition 4. [19] *Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function. The $\Phi-$modulus of a family Γ of paths in X is $M_\Phi(\Gamma) = \inf \|\rho\|_{L^\Phi(X)}$, where the infimum is taken over all Borel functions $\rho : X \to [0, +\infty]$ satisfying $\int\limits_\gamma \rho ds \geq 1$ for all locally rectifiable paths $\gamma \in \Gamma$.*

Definition 5. [19] *Let u be a real-valued function on a metric measure space X. A Borel function $g : X \to [0, +\infty]$ is called a $\Phi-$weak upper gradient of u if (1.3) holds for all compact rectifiable paths $\gamma : [0, 1] \to X$ except for a path family Γ_0 with $M_\Phi (\Gamma_0) = 0$ in X.*

The collection $\widetilde{N}^{1,\Phi}(X)$ of all functions $u \in L^\Phi (X)$ having a $\Phi-$weak upper gradient $g \in L^\Phi (X)$ is a vector space. For $u \in \widetilde{N}^{1,\Phi}(X)$ define $\|u\|_{1,\Phi} = \|u\|_{L^\Phi(X)} + \inf \|g\|_{L^\Phi(X)}$, where the infimum is taken over all $\Phi-$weak upper gradients $g \in L^\Phi (X)$ of u. Consider the equivalence relation $u \sim v \Leftrightarrow \|u - v\|_{1,\Phi} = 0$. Then $N^{1,\Phi}(X) = \widetilde{N}^{1,\Phi}(X)/ \sim$ is a Banach space with the norm $\|u\|_{N^{1,\Phi}} := \|u\|_{1,\Phi}$[19].

If $X = \Omega \subset \mathbb{R}^n$ is a domain and Φ is a doubling Young function, then $N^{1,\Phi}(X) = W^{1,\Phi}(\Omega)$ as Banach spaces and their norms are equivalent [19, Theorem 6.19].

We recall the $(1, p)-$Poincaré inequality and its generalization, the $(1, \Phi)-$Poincaré inequality.

Denote the mean value of a function $u \in L^1(A)$ over A by $u_A := \frac{1}{\mu(A)} \int\limits_B u d\mu$, where $0 < \mu(A) < \infty$.

Definition 6. [10] *Let Ω be an open subset of the metric measure space X. A pair formed by $u \in L^1_{loc}(\Omega)$ and a Borel measurable function $g : \Omega \to [0, \infty]$ is said to satisfy a weak $(1, p)-$Poincaré inequality, $1 \le p < \infty$, in Ω if there exist some constants $C_P > 0$ and $\tau \ge 1$ such that for every ball $B = B(x, r)$ satisfying $\tau B \subset \Omega$,, where $\tau B := B (x, \tau r)$*

$$\frac{1}{\mu(B)} \int\limits_B |u - u_B| \ d\mu \le C_P r \left(\frac{1}{\mu(\tau B)} \int\limits_{\tau B} g^p \ d\mu \right)^{1/p} .$$

It is said that Ω supports a $(1, p)-$Poincaré inequality if the above inequality holds for every $u \in L^1_{loc}(\Omega)$ and every upper gradient g of u, with fixed constants C_P and τ.

The weak $(1, p)-$ Poincaré inequality has been generalized for Orlicz-Sobolev spaces, as follows:

Definition 7. [19, Definition 5.2] *Let $\Phi : [0, \infty) \to [0, \infty)$ be a strictly increasing Young function and $\Omega \subset X$ an open set. We say that a function $u \in L^1_{loc}(\Omega)$ and a Borel measurable non-negative function g*

on Ω satisfy a $(1,\Phi)-$weak Poincaré inequality in Ω if there exist some constants $C_{P,\Phi} > 0$ and $\tau \geq 1$ such that

$$\frac{1}{\mu(B)} \int\limits_{B} |u - u_B|\, d\mu \leq C_{P,\Phi} r \Phi^{-1} \left(\frac{1}{\mu(\tau B)} \int\limits_{\tau B} \Phi(g) d\mu \right). \tag{1.4}$$

for each ball $B = B(x,r)$ satisfying $\tau B \subset \Omega$. It is said that Ω supports a weak $(1,\Psi)-$Poincaré inequality if the above inequality holds for each function $u \in L^1_{loc}(\Omega)$ and every upper gradient g of u, with fixed constants.

Remark 3. *If Φ is doubling, we may replace in the above definition upper gradients by $\Phi-$weak upper gradients.*

2. Mutual embeddings of Newtonian type and Hajłasz type Orlicz-Sobolev spaces

Shanmugalingam proved that $M^{1,p}(X)$ continuously embeds into $N^{1,p}(X)$ for every metric space X [18, Theorem 4.8] and that for a doubling metric measure space X supporting a $(1,q)-$Poincaré inequality for some $1 \leq q < p$, $N^{1,p}(X) = M^{1,p}(X)$ and the corresponding Banach spaces are isomorphic [18, Theorem 4.9]. Tuominen [19] extended these results to Orlicz-Sobolev spaces. It is proved that $M^{1,\Psi}(X)$ continuously embeds into $N^{1,\Psi}(X)$ whenever Ψ is a doubling $N-$function. If Ψ is a doubling $N-$function, the every function $u \in M^{1,\Psi}(X)$ with a Hajłasz gradient $g \in L^{\Psi}(X)$ has a representative that belongs to $N^{1,\Psi}(X)$ with $2g$ as a $\Psi-$ weak upper gradient. In particular, there is a continuous embedding $M^{1,\Psi}(X) \subset N^{1,\Psi}(X)$ with $\|u\|_{N^{1,\Psi}(X)} \leq 2\|u\|_{M^{1,\Psi}(X)}$ for each $u \in M^{1,\Psi}(X)$ [19, Theorem 6.22].

In general, $M^{1,\Psi}(X)$ is a smaller space than $N^{1,\Psi}(X)$. Conversely, $N^{1,\Psi}(X)$ embeds into $M^{1,\Psi}(X)$ under stronger assumptions on Ψ and X, as follows. If X supports a weak $(1,\Phi)-$ Poincaré inequality, where Φ is a strictly increasing Young function, and if Ψ is a doubling $N-$function, such that $\Psi \circ \Phi^{-1}$ is a $N-$function satisfying the ∇_2-condition, then $N^{1,\Psi}(X) \subset M^{1,\Psi}(X)$. A brief discussion on this embedding, not involving its continuity, is given in [19, 6.2].

In the following, we prove that $N^{1,\Psi}(X) \subset M^{1,\Psi}(X)$ under some slightly more relaxed assumptions and provide sufficient conditions for the continuity of this embedding.

We will need the following technical lemma, which is proved in [13].

Lemma 1. *Let Ψ be a doubling Young $N-$function such that $\Psi\circ\Phi^{-1}$ is a Young function and the Hardy-Littlewood maximal operator is bounded in $L^{\Psi\circ\Phi^{-1}}(X)$. If $g\in L^{\Psi}(X)$, then $\Phi^{-1}(\mathcal{M}(\Phi\circ g))\in L^{\Psi}(X)$. Moreover, there exists a strictly increasing continuous function $F:[0,\infty)\to[0,\infty)$ with $F(0)=0$ such that $\|\Phi^{-1}(\mathcal{M}(\Phi\circ g))\|_{L^{\Psi}(X)}\leq F(\|g\|_{L^{\Psi}(X)})$ for every $g\in L^{\Psi}(X)$. Assuming in addition that Φ and Φ^{-1} satisfy the $\Delta'-$condition and $\Psi\circ\Phi^{-1}$ is a $N-$function satisfying the ∇_2-condition, we may take in the previous estimate $F(t)=C't$ for all $t\geq 0$, where C' is a positive constant.*

Theorem 1. *Assume that the metric space X is equipped with a doubling measure and supports a weak $(1,\Phi)-$Poincaré inequality for some doubling Young function Φ. Let Ψ be a doubling $N-$function such that $\Psi\circ\Phi^{-1}$ is a Young function and the Hardy-Littlewood maximal operator is bounded in $L^{\Psi}(X)$ $L^{\Psi\circ\Phi^{-1}}(X)$. Then $N^{1,\Psi}(X)\subset M^{1,\Psi}(X)$. Moreover, if Ψ and Ψ^{-1} satisfy the $\Delta'-$condition and if $\Psi\circ\Phi^{-1}$ satisfies the ∇_2-condition, then the embedding $N^{1,\Psi}(X)\subset M^{1,\Psi}(X)$ is continuous.*

Proof. Let $u\in N^{1,\Psi}(X)$. Consider $g\in L^{\Psi}(X)$ a $\Psi-$weak upper gradient of u. Since X supports a weak $(1,\Phi)-$Poincaré inequality, the pair (u,g) satisfies a weak $(1,\Phi)-$Poincaré inequality. By [19, Lemma 5.15] the following Lipschitz-type estimate holds:

$$|u(x)-u(y)|\leq C\text{''}d(x,y)[\Phi^{-1}(\mathcal{M}\left(\Phi\circ g\right)(x))+\Phi^{-1}(\mathcal{M}\left(\Phi\circ g\right)(y)],$$

for $\mu-$almost all $x,y\in X$. The constant $C\text{''}>0$ depends only on the doubling constant C_d of μ and on the constant $C_{P,\Phi}$ of the $(1,\Phi)-$Poincaré inequality. Note that $\Phi^{-1}(\mathcal{M}\left(\Phi\circ g\right))$ is Borel measurable. It follows that $C\text{''}\ \Phi^{-1}(\mathcal{M}\left(\Phi\circ g\right))$ is a Hajłasz gradient of u and, according to Lemma 1, $C\text{''}\Phi^{-1}(\mathcal{M}(\Phi\circ g))\in L^{\Psi}(X)$. We proved that $u\in M^{1,\Psi}(X)$, therefore $N^{1,\Psi}(X)\subset M^{1,\Psi}(X)$.

There exists a sequence $g_i\in L^{\Psi}(X)$, $i\geq 1$, such that $\lim_{i\to\infty}\|g_i\|_{L^{\Psi}(X)}=\|u\|_{N^{1,\Psi}(X)}-\|u\|_{L^{\Psi}(X)}$. Let F be as in Lemma 1. Since $\|\Phi^{-1}(\mathcal{M}(\Phi\circ g_i))\|_{L^{\Psi}(X)}\leq F(\|g_i\|_{L^{\Psi}(X)})$, the definition of the norm of the space $M^{1,\Psi}(X)$ shows that $\|u\|_{M^{1,\Psi}(X)}\leq\|u\|_{L^{\Psi}(X)}+C\text{''}F(\|g_i\|_{L^{\Psi}(X)})$, for each $i\geq 1$. Letting $i\to\infty$ and taking into account the continuity of F, we obtain

$$\|u\|_{M^{1,\Psi}(X)}\leq\|u\|_{L^{\Psi}(X)}+C\text{''}F\left(\left(\|u\|_{N^{1,\Psi}(X)}-\|u\|_{L^{\Psi}(X)}\right)\right).$$

Assume now that Ψ and Ψ^{-1} satisfy the $\Delta'-$condition and that $\Psi\circ\Phi^{-1}$ satisfies the ∇_2-condition. By Lemma 1 we may take

$F(t) = C't$, $t \in [0, \infty)$. The above estimate implies $\|u\|_{M^{1,\Psi}(X)} \leq (1 + C'C") \|u\|_{N^{1,\Psi}(X)}$. Since $u \in N^{1,\Psi}(X)$ is arbitrary, the embedding $N^{1,\Psi}(X) \subset M^{1,\Psi}(X)$ is continuous in this case. ∎

Corollary 1. *Under the assumptions of Theorem 2, $N^{1,\Psi}(X) = M^{1,\Psi}(X)$ isomorphically as Banach spaces.*

Remark 4. *Taking $\Phi(t) = t^q$ and $\Psi(t) = t^p$, for $t \in [0, \infty)$, where $1 \leq q < p < \infty$, we obtain Theorem 4.9 from* [18] *by the above result.*

3. Mutual embeddings of Newtonian type and Cheeger type Orlicz-Sobolev spaces

First we introduce Orlicz-Sobolev spaces of Cheeger type on metric measure spaces.

Let $\Psi : [0, \infty) \to [0, \infty)$ be a Young function.

For $u \in L^{\Psi}(X)$ let

$$|u|_{1,\Psi} := \|u\|_{L^{\Psi}(X)} + \inf_{(g_i)} \liminf_{i \to \infty} \|g_i\|_{L^{\Psi}(X)}, \tag{3.1}$$

where the infimum is taken over all sequences $(g_i)_{i \geq 1}$, for which there exists a sequence $(u_i)_{i \geq 1}$ such that $u_i \to u$ in $L^{\Psi}(X)$ as $i \to \infty$ and $g_i \in L^{\Psi}(X)$ is an upper gradient of u_i, for each $i \geq 1$.

Definition 8. *For each Young function $\Psi : [0, \infty) \to [0, \infty)$ the Orlicz-Sobolev space of Cheeger type $H_{1,\Psi}(X)$ is the set of all equivalence class of functions $u \in L^{\Psi}(X)$ for which $|u|_{1,\Psi}$ is finite, where $u, v \in L^{\Psi}(X)$ are equivalent iff $|u - v|_{1,\Psi} = 0$.*

Remark 5. *For $\Psi(t) = t^p$, $p \geq 1$, $H_{1,\Psi}(X)$ is the Cheeger space $H_{1,p}(X)$ introduced in [4].*

Lemma 2. *The set $H_{1,\Psi}(X)$ is a vector space, equipped with the norm $|\cdot|_{1,\Psi}$.*

Proof. Let $u, v \in H_{1,\Psi}(X)$. Consider $(u_i)_{i \geq 1}$ and $(g_i)_{i \geq 1}$ for u, respectively $(v_i)_{i \geq 1}$ and $(h_i)_{i \geq 1}$, such that:

i) $u_i \to u$ and $v_i \to v$ in $L^{\Psi}(X)$ as $i \to \infty$;

ii) $g_i \in L^{\Psi}(X)$ is an upper gradient of u_i and $h_i \in L^{\Psi}(X)$ is an upper gradient of v_i, for each $i \geq 1$. . We have $u_i + v_i \to u + v$ in $L^{\Psi}(X)$ as $i \to \infty$ and $g_i + h_i \in L^{\Psi}(X)$ is an upper gradient of $u_i + v_i$ for each $i \geq 1$. Passing to subsequences we may assume that there exist the limits $\lim_{j\to\infty} \left\|g_{i_j}\right\|_{L^\Psi(X)} = \liminf_{i\to\infty} \|g_i\|_{L^\Psi(X)}$ and $\lim_{j\to\infty} \left\|h_{i_j}\right\|_{L^\Psi(X)} = \liminf_{i\to\infty} \|h_i\|_{L^\Psi(X)}$. Then

$$\liminf_{j\to\infty} \left\|g_{i_j} + h_{i_j}\right\|_{L^\Psi(X)} \leq \lim_{j\to\infty} \left\|g_{i_j}\right\|_{L^\Psi(X)} + \lim_{j\to\infty} \left\|h_{i_j}\right\|_{L^\Psi(X)} < +\infty \tag{3.2}$$

It follows that $u + v \in H_{1,\Psi}(X)$. It is easy to see that $\lambda u \in H_{1,\Psi}(X)$ and $|\lambda u|_{1,\Psi} = |\lambda| \cdot |u|_{1,\Psi}$ for every $\lambda \in \mathbb{R}$. We conclude that $H_{1,\Psi}(X)$ is a vector space. ∎

Obviously, $|u|_{1,\Psi} \geq 0$. The inequality (3.2) shows that

$$|u+v|_{1,\Psi} - \|u+v\|_{L^\Psi(X)} \leq \liminf_{i\to\infty} \|g_i\|_{L^\Psi(X)} + \liminf_{i\to\infty} \|h_i\|_{L^\Psi(X)}.$$

Taking the infimum over all sequences $(g_i)_{i\geq 1}$ and $(h_i)_{i\geq 1}$ as above, we get

$$|u+v|_{1,\Psi} - \|u+v\|_{L^\Psi(X)} \leq \left(|u|_{1,\Psi} - \|u\|_{L^\Psi(X)}\right) + \left(|v|_{1,\Psi} - \|v\|_{L^\Psi(X)}\right),$$

hence $|u+v|_{1,\Psi} \leq |u|_{1,\Psi} + |v|_{1,\Psi}$.

Note that $|u|_{1,\Psi} = 0$ implies $u = 0$ in $H_{1,\Psi}(X)$.

Remark 6. *In the definition (3.1) of the norm $|\cdot|_{1,\Psi}$ we can replace upper gradients by $\Psi-$weak upper gradients. This follows from the fact that for every $\Psi-$weak upper gradient $\widetilde{g} \in L^\Psi(X)$ of a function u and every $\varepsilon > 0$ there is an upper gradient $g \in L^\Psi(X)$ of u, such that $\|\widetilde{g} - g\|_{L^\Psi(X)} < \varepsilon$.* [19, Lemma 4.3].

In the case $\Psi(t) = t^p$, where $1 < p < +\infty$ it is proved [18, Theorem 4.10] that the Cheeger space $H_{1,p}(X)$ is isometrically equivalent to the Newtonian space $N^{1,p}(X)$. When $p = 1$, the space $N^{1,1}(X)$ embeds continuously into $H_{1,1}(X)$ by a norm non-increasing embedding, but it is not known if $H_{1,1}(X)$ embeds into $N^{1,1}(X)$. We will extend this results to the case of Orlicz-Sobolev spaces.

In order to extend [18, Theorem 4.10] to the case of Orlicz-Sobolev spaces, we will need the following convergence result, a Mazur-type lemma [19, Theorem 4.17].

Lemma 3. *Assume that X is a metric space and Ψ is a Young function.If $(u_j)_{j\geq 1}$ and $(g_j)_{j\geq 1}$ are a sequence of functions and a corresponding sequence of $\Psi-$weak upper gradients in $L^\Psi(X)$, such that*

$u_j \to u$ and $g_j \to g$ weakly in in $L^{\Psi}(X)$, then there are sequences $(\widetilde{u}_j)_{j\geq 1}$ and $(\widetilde{g}_j)_{j\geq 1}$ of convex combinations

$$\widetilde{u}_j = \sum_{k=j}^{n_j} \lambda_k u_k, \widetilde{g}_j = \sum_{k=j}^{n_j} \lambda_k g_k,$$

where $\lambda_k \geq 0$, $\sum_{k=j}^{n_j} \lambda_k = 1$, such that $\widetilde{u}_j \to u$ and $\widetilde{g}_j \to g$ in $L^{\Psi}(X)$. Moreover, g is a $\Psi-$weak upper gradient of a representative of u.

Recall that $L^{\Psi}(X)$ is reflexive if and only if Ψ satisfies the Δ_2- and ∇_2-conditions for large values of the variable, the measure μ being nonatomic [17].

Theorem 2. *Let $\Psi : [0,\infty) \to [0,\infty)$ be a Young function. Then $H_{1,\Psi}(X) \subset N^{1,\Psi}(X)$ as a continuous embedding. Moreover, if Ψ satisfies the Δ_2- and ∇_2-conditions for large values of the variable, then $H_{1,\Psi}(X)$ and $N^{1,\Psi}(X)$ are isometrically equivalent.*

Proof. If $u \in N^{1,\Psi}(X)$, then we take $u_i = u \in L^{\Psi}(X)$ and $g_i \in L^{\Psi}(X)$ an upper gradient of u, such that

$$\|g_i\|_{L^{\Psi}(X)} < \left(\|u\|_{N^{1,\Psi}(X)} - \|u\|_{L^{\Psi}(X)}\right) + \frac{1}{i}$$

for all $i \geq 1$. Then $\liminf_{i\to\infty} \|g_i\|_{L^{\Psi}(X)} \leq \left(\|u\|_{N^{1,\Psi}(X)} - \|u\|_{L^{\Psi}(X)}\right)$, hence $|u|_{1,\Psi} \leq \|u\|_{N^{1,\Psi}(X)}$. In particular, $u \in H_{1,\Psi}(X)$.

Assume now that Ψ satisfies the Δ_2- and ∇_2-conditions for large values of the variable. Then $L^{\Psi}(X)$ is reflexive. Let $u \in H_{1,\Psi}(X)$. Consider the sequences $(u_i)_{i\geq 1}$ and $(g_i)_{i\geq 1}$ such that $u_i \to u$ in $L^{\Psi}(X)$ as $i \to \infty$, $g_i \in L^{\Psi}(X)$ is an upper gradient of u_i, for each $i \geq 1$ and $\liminf_{i\to\infty} \|g_i\|_{L^{\Psi}(X)} < +\infty$. Passing to a subsequence we may assume that there exists $\lim_{i\to\infty} \|g_i\|_{L^{\Psi}(X)} < +\infty$. Then $(g_i)_{i\geq 1}$ is bounded in $L^{\Psi}(X)$. Since the Banach space $L^{\Psi}(X)$ is reflexive, $(g_i)_{i\geq 1}$ has a weakly convergent subsequence, which we denote again by $(g_i)_{i\geq 1}$. Let $g \in L^{\Psi}(X)$ such that $g_i \to g$ weakly in $L^{\Psi}(X)$. By Lemma 3, g is a $\Psi-$weak upper gradient of u, hence $u \in N^{1,\Psi}(X)$. Moreover, by the weak lower semicontinuity of the norm of $L^{\Psi}(X)$,

$$\|g\|_{L^{\Psi}(X)} \leq \liminf_{i\to\infty} \|g_i\|_{L^{\Psi}(X)}.$$

This inequality implies $\|u\|_{N^{1,\Psi}(X)} - \|u\|_{L^{\Psi}(X)} \leq \|u\|_{H_{1,\Psi}(X)} - \|u\|_{L^{\Psi}(X)}$, therefore

$$\|u\|_{N^{1,\Psi}(X)} = \|u\|_{H_{1,\Psi}(X)} .$$

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