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SUB-RIEMANNIAN STRUCTURES ASSOCIATED WITH A GENERALIZED LAGRANGE METRIC

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Abstract. We continue the investigations of GL-space with the study of sub-Riemannian structures which can be geometrically associated to a generalized Lagrange metric.

1. Introduction

Let M be a smooth manifold of dimension n, E a smooth manifold of dimension $n+m$ and $\pi : E \to M$ a surjective submersion.

For every $x \in M$, the fiber $\pi^{-1}(x)$ is a submanifold of dimension m in E.

Let $\pi_{*,u} : T_uE \to T_{\pi(u)=x}M$ be a linear tangent map. Its kernel $V_uE := ker\pi_{*,u}$ is called the vertical subspace in T_uE. Then, the mapping $V : u \to V_uE$ is a distribution on E that is called the vertical distribution (VE).

A distribution on E which is supplementary to the vertical distribution on E will be called a horizontal distribution on E.

We note by $VE = \bigcup_{u\in E} V_uE$ the vertical subbundle of TE and by $HE = \bigcup_{u\in E} H_uE$ the horizontal subbundle of TE.

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Then

$$T_uE = H_uE \oplus V_uE, \forall u \in E \tag{1.1}$$

and

$$HE \oplus VE = TE. \tag{1.2}$$

A horizontal distribution can be given by the local vector fields

$$\delta_i = \frac{\partial}{\partial x^i} - N_i^a(x,y)\frac{\partial}{\partial y^a} \tag{1.3}$$

and $\pi_*(\delta_i) = \frac{\partial}{\partial x^i}$ (local fields on M)

A section in the horizontal subbundle will be called a horizontal vector field on E and a section in the vertical subbundle will be called a vertical vector field on E. Any such fields can be given in the form $X^i(x,y)\delta_i$ or $Y^a\frac{\partial}{\partial y^a}$.

In [4] we studied distribution on the space of the π submersion. We give the structure equations and the Bianchi identities.

In the following, we will restrict the previews considerations on the surjective submersion $\tau : TM \to M$ which is the tangent bundle on the M manifold.

Let (x^i) be the local coordinates on M and (x^i, y^i) , $i,j = \overline{1, n = dimM}$ local coordinates on TM.

In [1] is introduced the notion of d-geometrical object as being a geometrical object on TM which on a change of coordinates $(x,y) \to (\tilde{x}, \tilde{y})$ on TM:

$$\begin{aligned} &\tilde{x}^i = \tilde{x}^i(x^1, ..., x^n) \\ &\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j}(x)y^j \\ &rang\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) = n \end{aligned} \tag{1.4}$$

acts as a geometric object on M this means that in it's transformation rule only appears $\left(\frac{\partial \tilde{x}^i}{\partial \tilde{x}^j}\right)$ although it's components depend only on x and y.

The generalized Lagrange space $GL = (M, g_{ij}(x,y))$ shortly GL-space is studied in [1], where $g_{ij}(x,y)$ is a generalized Lagrange metric that means a d-tensor field (0,2).

In [2] we studied a symmetric GL-metric with application in relativistic optics.

2. Main result

In this paragraph we define the notion of sub-Riemannian structure which can be geometrically associated to a generalized Lagrange metric.

Definition 2.1. A sub-Riemannian structure on a differential manifold X is a distribution on X that means a subbundle $H \subset TX$ together with a Riemannian metric g.

The notion of sub-Riemannian structure was introduced by *R.* Strichartz in 1986 [5] and now it is used in 2000 MSC.

A sub-Riemannian structure is in fact a nonholonomic Riemannian manifold which was studied by Gh. Vrănceanu since 1926 [6].

The notion of sub-Riemannian can be weakened by changing Riemannian metric g with a semi-Riemannian metric with constant signature.

On $X = TM$ we have the vertical distribution given by the kernel of the application $\tau_{*,u} u \in TM$ and we can also consider a supplementary distribution named horizontal.

We will prove that in the presence of a GL-metric $g_{ij}(x, y)$, these structures become sub-Riemannian structures.

Theorem 2.1. Let $GL^n = (M, g_{ij}(x, y))$ be a generalized Lagrange space and $TTM = HTM \oplus VTM$. If $(g_{ij}(x, y))$ is positively defined, then HTM and VTM can have Riemannian metrics so that these become sub-Riemannian structures on tangent manifold TM.

Proof. The condition for that $g_{ij}(x, y)$ is d-tensor field can be written in the form:

$$g_{ij} = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^h}{\partial x^j} \tilde{g}_{kh}(\tilde{x}, \tilde{y}) \tag{2.1}$$

The vertical distribution is generated by local fields $\left(\frac{\partial}{\partial y^i}\right)$ which can be viewed like local sections in vertical bundle.

We define bilinear symmetric application

$g^V : V_u TM \times V_u TM \to R, u \in TM$

$$g^V(A, B) = g_{ij}(x, y) A^i B^i, A = A^i \frac{\partial}{\partial y^i}, B = B^j \frac{\partial}{\partial y^j} \tag{2.2}$$

This application is positively defined if $(g_{ij}(x, y))$ is positively defined that means $g_{ij}(x, y) z^i z^j > 0, (z^i) \neq 0, \forall (x, y)$

The right term in (2.2) depends only on A, B used (2.1) and rule of change

$$\tilde{A}^i = \frac{\partial \tilde{x}^i}{\partial x^j} A^i \tag{2.3}$$

for the components of vertical fields.

So, g^V is a Riemannian metric in sub-bundle VTM and (VTM, g^V) is a sub-Riemannian structure.

If we give to VTM distribution the Pfaff system $\{dx^i\}$, we obtain

$$g^V = g_{ij}(x, y) dx^i \otimes dx^j. \tag{2.4}$$

Now, we consider the horizontal distribution HTM with local vector fields

(2.5) $$\delta_i = \frac{\partial}{\partial x^i} - N_i^j(x,y)\frac{\partial}{\partial y^j}$$

A section in the horizontal subbundle is $X^i\delta_i$.

We define the bilinear symmetric application

$g^H : H_uTM \times HTM \to R$ by

(2.6) $$g^H(X,Y) = g_{ij}(x,y)X^iY^j$$

$X = X^i\delta_i, Y = Y^j\delta_j$

used (2.1) and transformations $\delta_i = \frac{\partial \tilde{x}^j}{\partial x^i}\tilde{\delta}_j, \tilde{X}^i = \frac{\partial \tilde{x}^i}{\partial x^j}X^j$

This application is positively defined if $g_{ij}(x,y))$ is positively defined $\forall(x,y)$. So, g^H defines a Riemannian metric in the horizontal subbundle and (HTM, g^H) is a subRiemannian structure on TM.

Remark. If the horizontal distribution is given by 1-form $\{\delta y^i\}$, then we have

$g^H = g_{ij}\delta y^i \otimes \delta y^j$

and

$G = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j$

is in the hypothesis of theorem 2.1. a Riemannian metric on TM is Sasaki type.

The theorem 2.1. have a reciproc.

Theorem 2.2. 1) If (VTM, g^V) is a sub-Riemannian structure on TM, then M is a generalized Lagrange space (GL-space).

2) If (HTM, g^H) is a sub-Riemannian structure on TM, then M is a generalized Lagrange space.

Proof. We have the Riemannian metric g^V in the vertical subbundle and we have

$g^v_{ij}(x,y) = g^V\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right)$

and obtain a generalized Lagrange metric $(g^v_{ij}(x,y))$ and the structure $(M, g^v_{ij}(x,y))$ becomes a generalized Lagrange space.

Also, if (HTM, g^H) is a sub-Riemannian structure we define

$g^h_{ij}(x,y) = g^H(\delta_i, \delta_j)$

and we obtain the GL-metric $(g^h_{ij}(x,y))$.

So $(M, g^h_{ij}(x,y))$ becomes a GL-space.

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