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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 21 (2011), No. 1, 215 - 224

μ- SCALE INVARIANT LINEAR RELATIONS IN HILBERT SPACES

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Abstract. The concept of μ-scale invariant operator with respect to an unitary transformation in a separable Hilbert space is extended to the case of linear relations (multi-valued linear operators). It is shown that if S is a nonnegative linear relation which is μ-scale invariant for some $\mu > 0$, then its adjoint S^* and its extreme selfadjoint extensions S_F and S_N are also μ-scale invariant.

1. Introduction

Let U be a unitary operator in a separable Hilbert space $\mathfrak{H}$ and let $\mu \in \mathbb{C} \setminus \{0\}$ a complex number. In [9] K. A. Makarov and E. Tsekanovskii introduced and studied the concept of μ scale invariant (unbounded) operator with respect to U. The main goal of this note is to show how this concept can be extended to the case of linear relations (multi-valued linear operators) in Hilbert spaces.

The note is organized as follows. In the next section some results concerning linear relations in Hilbert spaces are presented. The third section contains some general information with respect to nonnegative selfadjoint linear relations in Hilbert spaces. In Section 4 the concept of μ-scale invariant relation with respect to an unitary transformation in a separable Hilbert space is introduced and studied. Some theoretical examples are presented in Section 5.

Keywords and phrases: Hilbert space, linear relation, nonnegative selfadjoint extension, unitary transformation, scale invariant relation.
(2000)Mathematics Subject Classification: Primary 47A06; Secondary 47A63, 47B25, 47B65.

2. Preliminaries

Let $(\mathfrak{H}, \langle \cdot \mid \cdot \rangle)$ be a separable complex Hilbert space. A typical element of the Cartesian product $\mathfrak{H} \times \mathfrak{H}$ is an ordered pair $\{x, y\}$ with x, $y \in \mathfrak{H}$. The product space $\mathfrak{H} \times \mathfrak{H}$ equipped with the usual Hilbert-space inner product:

$$\langle \{x_1, y_1\} \mid \{x_2, y_2\} \rangle = \langle x_1 \mid x_2 \rangle + \langle y_1 \mid y_2 \rangle, \tag{2.1}$$

where x_1, x_2, y_1, $y_2 \in \mathfrak{H}$ is called the Hilbert space $\mathfrak{H} \oplus \mathfrak{H}$.

2.1. **Linear relations in Hilbert spaces.** A linear relation (multi-valued linear operator) A in $\mathfrak{H}$ is a linear subspace of the space $\mathfrak{H} \times \mathfrak{H}$, The notations $\operatorname{dom} A$ and $\operatorname{ran} A$ denote the domain and the range of A, defined by

$$\operatorname{dom} A = \{\, x : \{x, y\} \in A \,\}, \quad \operatorname{ran} A = \{\, y : \{x, y\} \in A \,\}.$$

Furthermore, $\ker A$ and $\operatorname{mul} A$ denote the kernel and the multi-valued part of A, defined by

$$\ker A = \{\, x : \{x, 0\} \in A \,\}, \quad \operatorname{mul} A = \{\, y : \{0, y\} \in A \,\}.$$

A linear relation A is the graph of an operator if and only if $\operatorname{mul} A = \{0\}$ and the inverse A^{-1} is given by $\{\, \{y, x\} : \{x, y\} \in A \,\}$. The following identities express the duality between A and its inverse A^{-1}:

$$\operatorname{dom} A^{-1} = \operatorname{ran} A, \; \operatorname{ran} A^{-1} = \operatorname{dom} A,$$

$$\ker A^{-1} = \operatorname{mul} A, \; \operatorname{mul} A^{-1} = \ker A.$$

For linear relations A and B in $\mathfrak{H}$ the operator-like sum $A + B$ is the linear relation in $\mathfrak{H}$ defined by

$$A + B = \{\, \{x, y + z\} : \{x, y\} \in A, \{x, z\} \in B \,\}.$$

For $\lambda \in \mathbb{C}$ the linear relation λA in $\mathfrak{H}$ is defined by

$$\lambda A = \{\, \{x, \lambda y\} : \{x, y\} \in A \,\},$$

while $A - \lambda$ stands for $A - \lambda I$, where I is the identity operator on $\mathfrak{H}$. From $A - \lambda = \{\{x, y - \lambda x\} : \{x, y\} \in A\}$ it follows that

$$\ker (A - \lambda) = \{\, x : \{x, \lambda x\} \in A \,\}.$$

Furthermore, the following equality holds

$$(A - \lambda) + \lambda I = A \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{2.2}$$

For linear relations A and B in $\mathfrak{H}$ the product AB is defined as the relation

$$AB = \{\, \{x, y\} : \{x, z\} \in B, \{z, y\} \in A \text{ for some } z \in \mathfrak{H} \,\}.$$

The following inclusions hold true:

$$\operatorname{dom} AB \subset \operatorname{dom} B, \quad \operatorname{ran} AB \subset \operatorname{ran} A,$$

$$\ker A \subset AB, \quad \operatorname{mul} A \subset \operatorname{mul} AB.$$

Furthermore, the product of linear relations A and B and their inverses A^{-1} and B^{-1} are related as follows:

$$(AB)^{-1} = B^{-1}A^{-1} \tag{2.3}$$

For $\lambda \in \mathbb{C}$ the notation λA agrees in this sense with $(\lambda I)A$. The product of relations is clearly associative. Hence A^n, $n \in \mathbb{Z}$, is defined as usual with $A^0 = I$ and $A^1 = A$. Next result collects basic properties of powers of linear relations.

Lemma 2.1. *Let A be a linear relation in a Hilbert space $\mathfrak{H}$. Then for all $n \in \mathbb{N} \cup \{0\}$*

$$\operatorname{dom} A^{n+1} \subset \operatorname{dom} A^n, \quad \operatorname{ran} A^{n+1} \subset \operatorname{ran} A^n, \tag{2.4}$$

$$\ker A^{n+1} \supset \ker A^n, \quad \operatorname{mul} A^{n+1} \supset \operatorname{mul} A^n, \tag{2.5}$$

and for all $p, k \in \mathbb{N} \cup \{0\}$

$$\ker A^p \subset \operatorname{dom} A^k, \quad \operatorname{mul} A^p \subset \operatorname{ran} A^k. \tag{2.6}$$

2.2. **The adjoint of a linear relation.** The adjoint of a linear relation A in the Hilbert space $\mathfrak{H}$ is the closed linear relation A^* in $\mathfrak{H}$ defined by

$$A^* = \{\, \{f, f'\} \in \mathfrak{H} \times \mathfrak{H} : \langle f' \mid h \rangle = \langle f \mid h' \rangle \text{ for all } \{h, h'\} \in A \,\}.$$

It can be easily seen that $(\lambda A)^* = \bar{\lambda} A^*$ for all $\lambda \in \mathbb{C}$. Furthermore, observe that $(A^{-1})^* = (A^*)^{-1}$, so that $(\operatorname{dom} A)^\perp = \operatorname{mul} A^*$ and $(\operatorname{ran} A)^\perp = \ker A^*$. Clearly the double adjoint A^{**} is the closure of the relation A. A linear relation A in a Hilbert space $\mathfrak{H}$ is said to be symmetric if $A \subset A^*$, or equivalently, if $\langle f' \mid f \rangle \in \mathbb{R}$ for all $\{f, f'\} \in A$. A linear relation A in a Hilbert space $\mathfrak{H}$ is said to be nonnegative if $\langle f' \mid f \rangle \geq 0$ for all $\{f, f'\} \in A$. Furthermore, a linear relation A in a Hilbert space $\mathfrak{H}$ is said to be self-adjoint if $A^* = A$ (so that it is automatically closed).

If A and B are linear relations in $\mathfrak{H}$ then always

$$B^*A^* \subset (AB)^*. \tag{2.7}$$

The following result offers certain sufficient condition in order to have equality in (2.7).

Lemma 2.2. *Assume that A is a linear relation in $\mathfrak{H}$ and U a boundedly invertible operator in $\mathfrak{H}$, respectively. Then the following two identities hold:*

$$(UA)^* = A^*U^*, \quad (AU)^* = U^*A^*. \tag{2.8}$$

Proof. It follows from (2.7) that $A^*U^* \subset (UA)^*$. To prove the reverse inclusion let $\{x, x'\} \in (UA)^*$. Let also $\{a, a'\} \in A$, so that $\{a, Ua'\} \in UA$. This implies that $\langle x' \mid a\rangle = \langle x \mid Ua'\rangle$, which further leads to

$$\langle x' \mid a\rangle - \langle U^*x \mid a'\rangle = \langle x' \mid a\rangle - \langle x \mid Ua'\rangle = 0,$$

for all $\{a, a'\} \in A$. This implies that $\{U^*x, x'\} \in A^*$. Then $\{x, x'\} \in A^*U^*$ and the former identity in (2.8) is now proved. The latter identity in (2.8) can be now proved in the following way:

$$\begin{aligned}(AU)^* &= ((A^{-1})^{-1}(U^{-1})^{-1})^* = \left((U^{-1}A^{-1})^{-1}\right)^* = \left((U^{-1}A^{-1})^*\right)^{-1} \\ &= \left((A^{-1})^*(U^{-1})^*\right)^{-1} = \left((A^*)^{-1}(U^*)^{-1}\right)^{-1} = U^*A^*.\end{aligned}$$

□

3. Some general remarks concerning the Kreĭn-von Neumann and the Friedrichs extensions

This section contains general information concerning nonnegative self-adjoint extensions of a nonnegative relation S. In particular, the Kreĭn-von Neumann and the Friedrichs extensions are introduced.

A linear relation S in a Hilbert space $\mathfrak{H}$ is said to be semi-bounded from below if there exists a number $a \in \mathbb{R}$ such that $\langle f' \mid f\rangle \geq a\langle f \mid f\rangle$ for all $\{f, f'\} \in S$. Note that in this case the relation S is automatically symmetric and it has equal defect numbers. The largest number $a \in \mathbb{R}$ which serves this purpose is called the lower bound $m(S)$ of S. It is given by $m(S) = 0$ when S is purely multi-valued and by

$$m(S) = \inf\{\, \langle f', f\rangle : \{f, f'\} \in S, \|f\| = 1 \,\}$$

otherwise. Clearly, the lower bound of $\operatorname{clos} S$ is equal to the lower bound of S, where the notation $\operatorname{clos} S$ stands for the closure of S. When the lower bound is nonnegative the relation S is called nonnegative: $\langle f' \mid f\rangle \geq 0$, $\{f, f'\} \in S$. The fact that:

$$\langle \lambda f' + \mu g' \mid \lambda f + \mu g\rangle \geq 0, \quad \{f, f'\}, \{g, g'\} \in S, \quad \lambda, \mu \in \mathbb{C},$$

leads to the Cauchy inequality for nonnegative relations:

$$|\langle f' \mid g\rangle|^2 \leq \langle f' \mid f\rangle\langle g' \mid g\rangle, \quad \{f, f'\}, \{g, g'\} \in S. \tag{3.1}$$

Let S be a nonnegative linear relation and define on $\operatorname{dom} \mathfrak{t} = \operatorname{dom} S$

$$\mathfrak{t}[f,g] = \langle f' \mid g \rangle, \quad \{f,f'\}, \{g,g'\} \in S. \tag{3.2}$$

Then (3.2) gives rise to a nonnegative form since

$$\langle f' \mid g \rangle = \langle f \mid g' \rangle = \langle f'' \mid g \rangle \geq 0, \quad \{f,f'\}, \{f,f''\}, \{g,g'\} \in S.$$

In fact, the form $\mathfrak{t}$ in (3.2) is closable, cf. [5]. The closure $\bar{\mathfrak{t}}$ of the form $\mathfrak{t}$ in (3.2) is nonnegative and induces a nonnegative self-adjoint relation S_F which is the orthogonal sum of the self-adjoint operator induced by the form $\bar{\mathfrak{t}}$ in $\overline{\operatorname{dom}} S$ (cf. [5]) and the multi-valued part $\{0\} \times \operatorname{mul} S^*$ (cf. [7]). The nonnegative self-adjoint relation S_F is an extension of S and has the same lower bound as S, cf. [3]. By construction $\operatorname{mul} S_F = \operatorname{mul} S^*$, so that the Friedrichs extension is an operator if and only if S is densely defined (and necessarily an operator). For a nonnegative linear relation S introduce the space $\operatorname{dom}[S]$ as the set of all $f \in \mathfrak{H}$ for which there exists a sequence $(\{f_n, f_n'\}) \subset S$ such that

$$f_n \to f, \quad \langle f_n' - f_m', f_n - f_m \rangle \to 0 \quad m,\, n \to \infty.$$

It can be shown that $\operatorname{dom}[S] = \operatorname{dom}[S_F] = \operatorname{dom} S_F^{\frac{1}{2}}$, and that

$$S_F = \{\, \{f,f'\} \in S^* :\ f \in \operatorname{dom}[S]\,\}. \tag{3.3}$$

Moreover, the Friedrichs extension is the only self-adjoint extension of S whose domain is contained in $\operatorname{dom}[S]$.

If the linear relation S is nonnegative (self-adjoint), then likewise the formal inverse S^{-1} of S is nonnegative (self-adjoint). Hence the self-adjoint relation

$$S_N = ((S^{-1})_F)^{-1} \tag{3.4}$$

is also a nonnegative self-adjoint extension of S; in fact it is the Kreĭn-von Neumann extension of S, cf. [8], [1], [4]. In particular, S_N is the only selfadjoint extension of S whose range is contained in $\operatorname{ran}[S] := \operatorname{dom}[S^{-1}]$ and the following description holds

$$S_N = \{\, \{f,f'\} \in S^* :\ f' \in \operatorname{ran}[S]\,\}.$$

Notice also that $\ker S_N = \ker S^*$, and that $f' \in \operatorname{ran}[S]$ if and only if there exists a sequence $(\{f_n, f_n'\}) \subset S$, such that

$$f_n' \to f', \quad \langle f_n' - f_m', f_n - f_m \rangle \to 0, \quad m,\, n \to \infty.$$

The Kreĭn-von Neumann and the Friedrichs extensions are extreme nonnegative self-adjoint extensions of S in the following sense: if H is any nonnegative self-adjoint extension of S, then $S_F \leq H \leq S_N$ holds, where the inequalities are in the sense of resolvents or, equivalently, in the sense of the corresponding forms, cf. [7].

4. μ-scale invariant linear relations in Hilbert spaces

Definition 4.1. Let U be a unitary operator in a separable complex Hilbert space $(\mathfrak{H}, \langle \cdot \mid \cdot \rangle)$ and let $\mu \in \mathbb{C} \backslash \{0\}$. A linear relation S is said to be μ- scale invariant with respect to U if the following condition is satisfied:

$$\mu S \subset U S U^*. \tag{4.1}$$

It follows from the definition that

$$U^*(\operatorname{dom} S) \subset \operatorname{dom} S. \tag{4.2}$$

Indeed, if $x \in \operatorname{dom} S$ then $\{x, x'\} \in S$ for some $x' \in \mathfrak{H}$, so that $\{x, \mu x'\} \in \mu S \subset U S U^*$. This implies that $\{U^* x, \mu U^* x'\} \in S$. Then $U^* x \in \operatorname{dom} S$ which shows that (4.2) holds true. Furthermore, a similar invariance inclusion holds for the multi-valued part of S, namely

$$U^*(\operatorname{mul} S) \subset \operatorname{mul} S. \tag{4.3}$$

To prove this, let $m \in \operatorname{mul} S$, so that $\{0, m\} \in S$ which implies that $\{0, m\} \in \mu S$. Then $\{0, m\} \in U S U^*$, so that $\{0, x\} \in U^*$, $\{x, y\} \in S$ and $\{y, m\} \in U$ for some x, $y \in \mathfrak{H}$. Therefore $x = 0$ and $y = U^* m \in \operatorname{mul} S$, which shows that (4.3) holds true.

Some basic properties of μ- scale invariant linear relations are collected in the following result.

Lemma 4.2. *Assume that S is a linear relation in $\mathfrak{H}$ which is μ- scale invariant with respect to U. Then*

(i) *the inverse relation S^{-1} is μ^{-1}- scale invariant with respect to U;*
(ii) *the relation S is also μ- scale invariant with respect to the unitary transformation U^n, $n \in \mathbb{N}$. That is $\mu^n S \subset U^n S U^{*n}$, for all $n \in \mathbb{N}$;*
(iii) *the adjoint relation S^* is $\bar{\mu}$- scale invariant with respect to U;*

Proof. (i) It follows from (2.3) and (4.1) that

$$\mu^{-1} S^{-1} = (\mu S)^{-1} \subset (U S U^*)^{-1} = (U^*)^{-1} S^{-1} U^{-1} = U S^{-1} U^*,$$

so that S^{-1} is μ^{-1}- scale invariant with respect to U.

(ii) This follows by induction on $n \in \mathbb{N}$.

(iii) Taking into account the identities in (2.8) one has

$$\begin{aligned} USU^* &= (U^*)^* S^*U^* = (SU^*)^* U^* \\ &= (USU^*)^* \supseteq (\mu S)^* = \bar{\mu} S^*. \end{aligned}$$

This completes the proof. □

The main result of this note is now stated.

Theorem 4.3. *Assume that S is a nonnegative linear relation in $\mathfrak{H}$ which is μ- scale invariant with respect to U. Then*

(i) *the Friedrichs extension S_F of S is μ- scale invariant with respect to U;*
(ii) *the Krein-von Neumann extension S_F of S is μ- scale invariant with respect to U.*

Proof. (*i*) Let $\{f, f'\} \in S_F$. Then there exists a sequence $(\{f_n, f_n'\}) \subset S$ such that $f_n \to f$, and

$$(f_n' - f_m', f_n - f_m) \to 0, \quad \text{as} \quad m, n \to \infty.$$

It follows from $\{f_n, \mu f_n'\} \in \mu S \subseteq USU^*$ that

$$\{U^* f_n, \mu U^* f_n'\} \in S. \tag{4.4}$$

Furthermore,

$$U^* f_n \to U^* f, \tag{4.5}$$

and

$$(\mu U^* f_n' - \mu U^* f_m', f_n - f_m) \to 0, \quad \text{as} \quad m, n \to \infty. \tag{4.6}$$

Since $\{f, \mu f'\} \in \mu S_F \subset \mu S^* \subseteq US^*U^*$ it follows that

$$\{U^* f_n, \mu U^* f_n'\} \in S^*. \tag{4.7}$$

A combination of (4.5), (4.6) and (4.7) leads to $\{U^* f, \mu U^* f'\} \in S_F$, so that $\{f, \mu f'\} \in US_FU^*$. This implies that $\mu S_F \subset US_FU^*$.

(*ii*) Let $\{f, f'\} \in S_N$. Then there exists a sequence $(\{f_n, f_n'\}) \subset S$ such that $f_n' \to f'$, and

$$(f_n' - f_m', f_n - f_m) \to 0, \quad \text{as} \quad m, n \to \infty.$$

It follows from $\{f_n, \mu f_n'\} \in \mu S \subseteq USU^*$ that

$$\{U^* f_n, \mu U^* f_n'\} \in S. \tag{4.8}$$

Furthermore,

$$U^* f_n' \to U^* f', \tag{4.9}$$

and

$$(\mu U^* f'_n - \mu U^* f'_m, f_n - f_m) \to 0, \quad \text{as} \quad m, n \to \infty. \tag{4.10}$$

Since $\{f, \mu f'\} \in \mu S_N \subset \mu S^* \subseteq U S^* U^*$ it follows that

$$\{U^* f_n, \mu U^* f'_n\} \in S^*. \tag{4.11}$$

A combination of (4.9), (4.10) and (4.11) leads to $\{U^* f, \mu U^* f'\} \in S_N$, so that $\{f, \mu f'\} \in U S_N U^*$. This implies that $\mu S_N \subset U S_N U^*$. □

Remark 4.4. Since both μS_F and $U S_F U^*$ are selfadjoint linear relations and $\mu S_F \subset U S_F U^*$ it follows that in fact $\mu S_F = U S_F U^*$. Similar arguments imply that also $\mu S_N = U S_N U^*$.

5. Examples

5.1. **A purely multi-valued relation.** Let $\mathfrak{H}$ be a Hilbert space and assume that $\mathfrak{K}$ is a not necessarily closed linear subspace of $\mathfrak{H}$. Let U be a unitary operator in the Hilbert space $\mathfrak{H}$ and assume that $\mathfrak{K}$ is an invariant subspace of U^*, i.e. $U^*(\mathfrak{K}) \subset \mathfrak{K}$. Consider the purely multi-valued relation S in $\mathfrak{H}$ defined by $S = \{0\} \times \mathfrak{K}$. Then S is closed if and only if $\mathfrak{K}$ is closed. Furthermore, it is μ invariant with respect to U for any $\mu > 0$. To see this, let $\{0, k\} \in \mu S$ so that $\{0, k\} \in S$. Then $U^* k \in \operatorname{mul} S$ which implies that $\{0, U^* k\} \in S$, so that $\{0, k\} \in US$. Therefore, $\{0, k\} \in USU^*$. Thus, $\mu S \subset USU^*$. Furthermore, $S^* = \mathfrak{K}^\perp \times \mathfrak{H}$, $S_F = \{0\} \times \mathfrak{H}$ and $S_N = \mathfrak{K}^\perp \times \overline{\mathfrak{K}}$, where $\overline{\mathfrak{K}}$ is the closure of $\mathfrak{K}$ in $\mathfrak{H}$. It is easily seen that S^*, S_F and S_N are also μ scale invariant with respect to U.

5.2. **A purely kernel relation.** Let $\mathfrak{H}$ be a Hilbert space and assume that $\mathfrak{K}$ is a not necessarily closed linear subspace of $\mathfrak{H}$. Let U be a unitary operator in the Hilbert space $\mathfrak{H}$ and assume that $\mathfrak{K}$ is an invariant subspace of U, i.e. $U(\mathfrak{K}) \subset \mathfrak{K}$. Consider the purely kernel relation S in $\mathfrak{H}$ defined by $S = \mathfrak{K} \times \{0\}$. Then S is closed if and only if $\mathfrak{K}$ is closed. Furthermore, it is μ invariant with respect to U for any $\mu > 0$. To see this, let $\{k, 0\} \in \mu S$ so that $\{k, 0\} \in S$. Therefore $k \in \ker S$ which further implies that $Uk \in \ker S$. Thus $\{Uk, 0\} \in S$, so that $\{k, 0\} \in SU^*$. Therefore, $\{k, 0\} \in USU^*$. Thus, $\mu S \subset USU^*$. Furthermore, $S^* = \mathfrak{H} \times \mathfrak{K}^\perp$, $S_F = \mathfrak{H} \times \{0\}$ and $S_N = \overline{\mathfrak{K}} \times \mathfrak{K}^\perp$. All these three linear relations are also μ scale invariant with respect to U.

5.3. **The inverse of a graph of a differential operator.** Let $\mathfrak{H} = L^2(0,\infty)$ and denote the usual inner product by $\langle \cdot \mid \cdot \rangle$. Let also $\mu > 0$, $\mu \neq 1$, and consider U the unitary scaling transformation

$$(Uf)(x) = \mu^{-\frac{1}{4}} f\left(\mu^{-\frac{1}{2}} x\right), \quad f \in L^2(0,\infty).$$

Consider also $H^{2,2}(0,\infty)$ the corresponding Sobolev space and define the linear relation

$$S = \left\{ \left\{ -\frac{d^2 f}{dx^2}, f \right\} \,:\, f \in H^{2,2}(0,\infty),\ f(0) = 0,\ f'(0) = 0 \right\}.$$

Clearly S is a closed nonnegative linear relation with nontrivial multivalued part. Its adjoint is the linear relation given by

$$S^* = \left\{ \left\{ -\frac{d^2 f}{dx^2}, f \right\} \,:\, f \in H^{2,2}(0,\infty) \right\}.$$

Then the extreme extensions S_F and S_N are given by:

$$S_F = \left\{ \left\{ -\frac{d^2 f}{dx^2}, f \right\} \,:\, f \in H^{2,2}(0,\infty),\ f(0) = 0 \right\}$$

and by

$$S_N = \left\{ \left\{ -\frac{d^2 f}{dx^2}, f \right\} \,:\, f \in H^{2,2}(0,\infty),\ f'(0) = 0 \right\}.$$

respectively. A straightforward computation shows that all linear relations S, S^*, S_F and S_N are μ- scale invariant with respect to the transformation U. Any other nonnegative selfadjoint extension of S different from S_F and S_N can be parametrized as follows

$$S_\alpha = \left\{ \left\{ -\frac{d^2 f}{dx^2}, f \right\} \,:\, f \in H^{2,2}(0,\infty),\ f'(0) = \alpha f(0) \right\}.$$

for some $\alpha > 0$. It can be verified that S_α is not μ invariant with respect to U. Therefore the linear relation S admits only two μ scale invariant extensions, S_F and S_N.

6. Acknowledgments

The work of Adrian Sandovici was supported by AMPOSDRU, project number POSDRU/89/1.5/S/49944.

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