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OPTIMIZATION PROBLEMS ON QUASI-THRESHOLD GRAPHS

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Abstract. In this paper we characterize quasi-threshold graphs using the weakly decomposition, determine: density and stability number for quasi-threshold graphs.

1. Introduction

When searching for recognition algorithms, frequently appears a type of partition for the set of vertices in three classes A, B, C, which we call a *weakly decomposition*, such that: A induces a connected subgraph, C is totally adjacent to B, while C and A are totally non-adjacent.

The structure of the paper is the following. In Section 2 we present the notations to be used, in Section 3 we give the notion of weakly decomposition and in Section 4 we determine the clique number, the stability number and give some applications in optimization problems.

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2. General notations

Throughout this paper, $G = (V, E)$ is a connected, finite and undirected graph, without loops and multiple edges ([1]), having $V = V(G)$ as the vertex set and $E = E(G)$ as the set of edges. $\overline{G}$ is the complement of G. If $U \subseteq V$, by $G(U)$ we denote the subgraph of G induced by U. By $G - X$ we mean the subgraph $G(V - X)$, whenever $X \subseteq V$, but we simply write $G - v$, when $X = \{v\}$. If $e = xy$ is an edge of a graph G, then x and y are adjacent, while x and e are incident, as are y and e. If $xy \in E$, we also use $x \sim y$, and $x \not\sim y$ whenever x, y are not adjacent in G. A vertex $z \in V$ distinguishes the non-adjacent vertices $x, y \in V$ if $zx \in E$ and $zy \notin E$. If $A, B \subset V$ are disjoint and $ab \in E$ for every $a \in A$ and $b \in B$, we say that A, B are *totally adjacent* and we denote by $A \sim B$, while by $A \not\sim B$ we mean that no edge of G joins some vertex of A to a vertex from B and, in this case, we say that A and B are *non-adjacent*.

The *neighbourhood* of the vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$, while $N_G[v] = N_G(v) \cup \{v\}$; we simply write $N(v)$ and $N[v]$, when G appears clearly from the context. The neighbourhood of the vertex v in the complement of G will be denoted by $\overline{N}(v)$.

The neighbourhood of $S \subset V$ is the set $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = S \cup N(S)$. A *clique* is a subset Q of V with the property that $G(Q)$ is complete. The *clique number density* of G, denoted by $\omega(G)$, is the size of the maximum clique. A clique cover is a partition of the vertices set such that each part is a clique. $\theta(G)$ is the size of a smallest possible clique cover of G; it is called the *clique cover number* of G. A stable set is a subset X of vertices where every two vertices are not adjacent. $\alpha(G)$ is the number of vertices is a stable set o maximum cardinality; it is called the *stability number* of G. $\chi(G) = \omega\overline{(G)}$ and it is called *chromatic number*.

By P_n, C_n, K_n we mean a chordless path on $n \geq 3$ vertices, a chordless cycle on $n \geq 3$ vertices, and a complete graph on $n \geq 1$ vertices, respectively.

A graph is called *cograph* if it does not contain P_4 as an induced subgraph.

3. Preliminary results

3.1. Weakly decomposition

At first, we recall the notions of weakly component and weakly decomposition.

Definition 1. ([2], [5], [6]) *A set $A \subset V(G)$ is called a weakly set of the graph G if $N_G(A) \neq V(G) - A$ and $G(A)$ is connected. If A is a weakly set, maximal with respect to set inclusion, then $G(A)$ is called a weakly component. For simplicity, the weakly component $G(A)$ will be denoted with A.*

Definition 2. ([2], [5], [63]) *Let $G = (V, E)$ be a connected and non-complete graph. If A is a weakly set, then the partition $\{A, N(A), V - A \cup N(A)\}$ is called a weakly decomposition of G with respect to A.*

Below we remind a characterization of the weakly decomposition of a graph.

The name of "*weakly component*" is justified by the following result.

Theorem 1. ([3], [5], [6]) *Every connected and non-complete graph $G = (V, E)$ admits a weakly component A such that $G(V - A) = G(N(A)) + G(\overline{N}(A))$.*

Theorem 2. ([5], [6]) *Let $G = (V, E)$ be a connected and non-complete graph and $A \subset V$. Then A is a weakly component of G if and only if $G(A)$ is connected and $N(A) \sim \overline{N}(A)$.*

The next result, that follows from Theorem 1, ensures the existence of a weakly decomposition in a connected and non-complete graph.

Corollary 1. *If $G = (V, E)$ is a connected and non-complete graph, then V admits a weakly decomposition (A, B, C), such that $G(A)$ is a weakly component and $G(V - A) = G(B) + G(C)$.*

Theorem 2 provides an $O(n + m)$ algorithm for building a weakly decomposition for a non-complete and connected graph.

Algorithm for the weakly decomposition of a graph ([22])
Input: A connected graph with at least two nonadjacent vertices, $G = (V, E)$.
Output: A partition $V = (A, N, R)$ such that $G(A)$ is connected, $N = N(A)$, $A \not\sim R = \overline{N}(A)$.

begin
- $A :=$ any set of vertices such that
  $A \cup N(A) \neq V$
  $N := N(A)$
  $R := V - A \cup N(A)$
  while ($\exists n \in N$, $\exists r \in R$ such that $nr \notin E$) *do*
  - *begin*
    - $A := A \cup \{n\}$
      $N := (N - \{n\}) \cup (N(n) \cap R)$
      $R := R - (N(n) \cap R)$
  - *end*

end

3.2. Quasi-threshold graphs

In this subsection we remind some results on quasi-threshold graphs.

A graph G is called *quasi-threshold* graph if G is P_4-free and C_4-fre. Graphs obtained from a vertex by recursively applying the following operations: (i) adding a new vertex, (ii) adding a new vertex that is adjacent to all vertices, and (iii) disjoint union of two graphs are precisely the quasi-threshold graphs (see Yan et al. [7]).

4. New results on threshold graphs

4.1. Characterization of a quasi-threshold graph using the weakly decomposition

In this paragraph we give a new characterization of quasi-threshold graphs using the weakly decomposition.

Theorem 3. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices and (A,N,R) a weakly decomposition, with A the weakly component. G is a quasi-threshold graph if and only if:*
i) $A \sim N \sim R$;
ii) N is clique;
iii) $G(A)$, $G(R)$ *are quasi-threshold graphs.*

Proof. Let $G = (V, E)$ be a connected, uncomplete graph and (A, N, R) a weakly decomposition of G, with $G(A)$ as the weakly component.

At first, we assume that G is quasi-threshold. Then $N \sim R$ and $A \sim N$ also. Because $A \sim N \sim R$ and G is C_4-free, it follows that N is a clique.

Conversely, we suppose that i), ii) and iii) hold. Because $A \sim N \sim R$ and $G(A)$, $G(N)$, $G(R)$ are P_4-free, $G(A)$, $G(R)$ are quasi-threshold graphs and N is a clique it follows that G is P_4-free. $G(A)$, $G(N)$, $G(R)$ are C_4-free. $G(A \cup N)$ ia C_4-free because $G(A)$ is quasi-threshold, N is a clique and $A \sim N$. $G(N \cup R)$ ia C_4-free because $G(R)$ is quasi-threshold, N is a clique and $N \sim R$. $G(A \cup R)$ ia C_4-free because $G(A)$, $G(R)$ are quasi-threshold and $A \not\sim N$. If G is not C_4-free then N is not clique. So, G is quasi-threshold graph.

The above results lead to a recognition algorithm with the total execution time $O(n(n + m))$.

4.2. Determination of clique number and stability number for a quasi-threshold graph

We determine the stability number and the clique number for quasi-threshold graphs.

Theorem 4. *Let G=(V,E) be connected with at least two non-adjacent vertices and (A,N,R) a weakly decomposition with A the weakly component. If G is a threshold graph then*

$$\alpha(G) = \alpha(G(A)) + \alpha(G(R)) \text{ and}$$
$$\omega(G) = max\{\omega(G(A)), \omega(G(R))\} + |N|.$$

As a consequence of the above theorem, we give an algorithm that leads to a clique of maximal cardinal in a quasi-threshold graph.

Input: A quasi-threshold, connected graph with at least two non-adjacent vertices, $G = (V, E)$

Output: Determination of $\omega(G)$

begin

$Q = \emptyset$; $q := 0$; $i := 1$; $G_i := G$;

Determine the degree of vertices in G_i;

while $|V(G_i)| \geq 4$ *do*

Determine a weakly decomposition (A_i, N_i, R_i) of G_i, with N_i clique and $G(A_i)$, $G(N_i)$ quasi-threshold graphs

if (G_i is complete) *then*

$Q := Q \cup V(G_i)$, $q := q + |V(G_i)|$

else

$Q := Q \cup N_i$, $q := q + |N_i|$;

determine the degree vertices in $G(A_i)$;

// $(\forall v \in A_i : d_{G(A_i)}(v) = d_{G_i} - |N_i|)$

let q_1 be the number of vertices in A_i with maximum degree in $G(A_i)$;

// = the number of vertices in A_i with maximum degree in G_i;

determine the degree vertices in $G(R_i)$;

determine the connected components of $G(R_i)$;

let $G(R_i^c)$ the connected component of $G(R_i)$ which contains the vertices of maximum degree in $G(R_i)$;

let q_2 be the number of vertices in R_i of maximum degree in $G(R_i)$;

if $(q_1 > q_2)$ *then*

$G_{i+1} := G(A_i)$

else

$G_{i+1} := G(R_i^c)$

$i := i + 1$;

$\omega(G) := q$

end

As a consequence of the above theorem, we give an algorithm that leads to a stable set of maximal cardinal in a quasi-threshold graph.

Input: A quasi-threshold, connected graph with at least two non-adjacent vertices, $G = (V, E)$

Output: Determination of $\alpha(G)$

begin

$S = \emptyset$;

$L := \{G\}$;

while $L \neq \emptyset$ *do*

let H be in L ;

Determine a weakly decomposition (A, N, R) of H

if (H is complete) *then*

$S := S \cup \{v\}, \forall v \in V(H)$

else

enter $G(A)$, $G(R)$ in L;

$\alpha(G) := |S|$;

end

5. Some Applications in Optimization Problems

In this section we point some applications of threshold graphs in optimization problems.

Facility location analysis deals with the problem of finding optimal locations for one or more facilities in a given environment [4]. Location problems are classical optimization problems with many applications in industry and economy. The spatial location of the facilities often takes place in the context of a given transportation, communication, or transmission system. A first paradigme for location is based on the minimization of transportation cost.

According to their objective function, we can consider two types of location problems. The first type consists of those problems that use a minimax criterion. For example, if we want to determine the location of a hospital the main objective is to find a site that minimizes the maximum response time between the hospital and site of a possible emergency. More generally, the aim of the first problem type is to determine a location that minimizes the maximum distance to any other location in the network. The second type of location problems optimizes a "minimum of a sum" criterion, which is used in determining the location for a service facility like a shopping mall, for which we try to minimize the total travel time. The following centrality indices are defined in [4].

The eccentricity of a vertex u is $e_G(u) = max\{d(u,v)|v \in V\}$.

The radius is $r(G) = min\{e_G(u)|u \in V\}$.

The center of a graph G is $\mathcal{C}(G) = \{u \in V|r(G) = e_G(u)\}$.

We consider the second type of location problems. Suppose we want to place a service facility such that the total distance to all customers in the region is minimal. The problem of finding an appropriate location can be solved by computing the set of vertices with minimum total distance.

We denote the sum of the distances from a vertex u to any other vertex in a graph G=(V,E) as the total distance $s(u) = \sum_{v \in V} d(u,v)$. If the minimum total distance of G is denoted by $s(G) = min\{s(u)|u \in V\}$, the median $\mathcal{M}(G)$ of G is given by $\mathcal{M}(G) = \{u \in V|s(G) = s(u)\}$.

Our result concerning the center of a threshold graph is the following.

Theorem 5. *Let G=(V,E) be a connected graph with at least two nonadjacent vertices. If G is quasi-threshold then the center and the median are equal to N, the radius is 1.*

Proof. Because $A \sim N \sim R$, $A \not\sim R$, N is a clique it follows that $e_G(u) = 1$, $\forall u \in N$ and $e_G(u) = 2$, $\forall u \in A \cup R$. So $r(G) = 1$ and $\mathcal{C}(G) = N$.

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