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CHARACTERIZATIONS OF FAINT θ-RARE e-CONTINUITY

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Abstract. The object of this paper is to introduce and investigate the new notion of faint θ-rare e-continuity, that is more general that both rare continuity and faint θ-rare continuity.

1. Introduction

In 2008, Ekici [5] introduced e-open set as a generalization of open set. Also, Ekici [8, 9, 10] introduced the related sets with e-open sets as generalizations of open sets. In 1979, Popa [17] introduced the notion of rare continuity as a generalization of weak continuity [15]. Long and Herrington [16], Jafari [12, 13], Ekici and Jafari [11], Caldas and Jafari [1, 2, 3] further investigated the class of rarely continuous functions.

The second author [14] introduced the notion of faint θ-rare continuity as a generalization of faint continuity [16]. The purpose of the present paper is to introduce the concept of faint θ-rare e-continuity in topological spaces as a generalization of rare continuity and weak continuity. We investigate some of its fundamental properties. It turns out that faint θ-rare e-continuity is weaker than faint θ-rare continuity.

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2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply, X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. If A is any subset of a space X, then $Cl(A)$ and $Int(A)$ denote the closure and the interior of A respectively.

A set A is called regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$. A rare set is a set A such that $Int(A) = \emptyset$. The δ-closure (resp. θ-closure) of A denoted by $Cl_\delta(A)$ (resp. $Cl_\theta(A)$ is defined as $Cl_\delta(A) = \{x \in X \mid U \cap A \neq \emptyset$ for every regular open set U containing $x\}$ (resp. $Cl_\theta(A) = \{x \in X \mid Cl(U) \cap A \neq \emptyset$ for every open set U containing $x\}$). A subset A is called δ-closed (resp. θ-closed) if $Cl_\delta(A) = A$ (resp. $Cl_\theta(A) = A$). The complement of a δ-closed set (resp. θ-closed) is called δ-open (resp. θ-open) [19]. The family of all δ-open (resp. open, θ-open) sets will be denoted by $\delta O(X)$ (resp. $O(X)$, $\theta O(X)$). We set $\delta O(X, x) = \{U \mid x \in U \in \delta O(X)\}$, $O(X, x) = \{U \mid x \in U \in O(X)\}$ and $\theta O(X, x) = \{U \mid x \in U \in \theta O(X)\}$. The δ-interior of A denoted by $Int_\delta(A)$ is defined as
$Int_\delta(A) = \{x \in X \mid$ for some open subset U of $X, x \in U \subset Int(Cl(U)) \subset A\}$. The θ-interior of A denoted by $Int_\theta(A)$ and is defined as $Int_\theta(A) = \{x \in X \mid$ for some open subset U of $X, x \in U \subset Cl(U) \subset A\}$.

Jafari [14] introduced the notions:
(1) θ-rare set R as a set with $Int_\theta(R) = \emptyset$.
Since $Int_\theta(R) \subset Int(R)$, it follows that a rare set is θ-rare but the converse is not true.
(2) The θr-closure of A denoted by $\theta rCl(A)$ is defined as the set of all points $x \in X$ with the property that for every θ-open set U containing x there exists a θ-rare set R_U with $U \cap Cl(R_U) = \emptyset$ such that $(U \cup R_U) \cap A \neq \emptyset$.
A subset A is called θr-closed if $\theta rCl(A) = A$. The complement of a θr-closed set is called θr-open.

A set A is said to be e-open [5] if $A \subset Int(Cl_\delta(A)) \cup Cl(Int_\delta(A))$. The complement of an e-open set is said to be e-closed [5]. The intersection (resp. union) of all e-closed (resp. e-open) sets containing (resp. contained in) A in X is called the e-closure (resp. the e-interior) [5] of A and is denoted by $Cl_e(A)$ (resp. $Int_e(A)$). By $eO(X)$ or

$eO(X,\tau)$ (resp. $eC(X)$), we denote the collection of all e-open (resp. e-closed) sets of X. We set $eO(X,x) = \{U : x \in U \in eO(X)\}$.

The following lemma is given by Ekici [5, 7, 6, 9]:

Lemma 2.1. *The following properties holds for the e-closure of a set in a space X:*
(1) Arbitrary union (resp. intersection) of e-open (resp. e-closed) sets in X is e-open (resp. e-closed).
(2) A is e-closed in X if and only if $A = Cl_e(A)$.
(3) A is an e-open set if and only if $A = Int_e(A)$.
(4) $Cl_e(A) \subset Cl_e(B)$ whenever $A \subset B (\subset X)$.
(5) $Cl_e(A)$ (resp. $Int_e(A)$) is an e-closed set (resp. e-open set) in X.
(6) $Cl_e(A) = \{x \in X \mid U \cap A \neq \emptyset$ for every e-open set U containing $x\}$.
(7) $X - Int_e(A) = Cl_e(X - A)$ and $Int_e(X - A) = X - Cl_e(A)$.

We have the following diagram in which the converses of implications need not be true, as is shown in [5].

$$
\begin{array}{ccccccccc}
 & & \textit{open} & \rightarrow & & \alpha\textit{-open} & & \rightarrow & \textit{preopen} \\
 & \nearrow & & & & & & & \\
\textit{regular open} & & \uparrow & & & & & & \downarrow \\
 & \searrow & & & & & & & \\
 & & \delta\textit{-open} & & & & & & \delta\textit{-preopen} \\
 & & & \searrow & & & & \swarrow & \\
 & & & & \delta\textit{-semiopen} & \rightarrow & e\textit{-open} & &
\end{array}
$$

Definition 1. *A function $f : X \to Y$ is called:*
1) Weakly continuous [15] *if for each $x \in X$ and each open set G containing $f(x)$, there exists $U \in O(X,x)$ such that $f(U) \subset Cl(G)$.*
2) Rarely continuous [17] *if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ and $U \in O(X,x)$ such that $f(U) \subset G \cup R_G$.*
3) faintly θ-rare continuous [14] *if for each $x \in X$ and each $G \in \theta O(Y, f(x))$, there exist a θ-rare set R_G in Y with $G \cap Cl_\theta(R_G) = \emptyset$ and $U \in O(X,x)$ such that $f(U) \subset G \cup R_G$.*
4) Rarely e-continuous [4] *if for each $x \in X$ and each $G \in O(Y, f(x))$, there exist a rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ and $U \in eO(X,x)$ such that $f(U) \subset G \cup R_G$.*

5) e-continuous [5] *if the inverse image of every open set in Y is e-open in X.*
6) e-irresolute [7, 9] *if the inverse image of every e-open set in Y is e-open in X.*

3. Characterizations of faint θ-rare e-continuity

Definition 2. *A function $f : X \to Y$ is said to be faintly θ-rare e-continuous at $x \in X$ if for each $G \in \theta O(Y, f(x))$, there exist a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ and $U \in eO(X, x)$ such that $f(U) \subset G \cup R_G$. If f is faintly θ-rare e-continuous at every point of X, then it is called faintly θ-rare e-continuous.*

Example 3.1. *Let X and Y be the real line with indiscrete and discrete topologies respectively. The identity function is faintly θ-rarely e-continuous.*

Remark 3.2. *It is stated that faintly θ-rare e-continuity is weaker that faintly θ-rare continuity. Really, Assume that $f : X \to Y$ is faintly θ-rare continuous. For an arbitrary $x \in X$, let $G \in \theta O(Y, f(x))$. Since f is faintly θ-rare continuous at x, there exist a θ-rare set R_G in Y with $G \cap Cl_\theta(R_G) = \emptyset$ and $U \in O(X, x)$ such that $f(U) \subset G \cup R_G$. Since every open set is e-open, we have $U \in eO(X, x)$. On the other hand $Cl(A) \subset Cl_\theta(A)$ for every $A \subset Y$, hence $G \cap Cl(R_G) \subset G \cap Cl_\theta(R_G) = \emptyset$ and by therefore $G \cap Cl(R_G) = \emptyset$. Since for every $G \in \theta O(Y, f(x))$ there exist a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ and $U \in eO(X, x)$ such that $f(U) \subset G \cup R_G$, it follows that f is faintly θ-rare e-continuous.*

Note that every weakly continuous function is rarely continuous and every rarely continuous function is rarely e-continuous. We have the following implications :

$$\begin{array}{ccccccc}
\text{continuity} \Rightarrow & \text{weak continuity} & \Rightarrow & \text{rare continuity} & \Rightarrow & \text{rare } e\text{-continuity} \\
& \Downarrow & & \Downarrow & & \Downarrow \\
& faint\ cont. & \Rightarrow & faint\ \theta-rare-cont. & \Rightarrow & faint\ \theta-rare\ \ e-cont.
\end{array}$$

Example 3.3. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and let $f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is not continuous, but it is weakly continuous, rarely continuous and rarely e-Continuous.*

Example 3.4. *Let* $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ *and let* $f : (X, \tau) \to (X, \sigma)$ *be the identity function. Then* f *is faintly* θ*-rarely* e*-continuous but* f *is not rarely continuous. Therefore* f *is not weakly continuous.*

Theorem 3.5. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is faintly* θ*-rare* e*-continuous at* $x \in X$ *if and only if for each* $G \in \theta O(Y, f(x))$*, there exists a* θ*-rare set* R_G *in* Y *with* $G \cap Cl(R_G) = \emptyset$ *such that* $x \in Int_e(f^{-1}(G \cup R_G))$.

Proof. Necessity. Suppose that $G \in \theta O(Y, f(x))$. Then there exists a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ and $U \in eO(X, x)$ such that $f(U) \subset G \cup R_G$. It follows that $x \in U \subset f^{-1}(G \cup R_G)$. This implies that $x \in Int_e(f^{-1}(G \cup R_G))$.
Sufficiency. Let $x \in Int_e(f^{-1}(G \cup R_G))$. Put $U = Int_e(f^{-1}(G \cup R_G))$. We have $x \in U$, U is θ-open and $f(U) \subset G \cup R_G$. This show that f is faintly θ-rare e-continuous.

Theorem 3.6. *A function* $f : (X, \tau) \to (Y, \sigma)$ *is faintly* θ*-rare* e*-continuous at* $x \in X$ *if and only if for each* $G \in \theta O(Y, f(x))$*, there exists a* θ*-rare set* R_G *in* Y *with* $G \cap Cl(R_G) = \emptyset$ *such that* $f^{-1}(G) \subset Int_e(f^{-1}(G \cup R_G))$.

Proof. Necessity. Since f is faintly θ-rare e-continuous at $x \in X$, then by Theorem 3.4, there exists a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ such that $x \in Int_e(f^{-1}(G \cup R_G))$.
Sufficiency. Let x be any point of X. Suppose that for each $G \in \theta O(Y, f(x))$, there exists a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ such that $x \in f^{-1}(G) \subset Int_e(f^{-1}(G \cup R_G))$.Then f is faintly θ-rare e-continuous.

Theorem 3.7. *The following statements are equivalent for a function* $f : X \to Y$*:*
(1) f *is faintly* θ*-rare* e*-continuous.*
(2) $f(Cl_e(A)) \subset \theta rCl(f(A))$ *for every subset* $A \subset X$.
(3) $Cl_e(f^{-1}(B)) \subset f^{-1}(\theta rCl(B))$ *for every subset* $B \subset Y$.

Proof. (1) $\Rightarrow$ (2) : Let A be any subset of X. Let $x \in Cl_e(A)$ and G be any θ-open set containing $f(x)$. Since f is faintly θ-rare e-continuous, there exists a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ and $U \in eO(X, x)$ such that $f(U) \subset G \cup R_G$. Since $x \in Cl_e(A)$, $A \cap U \neq \emptyset$. Hence, we have $\emptyset \neq f(U) \cap f(A) \subset (G \cup R_G) \cap f(A)$. This shows that $f(x) \in \theta rCl(f(A))$.
(2) $\Rightarrow$ (3) : Let B any subset of Y. We have $f(Cl_e(f^{-1}(B)) \subset$

$\theta rCl(B))$. Then $Cl_e(f^{-1}(B)) \subset f^{-1}(\theta rCl(B))$.
(3) $\Rightarrow$ (1) : Let $x \in X$ and $G \in \theta O(Y, f(x))$ and $B = Y - (G \cup R_G)$, where R_G is a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$. Then $f(x) \notin \theta rCl(Y - (G \cup R_G))$ Thus $x \notin f^{-1}(\theta rCl(Y - (G \cup R_G)))$. By (3), it follow that $x \notin Cl_e(f^{-1}(Y - (G \cup R_G)))$. Hence, there exists $U \in eO(X, x)$ such that $U \cap f^{-1}(Y - (G \cup R_G)) = \emptyset$. Then we have $f(U) \subset G \cup R_G$. This shows that f is faintly θ-rare e-continuous.

Corollary 3.8. *If $f : X \to Y$ is faintly θ-rare e-continuous, then the following hold:*
(1) $f^{-1}(A)$ is e-closed in X for each θr-closed set A of Y.
(2) $f^{-1}(B)$ is e-open in X for each θr-open set B of Y.

Proof. (1) Suppose that A is a θr-closed set of Y. By Theorem 3.6(3), we have $Cl_e(f^{-1}(A)) \subset f^{-1}(A)$. This implies that $f^{-1}(A)$ is e-closed in X.
(2) It follows from (1).

Corollary 3.9. *A function $f : (X, \tau) \to (X, \sigma)$ is faintly θ-rare e-continuous if and only if $f^{-1}(Int_{\theta r}(E)) \subset Int_e(f^{-1}(E))$ for any subset E of Y.*

Proof. It is an immediate consequence of Theorem 3.6 and the fact that $Int_e(X - A) = X - Cl_e(A)$ and $Int_{\theta r}(X - A) = X - \theta rCl(A)$.

4. Other properties

Recall that a function $f : X \to Y$ is θ-open [18] if $f(A)$ is θ-open for each open set A in X.

Definition 3. *(1) Let $A = \{V_i\}$ be a class of θ-open subsets of X. By θ-rarely union sets of A,* [14] *we mean $\{G \cup R_{G_i}\}$, where each R_{G_i} is a θ-rare set such that each of $\{G_i \cap Cl(R_{G_i})\}$ is empty.*
(2) A subset $K \subset X$ is said to be rarely θ-compact relative to X [14] *if for every θ-open cover of K by θ-open sets of X, there exists a finite subfamily whose θ-rarely union sets cover K. A space X is said to be rarely θ-compact if the set X is rarely θ-compact relative to X*
(3) A subset $K \subset X$ is said to be e-compact relative to X [7, 9] *if for every e-open cover of K by e-open sets of X, there exists a finite subfamily which covers K. A space X is said to be e-compact* [7, 9] *if the set X is e-compact relative to X*

Remark 4.1. *If a space X is compact then it is obvious that it is rarely θ-compact.*

Theorem 4.2. *Let $f : X \to Y$ be faintly θ-rare e-continuous and K be an e-compact set in X. Then $f(K)$ is a rarely θ-compact subset of Y.*

Proof. Let Λ be a θ-open cover of $f(K)$. Let S be the set of all G in Λ such that $G \cap f(K) \neq \emptyset$. Thus for each $k \in K$ there is some $G_k \in S$ such that $f(k) \in G_k$. Since f is faintly θ-rare e-continuous, there exist a θ-rare set R_{G_k} in Y with $G_k \cap Cl(R_{G_k}) = \emptyset$ and an e-open set U_k containing k such that $f(U_k) \subset G_k \cup R_{G_k}$. Hence, there is a finite subfamily $\{U_k\}_{k\in\omega}$ which covers K, where ω is a finite subset of K. The subfamily $\{G_k \cup R_{G_k}\}_{k\in\omega}$ also covers $f(K)$. This shows that $f(K)$ is a rarely θ-compact subset of Y.

Theorem 4.3. *If $f : X \to Y$ is e-irresolute and $g : Y \to Z$ is faintly θ-rare e-continuous, then $g \circ f : X \to Z$ is faintly θ-rare e-continuous*

Proof. Suppose that $x \in X$ and $(g \circ f)(x) \in G$, where G is a θ-open set in Z. By hypothesis, g is faintly θ-rare e-continuous, therefore there exist a θ-rare set R_G in Y with $G \cap Cl(R_G) = \emptyset$ and an e-open set U containing $f(x) = y$ such that $g(U) \subset G \cup R_G$. Since f is e-irresolute, there exists an e-open set W containing x such that $f(W) \subset U$. Thus $g(f(W)) = (g \circ f)(U) \subset g(U) \subset G \cup R_G$. Hence the result.

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