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QUASI PERFECTLY CONTINUOUS FUNCTIONS AND THEIR FUNCTION SPACES

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Abstract. A new class of functions called 'quasi perfectly continuous functions' is introduced. Basic properties of quasi perfectly continuous functions are studied and their place in the hierarchy of variants of continuity, that already exist in the literature, is elaborated. The notion of quasi perfect continuity, in general is independent of continuity but coincides with perfect continuity (Indian J. Pure Appl. Math. 15(3) (1984), 241-250), a significantly strong form of continuity, if the range space is regular. The class of quasi perfectly continuous functions properly contains the class of δ-perfectly continuous functions (Demonstratio Math. 43(1) (2009), 221-231) and is strictly contained in the class of quasi cl-supercontinuous functions. Moreover, it is shown that if X is sum connected (e.g. connected or locally connected) and Y is Hausdorff, then the function space $P_q(X,Y)$ of all quasi perfectly continuous functions is closed in Y^X in the topology of pointwise convergence. Furthermore, certain known function spaces in the literature are shown to be identical with $P_q(X,Y)$ in case X is sum connected and Y is Hausdorff.

Keywords and phrases: perfectly continuous function, (almost) z-supercontinuous function, D_δ -supercontinuous function, strongly θ-continuous function, quasi -partition topology, Alexandroff space ($\equiv$ saturated space).
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1. Introduction

Variants of continuity arise naturally in almost all branches of mathematics and applications of mathematics. Certain of these variants of continuity are weaker than continuity while others are stronger than continuity and yet others, although in some cases their theories run parallel to continuity, are independent of continuity. The purpose of this paper is to introduce a new class of functions called 'quasi perfectly continuous functions' and to discuss their properties and to elaborate upon their place in the hierarchy of variants of continuity that already exist in the lore of mathematical literature. It turns out that the notion of quasi perfect continuity is independent of continuity but coincides with perfect continuity of Noiri [34], a significantly strong form of continuity, if the range is a regular space. The class of quasi perfectly continuous functions properly contains the class of δ-perfectly continuous functions [20] which in its turn contains all perfectly continuous functions and so includes all strongly continuous functions of Levine [25]. Moreover, the class of quasi perfectly continuous functions is strictly contained in the class of quasi cl-supercontinuous functions [11] which in turn is properly contained in the class of quasi z-supercontinuous functions [24].

The paper is organised as follows: Section 2 is devoted to preliminaries and basic definitions. In Section 3, we introduce the notion of a 'quasi perfectly continuous function' and discuss its interplay and interrelations with other variants of continuity that already exist in the mathematical literature. Therein examples are included to reflect upon distinctiveness of the notions so introduced with the ones which already exist in the literature. Basic properties of quasi perfectly continuous functions are elaborated in Section 4, wherein sufficient conditions are given for the preservation of quasi perfect continuity under (1) restriction and (2) passage to graph function. Moreover, sufficient conditions are formulated for the composition of two functions to be quasi perfectly continuous as well as perfectly continuous. Furthermore, it is shown that if a function $f : X \to \prod X_\alpha$ into a product space is quasi perfectly continuous, then its composition with each projection map is quasi perfectly continuous and converse holds in case X is an Alexandroff space [1]. The function space $P_q(X, Y)$ of quasi perfectly continuous functions from a space X into a space Y is considered in Section 5, wherein it is shown that if X is sum connected [8] (e.g. connected or locally connected) and Y is Hausdorff then $P_q(X, Y)$ is closed in Y^X in the topology of pointwise convergence. Furthermore, it is shown that under the same hypothesis on X and Y, the function space $P_q(X,Y)$ coincides with certain other known function spaces in the literature.

2. Preliminaries and basic definitions

Throughout the paper X is a topological space. A collection β of subsets of a space X is called an ***open complementary system*** [7] if β consists of open sets such that for every $B \in \beta$, there exist $B_1, B_2, ., \in \beta$, with $B = \cup\{X/B_i : i \in N\}$. A subset A of a space X is called a ***strongly open*** F_σ***-set*** [7] if there exists a countable open complementary system $\beta(A)$ with $A \in \beta(A)$. The complement of a strongly open F_σ-set is called ***strongly closed*** G_δ***-set***. A subset A of a space X is called a **regular** G_δ**-set** [30] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap\limits_{n=1}^{\infty} F_n = \bigcap\limits_{n=1}^{\infty} F_n^o$, where each F_n is a closed subset of X (here F_n^o denote the interior of F_n). The complement of a regular G_δ-set is called a **regular** F_σ**-set**. A point $x \in X$ is called a θ***-adherent point*** [45] of A $\subset$ X if every closed neigbourhood of x intersects A. Let $cl_\theta\ A$ denote the set of all θ-adherent points of A. The set A is called θ***-closed*** if $A = cl_\theta A$. The complement of a θ-closed set is referred to as a θ***-open set***. A subset A of a space X is said to be ***regular open*** if it is the interior of its closure, i.e., $A = \overline{A}^{\circ}$. The complement of a regular open set is referred to as a ***regular closed set***. A union of regular open sets is called δ**-open** [45]. The complement of a δ-open set is referred to as a δ***-closed set***. A subset A of X is called ***zero set*** of X if there exists a continuous real valued function f on X such that A=$\{x \in X : f(x)=0\}$. The complement of zero set is called ***cozero set***.

2.1 Lemma([12] [14])**:** *A subset U of a space X is θ-open if and only if for each $x \in U$ there exists an open set V containing x such that $\overline{V} \subset U$.*

An open subset U of X is said to be ***r-open*** [22] if for each $x \in U$ there exists a closed set B such that $x \in B \subset$ U or equivalently, if U is expressible as a union of closed sets. An open subset W of X is said to be ***F-open*** [22] if for each $x \in W$ there exists a zero set Z such that $x \in Z \subset$ W, or equivalently if W is expressible as a union of zero sets. Now we give the definitions of strong variants of continuity related to the theme of the present paper.

2.1. **Definitions.** A function $f : X \to Y$ *from a topological space* X *into a topological space* Y *is said to be*
(a) ***strongly continuous*** [25] *if* $f(\overline{A}) \subset f(A)$ *for each subset* A *of* X.
(b) ***perfectly continuous*** ([34] [23]) *if* $f^{-1}(V)$ *is clopen in* X *for every open set* $V \subset Y$.
(c) ***cl-supercontinuous*** [42] ($\equiv$ ***clopen continuous*** [37]) *if for each* $x \in X$ *and for each open set* V *containing* $f(x)$ *there is a clopen set* U *containing* x *such that* $f(U) \subset V$.
(d) ***z-supercontinuous*** [15] *if for each* $x \in X$ *and for each open set open set* V *containing* $f(x)$ *there is a cozero set* U *containing* x *such that* $f(U) \subset V$.
(e) D_δ***-supercontinuous*** [16] *if for each* $x \in X$ *and for each open set* V *containing* $f(x)$, *there exists a regular* F_σ *-set* U *containing* x *such that* $f(U) \subset V$.
(f) ***(almost) strongly*** θ***-continuous*** ([35] [27]) *if for each* $x \in X$ *and for each (regular open) open set* V *containing f(x), there exists an open set* U *containing* x *such that* $f(\overline{U}) \subset V$.
(g) ***supercontinuous*** [31] *if for each* $x \in X$ *and for each open set* V *containing f(x), there exists a regular open set* U *containing* x *such that* $f(U) \subset V$.
(h) ***completely continuous*** [2] *if* $f^{-1}(V)$ *is a regular open set in* X *for every open set* $V \subset Y$.
Next we give definitions of weak variants of continuity that will crop up in our discussion.

2.2. **Definitions.** A function $f : X \to Y$ *from a topological space* X *into a topological space* Y *is said to be*
(a) D_δ ***-continuous*** [17] *if for each point* $x \in X$ *and each regular* F_σ *- set* V *containing f(x) there is an open set* U *containing* x *such that* $f(U) \subset V$.
(b) ***D*-continuous*** [41] *if for each point* $x \in X$ *and each strongly open* F_σ*- set* V *containing f(x) there is an open set* U *containing* x *such that* $f(U) \subset V$.
(c) ***D-continuous*** [9] *if for each* $x \in X$ *and each open* F_σ*- set* V *containing f(x) there is an open set* U *containing* x *such that* $f(U) \subset V$.
(d) ***z-continuous*** [39] *if for each* $x \in X$ *and each cozero set* V *containing f(x) there is an open set* U *containing* x *such that* $f(U) \subset V$.
(e) ***almost continuous*** [40] *if for each* $x \in X$ *and for each regular open set* V *containing f(x), there exists an open set* U *containing* x *such that* $f(U) \subset V$.

(f) ***faintly continuous*** [28] *if for each $x \in X$ and for each θ -open set V containing f(x), there exists an open set U containing x such that $f(U) \subset V$.*
(g) θ ***-continuous*** [6] *if for each $x \in X$ and each open set V containing f(x), there exists an open set U containing x such that $f(\overline{U}) \subset \overline{V}$.*
(h)***F-continuous (R-continuous)*** [22] *If for each $x \in X$ and for each F-open (r-open) set V in Y containing f(x) there exists an open set U containing x such that $f(U) \subset V$.*
(i) ***weakly continuous*** [26] *if for each $x \in X$ and each open set V containing f(x) there exists an open set U containing x such that $f(U) \subset \overline{V}$.*
(j) ***quasi θ-continuous function*** [36] *if for each $x \in X$ and each θ -open set V containing f(x) there exists an θ-open set U containing x such that $f(U) \subset V$.*
In the following we give definitions of variants of continuity which are independent of continuity and are related to the paper.

2.3. **Definitions.** *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be*
(a) ***δ-perfectly continuous*** [20] *if for each δ -open set V in Y, $f^{-1}(V)$ is a clopen set in X.*
(b) ***almost perfectly continuous*** [43]*(≡ **regular set connected**) [4]* if $f^{-1}(V)$ is clopen for every regular open set V in Y.
(c) ***almost cl-supercontinuous*** [19] (≡ ***almost clopen continuous*** [5]) *if for each $x \in X$ and each regular open set V containing f(x) there is a clopen set U containing x such that $f(U) \subset V$.*
(d) ***almost completely continuous*** [21] (≡ ***R-maps*** [3]) *if $f^{-1}(V)$ is a regular open set in X for every regular open set $V \subset Y$.*
(e) ***quasi z-supercontinuous function*** [24] *if for each $x \in X$ and each θ-open set V containing f(x) there exists an cozero set U containing x such that $f(U) \subset V$.*
(f) ***quasi cl-supercontinuos function*** [11] *if for each $x \in X$ and each θ-open set V containing f(x) there exists a clopen set U containing x such that $f(U) \subset V$.*
(g) δ***-continuous*** [33] *if for each $x \in X$ and for each regular open set V containing f(x), there exists a regular open set U containing x such that $f(U) \subset V$.*

2.4. **Definitions.** A space X is said to be
(a) ***quasi zero dimensional*** [10] *if for each $x \in X$ and each θ-open set U containing x there exists a clopen set V containing x such that*

$V \subset U$.
(b) *endowed with a **quasi partition topology** if every θ-open set in X is closed; or equivalently every θ-closed set in X is open.*
(c) ***almost regular*** [38] *if for each $x \in X$ and each regular closed set F not containing x there exist disjoint open sets U and V containing x and F, respectively.*

3. Quasi perfectly continuous functions

We call a function $f : X \to Y$ from a topological space X into a topological space Y ***quasi perfectly continuous*** if $f^{-1}(V)$ is clopen in X for every θ-open set V in Y.
The following two diagrams well exhibit the interrelations that exist among quasi perfect continuity and other variants of continuity that already exist in the literature and are closely related to the theme of the present paper and so well exhibit the place of quasi perfect continuity in the hierarchy of known variants of continuity.

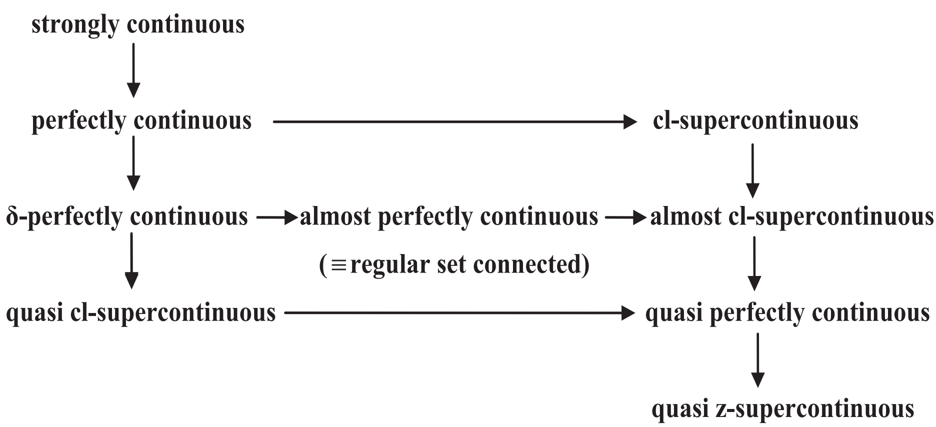

Figure 1

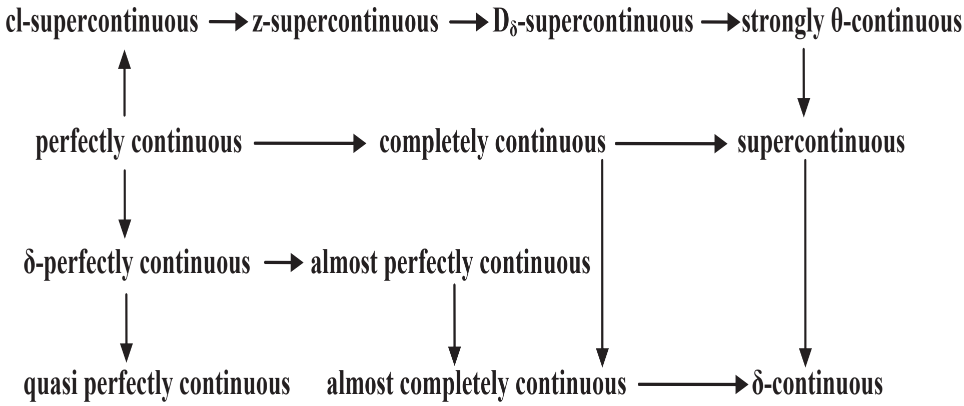

Figure 2

Observation and Examples

3.1 A space X is endowed with a quasi partition topology if and only if every quasi θ-continuous functions f: $X \to Y$ is quasi perfectly continuous. Necessity is obvious in view of definitions. To prove sufficiency assume contrapositive and let V be a θ-open set in X which is not clopen. Then the identity mapping defined on X is quasi θ-continuous but not quasi perfectly continuous.

3.2 Let X be endowed with a quasi zero dimensional topology which is not a quasi partition topology. Then every quasi θ-continuous function f: $X \to Y$ is quasi cl-supercontinuous but not necessarily quasi perfectly continuous.

3.3 Let X be the real line endowed with usual topology. Then the identity map defined on X is continuous but not quasi perfectly continuous.

3.4 Let X denote the real line with usual topology and let Y be the real line with cofinite (or co-countable) topology. Then the identity function f: $X \to Y$ is quasi perfectly continuous but not continuos.

3.5 If X is a zero dimensional space, then every faintly continuous function f:$X \to Y$ is quasi cl-supercontinuous but not necessarily quasi perfectly continuous.

3.6 If Y is a regular space, then every quasi perfectly continuous function f: $X \to Y$ is perfectly continuous.
Since in view of Lemma 2.1 in a regular space every open set is θ-open.

3.7 If Y is an almost regular space, then every quasi perfectly continuous function f:$X \to Y$ is δ-perfectly continuous. This is true since in an almost regular space, every δ-open set is θ-open.

3.8 If X is endowed with a quasi partition topology, then every strongly θ-continuous function is perfectly continuous.
3.9 If X is equipped with a quasi partition topology, then every almost strongly θ-continuous function is almost perfectly continuous.

4. Basics properties of quasi perfectly continuous functions

4.1 **Theorem**: *If f:X $\to$ Y is a quasi perfectly continuous function and g: Y $\to$ Z is a quasi θ-continuous function, then gof is a quasi perfectly continuous function. In particular composition of two quasi perfectly continuous functions is quasi perfectly continuous.*
Proof: Let W be a θ-open set in Z. Since g is quasi θ-continuous, $g^{-1}(W)$ is a θ-open set in Y. In view of quasi perfect continuity of f, $f^{-1}(g^{-1}(W)) = (\text{gof})^{-1}(W)$ is a clopen set in X and so gof is quasi perfectly continuous.
4.2 **Remark**: The hypothesis 'g is quasi θ-continuous' in Theorem 4.1 can be replaced by any one of the weak variants of continuity depicted in the following diagram, since each one of them implies quasi θ-continuity.

$$\begin{array}{c} \text{continuous} \\ \downarrow \\ \text{almost continuous} \\ \downarrow \\ \theta \text{ - continuous} \\ \downarrow \\ \text{quasi } \theta \text{ - continuous} \end{array}$$

Figure 3

4.3 **Theorem**: *Let f: X $\to$ Y be a quasi perfectly continuous function and let g:Y $\to$ Z be a strongly θ-continuous function. Then gof is perfectly continuous.*
Proof: Let W be an open set in Z. In view of strong θ-continuity of the function g, $g^{-1}(W)$ is a θ-open set in Y. Since f is quasi perfectly continuous, $f^{-1}(g^{-1}(W)) = (\text{gof})^{-1}(W)$ is a clopen set in X and so gof is perfectly continuous.
4.4 **Remark**: The hypothesis of 'strong θ-continuity' of the function g in Theorem 4.3 can be traded of with any one of the strong variants

of continuity in the following diagram, since each one of them implies strong θ-continuity.

strongly continuous

$\downarrow$

perfectly continuous

$\downarrow$

cl - supercontinuous

$\downarrow$

z- supercontinuous

$\downarrow$

D_δ - supercontinuous

$\downarrow$

strongly θ - continuous

Figure 4

4.5 **Theorem**: *Let f: $X \to Y$ be a z-continuous function and let g: $Y \to Z$ be a quasi perfectly continuous function, then gof is a quasi perfectly continuous function.*

Proof: Let W be a θ-open set in Z. Since g is quasi perfectly continuous, $g^{-1}(W)$ is a clopen set in Y. Since f is z-continuous and since a clopen set is both a zero set and a cozeroset, in view of [39,Theorem 2.3 and Corollary 2.4] $f^{-1}(g^{-1}(W)) = (\text{gof})^{-1}(W)$ is both a zero set and a cozero set and so a clopen set in X. Thus gof is quasi perfectly continuous.

4.6 **Remark**: The hypothesis of 'z-continuity of f' in Theorem 4.5 can be replaced by any one of the weak variants of continuity in the following diagram, since each one of them is stronger than z-continuity.

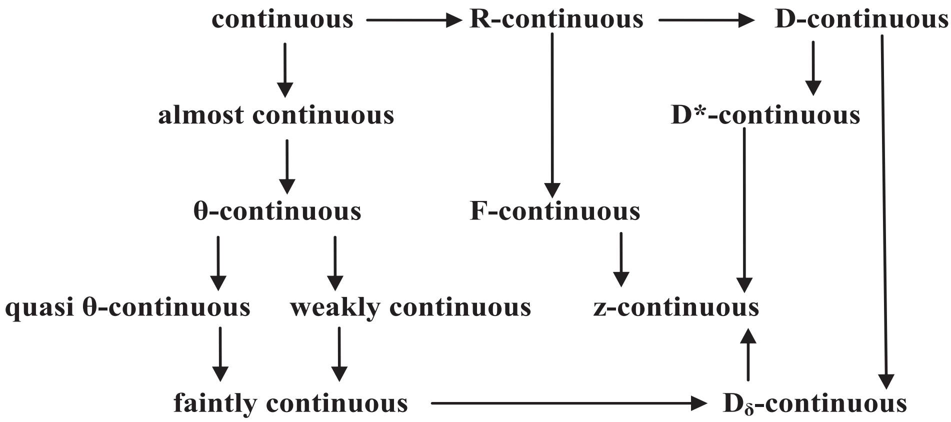

Figure 5

4.7 **Theorem**: *Let f: $X \to Y$ be a function and g: $X \to X \times Y$, defined by g(x) = (x, f(x)) for each $x \in X$, be the graph function. If g is quasi perfectly continuous, then so is f and the space X is endowed with a quasi partition topology.*

Proof: Suppose that the graph function g: $X \to X \times Y$ is quasi perfectly continuous. Consider the projection map $p_y : X \times Y \to Y$. Since it is continuous, so it is θ-continuous. Hence in view of Theorem 4.1 the function f = $p_y og$ is quasi perfectly continuous. To prove that the space X possesses a quasi partition topology, let U be a θ-open set in X. Then $U \times Y$ is a θ-open set in $X \times Y$. Since g is quasi perfectly continuous, $g^{-1}(U \times Y) = U$ is clopen in X and so the topology of X is a quasi partition topology.

The following result embodies sufficient conditions for the domain and / or range of a quasi perfectly continuous function to be equipped with a quasi partition topology.

4.8 **Theorem**: *Let f: $X \to Y$ be a quasi perfectly continuous surjection which maps clopen sets to closed (open) sets. Then Y is endowed with a quasi partition topology. Moreover, if f is a bijection which maps θ-open (θ-closed) sets to θ-open (θ-closed) sets, then X is also equipped with a quasi partition topology.*

Proof: Suppose f maps clopen sets to closed (open) sets. Let V be a θ-open (θ-closed) set in Y. In view of quasi perfect continuity of f, $f^{-1}(V)$ is a clopen set in X. Again, since f is a surjection which maps clopen sets to closed (open) sets, the set f($f^{-1}(V)$) = V is closed (open) in Y and hence clopen. Thus Y is endowed with a quasi partition topology.

To prove the last part of the theorem assume that f is a bijection which maps θ-open (θ-closed) sets to θ-open (θ-closed) sets and let U be a θ-open (θ-closed) set in X. Then f(U) is a θ-open (θ-closed) set in Y. Since f is a quasi perfectly continuous bijection, f($f^{-1}(U)$)= U is a clopen set in X and so Thus X is endowed with a quasi partition topology.

4.9 **Theorem**: *If f: $X \to Y$ is a surjection which maps clopen sets to open sets and g: $Y \to Z$ is a function such that gof is quasi perfectly continuous, then g is a faintly continuous function. Moreover, if f maps clopen sets to clopen sets, then g is a quasi perfectly continuous function.*

Proof: Let V be a θ-open set in Z. Since gof is quasi perfectly continuous, $(gof)^{-1}(V) = f^{-1}(g^{-1}(V))$ is clopen set in X. Again, since f is a surjection which maps clopen sets to open sets,$f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is open in Y and so g is a faintly continuous function. The last assertion is immediate, since in this case $g^{-1}(V)$ is a clopen set in Y.

4.10 **Theorem**: *Let f: $X \to Y$ be a function and let $\boldsymbol{Q} = \{X_\alpha : \alpha \in \Lambda\}$ be a locally finite clopen cover of X. For each $\alpha \in \Lambda$, let $f_\alpha = f|X_\alpha$: $X_\alpha \to Y$ denote the restriction map. Then f is quasi perfectly continuous if and only if each f_α is quasi perfectly continuous.*

Proof: Necessity is immediate in view of the fact that quasi perfect continuity is preserved under the restriction of domain. To prove sufficiency, let V be a θ open set in Y. Then $f^{-1}(V) = \bigcup\limits_{\alpha\in\Lambda}(f|X_\alpha)^{-1}(V) = \bigcup\limits_{\alpha\in\Lambda}(f^{-1}(V) \cap X_\alpha)$. Since each $f^{-1}(V) \cap X_\alpha$ is clopen in X_α and hence in X. Thus $f^{-1}(V)$ is open being the union of clopen sets. Moreover,since the collection **Q** is locally finite, the collection $\{f^{-1}(V) \cap X_\alpha : \alpha \in \Lambda\}$ is a locally finite collection of clopen sets. Since the union of a locally finite collection of closed sets is closed, $f^{-1}(V)$ is also closed and hence clopen.

In the following result we formulate a sufficient condition for the invariance of quasi perfect continuity under the shrinking of range. First we quote the following definition from [13].

4.11 **Definition** [13]: *A subset S of a space X is said to be θ-**embedded** in X if every θ-open set in S is the intersection of a θ-open set in X with S; or equivalently every θ-closed set in S is the intersection of a θ-closed set in X with S.*

4.12 **Theorem**: *Let f: $X \to Y$ be a quasi perfectly continuous function. If f(X) is θ-embedded in Y, then f: $X \to f(X)$ is quasi perfectly continuous.*

Proof: Let V_1 be a θ-open set in f(X). Since f(X) is θ-embedded in Y, there exists a θ-open set V in Y such that $V_1 = V \cap f(X)$. In view of quasi perfect continuity of f, $f^{-1}(V)$ is clopen in X. Now $f^{-1}(V_1) = f^{-1}(V \cap f(X)) = f^{-1}(V) \cap f^{-1}(f(X)) = f^{-1}(V)$ and hence the result.

In contrast to Theorem 4.12 it is easily verified that quasi perfect continuity is preserved under the expansion of range.

4.13 **Definition** [1]: *A topological space X is called an* ***Alexandroff space*** *if any intersection of open sets in X is itself an open in X, or equivalently any union of closed sets in X is closed in X.*

Alexandroff spaces have been referred to as ***saturated spaces*** by Lorrain in [29].

4.14 **Theorem**: *For each $\alpha \in \Lambda$, let $f_\alpha : X \to X_\alpha$ be a function and let $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$ for each $x \in X$. If f is quasi perfectly continuous,then each f_α is quasi perfectly continuous. Further, if X is a Alexandroff space and each f_α is quasi perfectly continuous,then f is quasi perfectly continuous.*

Proof: Let f be quasi perfectly continuous. Now for each α, $f_\alpha = \Pi_\alpha \circ f$ where Π_α denotes the projection map $\Pi_\alpha : \prod X_\alpha \to X_\alpha$. Since each projection map Π_α is continuous and hence quasi θ continuous, in view of Theorem 4.1 it follows that each f_α is quasi perfectly continuous.

Conversely, suppose that X is an Alexandroff space and each f_α is quasi perfectly continuous.To show that the function f is quasi perfectly continuous, it is sufficient to show that $f^{-1}(U)$ is clopen for each θ open set U in the product space $\prod_{\alpha\in\Lambda} X_\alpha$. Since X is a Alexandroff space, it suffices to prove that $f^{-1}(S)$ is clopen for every subbasic θ open set S in the product space $\prod_{\alpha\in\Lambda} X_\alpha$. Let $U_\beta \times \prod_{\alpha\in\Lambda} X_\alpha$ be a subbasic θ open set in $\prod_{\alpha\in\Lambda} X_\alpha$, where U_β is a θ open set in X_β. Then $f^{-1}(U_\beta \times \prod_{\alpha\neq\beta} X_\alpha) = f^{-1}(\Pi_\beta^{-1}(U_\beta)) = f_\beta^{-1}(U_\beta)$ is clopen in X. Hence f is quasi perfectly continuous.

We may recall that a space X is said to be ***ultranormal*** [44] if each pair of disjoint closed sets are contained in disjoint clopen sets.

4.15 **Theorem**: *Let f: $X \to Y$ be a closed, quasi perfectly continuous injection into a normal space Y. Then X is an ultranormal space.*

Proof: Let A and B be any two disjoint closed sets in X. Since f is closed and injective, f(A) and f(B) are disjoint closed subsets of Y. Again, since Y is normal, by Urysohn's Lemma there exists a continuous function φ: $Y \to [0,1]$ such that $\varphi(f(A)) = 0$ and $\varphi(f(B))$ $= 1$. Then V $= \varphi^{-1}([0,1/2))$ and W $= \varphi^{-1}((1/2,1])$ are disjoint

cozero sets in Y containing f(A) and f(B), respectively. Since every cozero set is a θ-open, $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint clopen sets containing A and B, respectively and so X is an ultranormal space.
We may recall that a space X is said is said to be ***ultraregular*** [44] if disjoint points and closed sets are contained in disjoint clopen sets. Equivalently, if X has a basis consisting of clopen sets. Ultraregular spaces are usually referred to as a zero dimensional spaces in literature.
4.16 **Theorem**: *Let f: $X \to Y$ be a closed quasi perfectly continuous injection into a regular space Y. Then X is a zero dimensional ($\equiv$ ultraregular) space.*
Proof: Let F be a closed set in X and let $x \notin X$. Since f is closed and injective, $f(x) \notin f(F)$ and f(F) is a closed set in Y. Now since Y is a regular space, there exist disjoint open sets U and V containing f(x) and f(F), respectively. Since in view of Lemma 2.1 every open set in a regular space is θ-open, U and V are θ-open sets in Y. So $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint clopen sets in X containing x and F, respectively.

5. Function spaces and quasi perfectly continuous functions

A topological space X is said to be **sum connected** [8] if each $x \in X$ has a connected neighbourhood, or equivalently each component of X is open in X. The category of sum connected spaces properly includes the class of connected spaces as well as the class of locally connected spaces and is precisely the coreflective hull of the category of connected spaces (see [8]). The product of topologist's sine curve with a nondegenrate discrete space is a sum connected space which is neither connected nor locally connected. It is well known that in general the set of all continuous functions from a space X into a space Y is not closed in Y^X in the topology of pointwise convergence. In contrast, Naimpally [32] showed that if X is a locally connected space and Y is Hausdorff, then the set S(X,Y) of all strongly continuous functions from X into Y is closed in Y^X in the topology of pointwise convergence. In [18] we extended Naimpally's result to a larger framework of sum connected spaces and further proved that P(X,Y) the set of all perfectly continuous functions as well as L(X,Y) the set of all cl-supercontinuous functions is closed in Y^X in the topology of pointwise convergence. In ([20] [43]) these results are further extended to $P_\Delta(X,Y)$ the set of all δ-perfectly continuous functions and $P_\delta(X,Y)$ the set of all almost perfectly continuous functions from X into Y. Herein we further strengthen these results to show that if X is a sum connected space and Y is a Hausdorff space, then all the six classes of functions are identical, i.e. S(X,Y) = P(X,Y) = L(X,Y) = $P_\Delta(X,Y) = P_\delta(X,Y) = P_q(X,Y)$ and are closed in Y^X in the topology of pointwise convergence.

5.1 **Theorem**: *Let f: $X \to Y$ be a quasi perfectly continuous function into a Hausdorff space Y. Then f is constant on each connected subset of X. In particular, if X is connected, then f is constant on X and hence strongly continuous.*

Proof: Assume contrapositive and let C be the connected subset of X such that f(C) is not a singleton. Let $f(x), f(y) \in f(C)$, $f(x) \neq f(y)$. Since every compact space in a Hausdorff space is θ-closed, V= $X \setminus f(y)$ is a θ-open set in Y containing f(x) but not f(y). Since f is a quasi perfectly continuous, $f^{-1}(V) \cap C$ is a non empty proper clopen subset of C, contradicting the fact that C is connected. The last part of the theorem is immediate, since every constant function is strongly continuous.

5.2 **Remark**: The hypothesis of 'Hausdorffness' cannot be omitted in Theorem 5.1. For let X be the real line with usual topology and let Y denote the real line endowed with indiscrete topology. Let f denote the identity mapping from X into Y. Then clearly f is a nonconstant quasi perfectly continuous function.

5.3 **Corollary**: *Let f: $X \to Y$ be a quasi perfectly continuous function from a sum connected space X into a Hausdorff space Y. Then f is constant on each component of X and hence strongly continuous.*

Proof: Clearly, in view of Theorem 5.1 f is constant on each component of X. Since X is a sum connected space, each component of X is clopen in X. Hence it follows that any union of components of X and the complement of this union are complementary clopen sets in X. Thus f is constant on each component on X. Therefore, for every subset A of Y, $f^{-1}(A)$ and $X \setminus f^{-1}(A)$ are complementary clopen sets in X being the union of component of X. So f is strongly continuous.

We may recall that a space X a δT_0***-space*** [19] if for each pair of distinct points x and y in X there exists a regular open set containing one of the points x and y but not the other. In particular, every Hausdorff space is a δT_0-space.

Next, we quote the following results from ([20] [43]).

5.4 **Theorem** [20, Theorem 5.3]: *Let f: $X \to Y$ be a function from*

a sum connected space X into a δT_0-space Y. Then the following statements are equivalent.
a) *f is strongly continuous.*
b) *f is perfectly continuous.*
c) *f is cl-supercontinuous.*
d) *f is δ -perfectly continuous.*
5.5 **Theorem** [43, Theorem 4.5]: *Let f: $X \to Y$ be a function from a sum connected space X into a δT_0 -space Y. Then the following statements are equivalent.*
a) *f is strongly continuous.*
b) *f is perfectly continuous.*
c) *f is cl-supercontinuous.*
d) *f is δ -perfectly continuous.*
e) *f is almost perfectly continuous.*
5.6 **Theorem**: *Let f: $X \to Y$ be a function from a sum connected space X into a Hausdorff space Y. Then the following statements are equivalent.*
a) *f is strongly continuous.*
b) *f is perfectly continuous.*
c) *f is cl-supercontinuous.*
d) *f is δ -perfectly continuous.*
e) *f is almost perfectly continuous.*
f) *f is quasi perfectly continuous.*
Proof: The equivalence of the assertions (a) - (e) is a consequence of Theorem 5.5. The implications $(a) \Rightarrow (b) \Rightarrow (d) \Rightarrow (f)$ are trivial and the implication $(f) \Rightarrow (a)$ is embodied in Corollary 5.3.
5.7 **Theorem**: *Let X be a sum connected space and let Y be a Hausdorff space. Then S(X,Y) = P(X,Y) = L(X,Y) = $P_\Delta(X,Y)$ = $P_\delta(X,Y)$ =$P_q(X,Y)$ is closed in Y^X in the topology of pointwise convergence.*
Proof: It is immediate from Theorem 5.6 that the above six classes of functions are identical and its closedness in Y^X in the topology of pointwise convergence follows either from [20, Theorem 5.4] or [43, Theorem 4.6].
The above results are important from applications view point since in particular it follows that if X is sum connected (e.g. connected or locally connected) and Y is Hausdorff, then the pointwise limit of a sequence $\{f_n : X \to Y \ , n \in N\}$ of quasi perfectly continuous functions is quasi perfectly continuous. Moreover, if X is a sum connected space and Y is a compact Hausdorff space, then the

function space $P_q(X, Y)$ is compact Hausdorff in the topology of pointwise convergence.

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