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Scientific Studies and Research
Series Mathematics and Informatics
Vol. 21 (2011), No. 2, 41 - 44

A CARLEMAN'S INEQUALITY REFINEMENT NOTE

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Abstract. The aim of this note is to give an improvement of Carleman's inequality. The proof is elementary and our new inequality refines results stated by Bicheng and Debnath [Some inequlities involving the constant e and an application to Carleman's inequlity. J. Math. Anal. Appl. 223 (1998) 347–353], Xie and Zhong [A best approximation for constant e and an improvement to Hardy's inequality J. Math. Anal. Appl. 252 (2000) 994–998.], Ping and Guozheng [A Strengthened Carleman's inequality. J. Math. Anal. Appl. 240 (1999) 290–293] and Yang [On Carleman's inequality J. Math. Anal. Appl. 253 (2001) 691–694].

1. Introduction

The following Carleman inequality [2] is well-known

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n,$$

whenever $a_n \geq 0$, $n = 1, 2, 3, \ldots$, with $0 < \sum_{n=1}^{\infty} a_n < \infty$. Here

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

Keywords and phrases: Inequalities; Carleman inequality' approximations.
(2010)Mathematics Subject Classification: 26A06; 26A24

The constant e is sharp in the sence that it cannot be replaced by a smaller one.

One possible approach is the following inequality

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n a_n, \tag{1.1}$$

see e.g. [1]. As $(1+1/n)^n<e$, Carleman's inequality follows.

In the recent past, many authors established better upper bounds for $(1+1/n)^n$ to obtain new improved forms of Carleman's inequality.

We refer here to the following increasingly better results

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}\left(1-\frac{1}{2n+2}\right)a_n \tag{1.2}$$

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}\left(1-\frac{6}{12n+11}\right)a_n \tag{1.3}$$

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}\left(1+\frac{1}{n+1/5}\right)^{-1/2}a_n \tag{1.4}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} \\ <\;& e\sum_{n=1}^{\infty}\left(1-\frac{1}{2(n+1)}-\frac{1}{24(n+1)^2}-\frac{1}{48(n+1)^3}\right)a_n\end{aligned} \tag{1.5}$$

stated by Bicheng and Debnath [1], Xie and Zhong [4], Ping and Guozheng [3] and Yang [5], respectively.

Moreover, Yang [5] proposed

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{6}\frac{b_k}{(n+1)^k}\right)a_n,$$

where

$$b_1=\frac{1}{2},\quad b_2=\frac{1}{24},\quad b_3=\frac{1}{48},\quad b_4=\frac{73}{5760},\quad b_5=\frac{11}{1280}$$

and conjectured that if

$$\left(1+\frac{1}{x}\right)^x=e\left[1-\sum_{k=1}^{\infty}\frac{b_k}{(n+1)^k}\right]a_n,\quad x>0,$$

then $b_k>0$, $k=1,2,\ldots$. This open problem was solved to some extend by Hu [6].

We improve in this paper (1.2)-(1.5) by stating the following

Theorem 1. *For every* $a_n \geq 0$, $n = 1, 2, 3, \ldots,$ *with* $0 < \sum_{n=1}^{\infty} a_n < \infty$, *it holds*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{\frac{1}{2}}{n + \frac{11}{12}} - \frac{\frac{5}{288}}{\left(n + \frac{11}{12}\right)^3}\right) a_n. \tag{1.6}$$

Inequality (1.6) is stronger than (1.2)-(1.5), since

$$\begin{aligned} 1 - \frac{\frac{1}{2}}{n + \frac{11}{12}} - \frac{\frac{5}{288}}{\left(n + \frac{11}{12}\right)^3} &= \left(1 - \frac{1}{2(n+1)} - \frac{1}{24(n+1)^2} - \frac{1}{48(n+1)^3}\right) \\ &\quad - \frac{734n + 384n^2 + 351}{48(n+1)^3 (12n+11)^3}. \end{aligned}$$

2. The Proof

Theorem 1 easily follows using (1.1) and from

Lemma 1. *For every integer* $n \geq 1$, *we have*

$$\left(1 + \frac{1}{n}\right)^n < e \left(1 - \frac{\frac{1}{2}}{n + \frac{11}{12}} - \frac{\frac{5}{288}}{\left(n + \frac{11}{12}\right)^3}\right).$$

Proof. By letting the logarithm, the requested inequality can be equivalently written as

$$n \ln \left(1 + \frac{1}{n}\right) < 1 + \ln \left(1 - \frac{\frac{1}{2}}{n + \frac{11}{12}} - \frac{\frac{5}{288}}{\left(n + \frac{11}{12}\right)^3}\right),$$

so it suffices to show that the function

$$f(x) = x \ln \left(1 + \frac{1}{x}\right) - 1 - \ln \left(1 - \frac{\frac{1}{2}}{x + \frac{11}{12}} - \frac{\frac{5}{288}}{\left(x + \frac{11}{12}\right)^3}\right)$$

is negative on $[1, \infty)$. In this sense, note that

$$f'(x) = \ln \left(1 + \frac{1}{x}\right) - \frac{16\,117 + 66\,192x + 105\,408x^2 + 76\,032x^3 + 20\,736x^4}{(x+1)(12x+11)(2772x + 3888x^2 + 1728x^3 + 575)}$$

and

$$f''(x) = -\frac{P(x)}{x(x+1)^2 (12x+11)^2 (2772x + 3888x^2 + 1728x^3 + 575)^2},$$

where

$$\begin{aligned} P(x) &= 227\,091\,936x + 510\,277\,248x^2 + 568\,517\,184x^3 \\ &\quad + 314\,502\,912x^4 + 69\,175\,296x^5 + 40\,005\,625. \end{aligned}$$

Now f is concave, with $f(\infty) = 0$ and

$$f(1) = \ln 2 - \ln \frac{8963}{12\,167} - 1 = -1.\,230\,4 \times 10^{-3} < 0,$$

thus $f(x) < 0$, for every $x \in [1, \infty)$. The proof is complete. ∎

Acknowledgements. 1. The results made in this note were completed using Maple software for symbolic computations.

2. The work of the first author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS – UEFISCDI, project number PN-II-ID-PCE-2011-3-0087.

References

[1] Y. Bicheng and L. Debnath, **Some inequlities involving the constant** e and an application to Carleman's inequlity. J. Math. Anal. Appl., 223 (1998), 347–353.

[2] G.H. Hardy, J.E. Littlewood and G. Polya,**Inequalities, Cambridge Univ. Press, London, 1952.**

[3] Y. Ping and S. Guozheng,**A Strengthened Carleman's inequality. J. Math. Anal. Appl., 240 (1999), 290–293.**

[4] Z. Xie and Y. Zhong, **A best approximation for constant** e and an improvement to Hardy's inequality, J. Math. Anal. Appl. 252 (2000), 994–998.

[5] X.Yang,**On Carleman's inequality, J. Math. Anal. Appl., 253 (2001), 691–694.**

[6] H. Yue, **A strenghtened CArleman's inequality, Comm. Math. Anal., 1(2006), no. 2, 115-119.**

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