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ON A SUBCLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS ASSOCIATED TO AN INTEGRAL OPERATOR INVOLVING HURWITZ-LERCH ZETA FUNCTION

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Abstract. Making use of an integral operator involving the Hurwitz-Lerch zeta function, we introduce a new subclass of analytic functions $Q^{*\alpha}_{s,b}(\delta, \beta)$ defined in the open unit disk and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $Q^{*\alpha}_{s,b}(\delta, \beta)$.

1. Introduction and preliminaries

Let A denote the class of all analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\},$$

of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1}$$

For functions $f \in A$ given by (1) and $g \in A$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

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The authors [1] have recently introduced a new generalized integral operator $\Im^{\alpha}_{s,b}f(z)$ as we will show in the following:

Definition 1.1. *(Srivastava and Choi [2]) A general Hurwitz –Lerch Zeta function* $\Phi(z,s,b)$ *defined by*

$$\Phi(z,s,b)=\sum_{n=0}^{\infty}\frac{z^n}{(n+b)^s},$$

where $(s\in\mathbb{C},\ b\in\mathbb{C}-\mathbb{Z}_0^-)$ *when* $(|z|<1)$, *and* $(\Re(b)>1)$ *when* $(|z|=1)$.

We define the function :

$$\Phi^*(z,s,b)=(b^s z\Phi(z,s,b))*f(z),$$

then

$$\Phi^*(z,s,b)=z+\sum_{n=2}^{\infty}\frac{a_n z^n}{(n+b-1)^s}.$$

Definition 1.2. *(see [3] ,[4]) Let the function* f *be analytic in a simply connected domain of the z-plane containing the origin. The fractional derivative of* f *of order*α *is defined by*

$$D_z^{\alpha}f(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z\frac{f(t)}{(z-t)^{\alpha}}dt,\qquad (0\le\alpha<1),$$

where the multiplicity of $(z-t)^{-\alpha}$ *is removed by requiring* $log(z-t)$ *to be real when* $(z-t)>0$.

Using Definition 1.2 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [3] introduced the operator $\Omega^{\alpha}:A\to A$, which is known as an extension of fractional derivative and fractional integral, as follows:

$$\Omega^{\alpha}f(z)=\Gamma(2-\alpha)z^{\alpha}D_z^{\alpha}f(z),\qquad (\alpha\neq 2,3,4,\cdots),$$

$$=z+\sum_{n=2}^{\infty}\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}a_n z^n,\quad (z\in\mathbb{U}).$$

For $s \in \mathbb{C}$, $b \in \mathbb{C} - \mathbb{Z}_0^-$, and $0 \leq \alpha < 1$, the generalized integral operator $(\Im_{s,b}^{\alpha} f) : A \rightarrow A$, is defined by

$$\begin{aligned} \Im_{s,b}^{\alpha} f(z) &= \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^*(z,s,b), \quad (\alpha \neq 2,3,4,\cdots) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left(\frac{b}{n-1+b} \right)^s a_n z^n \quad (z \in \mathbb{U}). \end{aligned} \tag{2}$$

Note that : $\Im_{0,b}^{0} f(z) = f(z)$.

Special cases of this operator include:

- $\Im_{0,b}^{\alpha} f(z) \equiv \Omega^{\alpha} f(z)$ is Owa and Srivastava operator [3].
- $\Im_{s,b+1}^{0} f(z) \equiv J_{s,b} f(z)$ is the Srivastava and Attiya integral operator [5].
- $\Im_{1,1}^{0} f(z) \equiv A(f)(z)$ is the Alexander integral operator [6].
- $\Im_{s+1,1}^{0} f(z) \equiv L(f)(z)$ is the Libera integral operator [7].
- $\Im_{1,\delta}^{0} f(z) \equiv L_{\delta}(f)(z)$ is the Bernardi integral operator [8].
- $\Im_{\sigma,2}^{0} f(z) \equiv I^{\sigma} f(z)$ is the Jung– Kim– Srivastava integral operator [9].

It is easily verified from the above definition of the operator $\Im_{s,b}^{\alpha} f(z)$ that:

$$z(\Im_{s+1,b}^{\alpha} f(z))'^{\alpha}_{s+1,b} f(z) + b \Im_{s,b}^{\alpha} f(z).$$

Making use of the operator defined above, we introduce a new subclass of analytic functions with negative coefficients, and discuss some properties of geometric function theory in relation to this subclass.

For $(0 \leq \delta < 1)$, $(0 < \beta \leq 1)$ and $(\frac{1}{2} < \gamma \leq 1)$ if $\delta = 0$, and $(\frac{1}{2} < \gamma \leq \frac{1}{2\delta})$if $\delta = 0$, we let $Q_{s,b}^{*\alpha}(\delta, \beta)$ be the subclass of A consisting of functions of the form (1) and satisfying the inequality

$$\left| \frac{(\Im_{s,b}^{\alpha} f(z))' - 1}{2\gamma((\Im_{s,b}^{\alpha} f(z))'^{\alpha}_{s,b} f(z))' - 1)} \right| < \beta.$$

We further let

$$Q_{s,b}^{*\alpha}(\delta, \beta) = Q_{s,b}^{\alpha}(\delta, \beta) \cap T,$$

where

(3)

$$T := \left\{ f \in A : f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad where \ a_n \geq 0 \ \ for \ all \ \ n \geq 2 \right\},$$

is a subclass of A introduced and studied by Silverman [10].

In the following section we obtain coefficient bounds and extreme points for the subclass $Q_{s,b}^{*\alpha}(\delta, \beta)$.

2. Coefficient bounds

Theorem 2.1. *Let the function f be defined by (3). Then $f \in Q_{s,b}^{*\alpha}(\delta, \beta)$ if and only if*

(4)

$$\sum_{n=2}^{\infty} n\left[1 + \beta(2\gamma - 1)\right] \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \right) \left| \left(\frac{b}{n-1+b} \right)^{s} \right| a_n \leq 2\beta\gamma(1-\delta).$$

The result is sharp for the function

(5)

$$f(z) = z - \frac{2\beta\gamma(1-\delta)}{n\left[1 + \beta(2\gamma - 1)\right] \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \right) \left| \left(\frac{b}{n-1+b} \right)^{s} \right|} z^n, \quad (n \geq 2).$$

Proof: Assume that the inequality (4) holds and let $|z| = 1$. Then by hypothesis, we have

$$\left| (\Im_{s,b}^{\alpha} f(z))' - 1 \right| - \beta \left| 2\gamma((\Im_{s,b}^{\alpha} f(z))_{s,b}'^{\alpha} f(z))' - 1) \right|$$

$$= \left| - \sum_{n=2}^{\infty} n \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \right) \left(\frac{b}{n-1+b} \right)^{s} a_n z^{n-1} \right| - \beta$$

$$\left| 2\gamma(1-\delta) - \sum_{n=2}^{\infty} n \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \right) \left(\frac{b}{n-1+b} \right)^{s} (2\gamma - 1) a_n z^{n-1} \right|$$

$$\leq \sum_{n=2}^{\infty} n \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \right) \left| \left(\frac{b}{n-1+b} \right)^{s} \right| a_n - 2\beta\gamma(1-\delta) +$$

$$\sum_{n=2}^{\infty} n \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \right) \left| \left(\frac{b}{n-1+b} \right)^{s} \right| \beta(2\gamma - 1) a_n,$$

$$= \sum_{n=2}^{\infty} n\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^{s}\right| a_n -$$

$$2\beta\gamma(1-\delta) \leq 0.$$

Hence, by the maximum modulus theorem $f \in Q^{*\alpha}_{s,b}(\delta,\beta)$.

In order to prove the sufficiency, assume that $f \in Q^{*\alpha}_{s,b}(\delta,\beta)$.

$$\left|\frac{(\Im^{\alpha}_{s,b}f(z))'-1}{2\gamma((\Im^{\alpha}_{s,b}f(z))'^{\alpha}_{s,b}f(z))'-1)}\right|$$

$$= \left|\frac{-\sum_{n=2}^{\infty} n\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left(\frac{b}{n-1+b}\right)^{s} a_n z^{n-1}}{2\gamma(1-\delta)-\sum_{n=2}^{\infty} n\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left(\frac{b}{n-1+b}\right)^{s}(2\gamma-1)a_n z^{n-1}}\right| \leq \beta,$$

which implies,

(6)

$$\Re\left\{\frac{\sum_{n=2}^{\infty} n\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^{s}\right| a_n z^{n-1}}{2\gamma(1-\delta)-\sum_{n=2}^{\infty} n\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^{s}(2\gamma-1)\right| a_n z^{n-1}}\right\} \leq \beta,$$

since $\Re(z) \leq |z|$ for all z.

Letting $z \to 1$ through real values in (6) we obtain the desired inequality (4).

Corollary 2.2. *Let the function f be defined by (3),and $f \in Q^{*\alpha}_{s,b}(\delta,\beta)$. Then*

$$a_n \leq \frac{2\beta\gamma(1-\delta)}{n\left[1+\beta(2\gamma-1)\right]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n-1+b}\right)^{s}\right|}, \quad (n \geq 2), \tag{7}$$

with equality only for functions of the form (5).

Theorem 2.3 (Extreme Points)**.** *Let $f_1(z) = z$ and,*

$$f_n(z) = z - \frac{2\beta\gamma(1-\delta)}{n\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^{s}\right|} z^n, \quad (n \geq 2),$$

for $(0 \leq \delta < 1)$, $(0 < \beta \leq 1)$ *and* $(\frac{1}{2} < \gamma \leq 1)$ *if* $\delta \neq 0$, *and* $(\frac{1}{2} < \gamma \leq \frac{1}{2\delta})$*if* $\delta = 0$. *Then* f *is in the class* $Q^{*\alpha}_{s,b}(\delta,\beta)$, *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z), \tag{8}$$

where $\omega_n \geq 0$ *and* $\sum_{n=1}^{\infty} \omega_n = 1$.

Proof: Suppose f can be written as in (8). Then

$$f(z) = z - \sum_{n=2}^{\infty} \omega_n \frac{2\beta\gamma(1-\delta)}{n\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|} z^n.$$

Now,

$$\sum_{n=2}^{\infty} \frac{n\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)} \times$$

$$\times \omega_n \frac{2\beta\gamma(1-\delta)}{n\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|},$$

$$= \sum_{n=1}^{\infty} \omega_n = 1 - \omega_1 \leq 1.$$

Then (4) holds, thus $f \in Q^{*\alpha}_{s,b}(\delta,\beta)$, by Theorem 2.1 (Sufficiency).

Conversely, assume that f defined by (3) belongs to class $Q^{*\alpha}_{s,b}(\delta,\beta)$. Then by using (7), we set

$$\omega_n = \frac{n\left[1+\beta(2\gamma-1)\right]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)} a_n, \qquad (n \geq 2),$$

and $\omega_1 = 1 - \sum_{n=1}^{\infty} \omega_n$. Then we have $f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z)$, and

hence this completes the proof.

3. Distortion bounds

In this section we obtain distortion bounds for the class $Q^{*\alpha}_{s,b}(\delta,\beta)$.

Theorem 3.1. *If $f \in Q^{*\alpha}_{s,b}(\delta,\beta)$, then the following inequalities hold whenever $|z| = r$, where $0 < r < 1$*

$$r - \frac{2\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}r^2 \le |f(z)|$$

$$\le r + \frac{2\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}r^2,$$

and

$$1 - \frac{4\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}r \le |f'(z)|$$

$$\le 1 + \frac{4\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}r. \tag{9}$$

The result is sharp for the function

$$f(z) = z - \frac{2\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}z^2.$$

Proof: Since $f(z) \in Q^{*\alpha}_{s,b}(\delta,\beta)$, and in view of inequality (4) of Theorem 2.1, we have

$$[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|\sum_{n=2}^{\infty}a_n \le$$

$$\sum_{n=2}^{\infty}[1+\beta(2\gamma-1)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n-1+b}\right)^s\right|a_n \le 2\beta\gamma(1-\delta).$$

$$\sum_{n=2}^{\infty}a_n \le \frac{2\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n-1+b}\right)^s\right|}. \tag{10}$$

After the inequality obtained by (3) and (10), to assume that $|z| = r$, in order to get the next inequality.

$$|z| - |z|^2 \sum_{n=2}^{\infty} a_n \leq |f(z)|$$

$$\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n,$$

$$r - r^2 \frac{2\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|} \leq |f(z)|$$

$$\leq r + r^2 \frac{2\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right)\left|\left(\frac{b}{n-1+b}\right)^s\right|}.$$

Further,

$$\sum_{n=2}^{\infty} n a_n \leq \frac{4\beta\gamma(1-\delta)}{[1+\beta(2\gamma-1)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n-1+b}\right)^s\right|}.$$

Hence (9) follows from

$$1 - r\sum_{n=2}^{\infty} a_n \leq |f'(z)| \leq 1 + r\sum_{n=2}^{\infty} a_n.$$

4. Radius of starlikeness and convexity

In the next theorems, we will find the radius of starlikeness, convexity and close-to-convexity for the class $Q^{*\alpha}_{s,b}(\delta,\beta)$.

Theorem 4.1. *Let the function f be defined by (3), belong to the class $Q^{*\alpha}_{s,b}(\delta,\beta)$. Then f is close-to-convex of order λ, $(0 \leq \lambda < 1)$ in the disk $|z| < r$, where*

$$r := \inf_{n\geq 2}\left(\frac{(1-\lambda)[1+\beta(2\gamma-1)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)}\right)^{\frac{1}{n-1}}.$$

The result is sharp, with extremal function f given by (5).

Proof: Given $f \in T$, f is close-to-convex of order λ in the disk $|z| < r$ if and only if

$$|f'(z) - 1| < 1 - \lambda, \quad whenever \quad |z| < r. \tag{11}$$

For the left hand side of (11) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Then (11) is implied by

$$\sum_{n=2}^{\infty} \frac{n}{1-\lambda} a_n |z|^{n-1} < 1.$$

Using the fact that $f(z) \in Q_{s,b}^{*\alpha}(\delta, \beta)$, if and only if

$$\sum_{n=2}^{\infty} \frac{n\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)} a_n \leq 1,$$

it follows that(11) is true if

$$\frac{n}{1-\lambda}|z|^{n-1} \leq \frac{n\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)},$$

whenever $|z| < r$ *and* $n \geq 2$.

We obtain

$$r := \inf_{n \geq 2} \left(\frac{(1-\lambda)\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)} \right)^{\frac{1}{n-1}}.$$

This completes the proof.

Theorem 4.2. *Let the function f be defined by (3) belong to the class $Q_{s,b}^{*\alpha}(\delta, \beta)$. Then*

(I) f is starlike of order λ, $(0 \leq \lambda < 1)$ in the disk $|z| < r$, that is,

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \lambda, \quad (|z| < r, 0 \leq \lambda < 1),$$

where

$$r := \inf_{n \geq 2} \left(\frac{n(1-\lambda)\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)(n-\lambda)} \right)^{\frac{1}{n-1}}.$$

(II) f is convex of order λ, $(0 \leq \lambda < 1)$ in the disk $|z| < r$, that is,

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \lambda, \quad (|z| < r, 0 \leq \lambda < 1),$$

where

$$r := \inf_{n \geq 2} \left(\frac{(1-\lambda)\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)(n-\lambda)} \right)^{\frac{1}{n-1}}.$$

Each of these results is sharp for the extremal function given by (5).

Proof: (I) Given $f \in T$ and f is starlike of order λ, in the disk $|z| < r$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \lambda \text{ whenever } |z| < r. \tag{12}$$

For the left hand side of (12) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n|z|^{n-1}}.$$

Then (12) is implied by

$$\sum_{n=2}^{\infty} \frac{n-\lambda}{1-\lambda} a_n|z|^{n-1} < 1.$$

Using the fact that $f(z) \in Q_{s,b}^{*\alpha}(\delta, \beta)$, if and only if

$$\sum_{n=2}^{\infty} \frac{n\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)} a_n \leq 1.$$

(12) is true for every z in the disk $|z| < r$ if

$$\frac{n-\lambda}{1-\lambda}|z|^{n-1} \leq \frac{n\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)(n-\lambda)}.$$

Thus

$$r := \inf_{n \geq 2} \left(n\frac{(1-\lambda)\left[1 + \beta(2\gamma - 1)\right] \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left|\left(\frac{b}{n-1+b}\right)^s\right|}{2\beta\gamma(1-\delta)(n-\lambda)} \right)^{\frac{1}{n-1}}.$$

This completes the proof.

(II) Using the fact that f is convex of order λ if and only if $zf'(z)$ is starlike of order λ, we can prove (II) using similar methods to the proof of (I).

5. Conclusions

The work presented here is the generalization of some work done by earlier researchers. Further the research can be continued by using fractional calculus operators for this class. For example, we can also study the Cesàro means like the one given by Darus and Ibrahim [11].

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