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A GENERAL DECOMPOSITION THEOREM FOR CLOSED FUNCTIONS

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Abstract. Quite recently, the present authors [16] have defined and investigated the notion of mg^*-closed sets as a modification of g-closed sets due to Levine [10]. In this paper, we introduce the notion of mg^*-closed functions and obtain a general decomposition of closed functions in topological spaces.

1. Introduction

Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the research of generalizations of closed functions in topological spaces. By using these sets, many authors introduced and studied various types of modifications of closed functions. The notion of g-closed sets was introduced by Levine [10]. The present authors [16] introduced a new class of sets called mg^*-closed sets. The mg^*-closed sets place between closed sets and g-closed sets.

In this paper, we introduce the notion of mlc-sets as a general form of locally closed sets. By using mg^*-closed sets and mlc-sets, we introduce the notions of mg^*-closed functions and mlc-closed functions, respectively, and obtain a general decomposition of closed functions. Furthermore, we provide a sufficient condition for a mg^*-closed function to be closed. In the last section, we consider new forms of decomposition of closed functions.

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2. Preliminaries

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1. A subset A of a topological space (X, τ) is said to be *semi-open* [9] (resp. *preopen* [12], *α-open* [14], *β-open* [1]) if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}(A))$, $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$, $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A))))$.

The family of all semi-open (resp. preopen, α-open, β-open) sets in (X, τ) is denoted by $\mathrm{SO}(X)$ (resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$).

Definition 2.2. Let (X, τ) be a topological space. A subset A of X is said to be *g-closed* [10] (resp. *sg^*-closed*, *pg^*-closed*, *αg^*-closed*, *βg^*-closed* [13]) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is open (resp. semi-open, preopen, α-open, β-open) in (X, τ).

Remark 2.1. (1) An sg^*-closed set is also called ω-closed [22], semi-star-closed [19], or $\hat{g}$-closed [8].

(2) By the definitions, we obtain the following diagram:

DIAGRAM I

$$\begin{array}{ccccc} g\text{-closed} & \Leftarrow & \alpha g^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & & \\ & & \Uparrow & & \Uparrow & & \\ & & sg^*\text{-closed} & \Leftarrow & \beta g^*\text{-closed} & \Leftarrow & \text{closed} \end{array}$$

Throughout the present paper, (X, τ) and (Y, σ) always denote topological spaces and $f : (X, \tau) \to (Y, \sigma)$ presents a function.

Definition 2.3. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *g-closed* (resp. *sg^*-closed*, *pg^*-closed*, *αg^*-closed*, *βg^*-closed*) if $f(F)$ is g-closed (resp. sg^*-closed, pg^*-closed, αg^*-closed, βg^*-closed) for each closed set F of (X, τ).

Remark 2.2. For a function $F : (X, \tau) \to (Y, \sigma)$, we obtain the similar diagram with Diagram I. In this new diagram, the implications are strict. For example, every sg^*-closed function is not closed as shown by the following example.

Example 2.1. Let $X = \{p, q\}, \tau =$ discrete topology, $Y = \{a, b, c\}$, and $\sigma = \{X, \emptyset, \{a\}, \{b, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(p) = b, f(q) = c$. Then f is an sg^*-closed function which is not closed. Because $\{p\}$ is a closed set of (X, τ) and $f(\{p\}) = \{b\}$ is an sg^*-closed set which is not closed in (Y, σ).

Definition 2.4. A subset A of a topological space (X,τ) is called a *LC-set* [5], [6] (resp. an *slc*-set* [21] or an *slc*-set [4], a *plc-set* [4], an αlc [2], a βlc*-set* [4]) if $A = U \cap F$, where $U \in \tau$ (resp. $U \in \mathrm{SO}(X)$, $U \in \mathrm{PO}(X)$, $U \in \alpha(X)$, $U \in \beta(X)$) and F is closed in (X,τ).

Remark 2.3. It is known in Remark 4.5 of [21] that every closed set is an *slc*∗-set but not conversely and that an sg^*-closed set and an *slc*∗-set are independent.

Definition 2.5. A function $f : (X,\tau) \to (Y,\sigma)$ is said to be *LC-closed* (resp. *slc-closed*, *plc-closed*, αlc*-closed*, βlc*-closed*) if $f(F)$ is an *LC*-set (resp. an *slc*-set, a *plc*-set, an αlc-set, a βlc-set) of (Y,σ) for each closed set F of (X,τ).

Remark 2.4. Every closed function is *slc**-closed but not conversely as shown by the following example.

Example 2.2. Let $X = Y = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$, and $\sigma = \{\emptyset, Y, \{a\}\}$. Let $f : (X,\tau) \to (Y,\sigma)$ be the identity function. Then f is an *slc**-closed function which is not closed. Because $\{a\}$ is a closed set of (X,τ) and $f(\{a\}) = \{b\}$ is an *slc**-closed set which is not closed in (Y,σ).

3. m-Structures and mg^*-closed sets

Definition 3.1. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) [17], [18] on X if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be m_X*-open* and the complement of an m_X-open set is said to be m_X*-closed*.

Definition 3.2. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [11] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 3.1. Let (X,τ) be a topological space. Then the families τ, $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, and $\beta(X)$ are all m-structures with property $\mathcal{B}$.

Definition 3.3. Let (X,τ) be a topological space and m_X an m-structure on X. A subset A is said to be mg^**-closed* [16] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in m_X$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$) and A is mg^*-closed, then A is g-closed (resp. sg^*-closed, pg^*-closed, αg^*-closed, βg^*-closed).

Lemma 3.1. (Noiri and Popa [16]). *Let (X, τ) be a topological space and m_X an m-structure on X such that $\tau \subset m_X$. Then the following implications hold:*

$$\textit{closed} \Rightarrow mg^* \text{ - } \textit{closed} \Rightarrow g\textit{-closed}$$

Lemma 3.2. (Noiri and Popa [16]). *If A is mg^* - closed and m_X-open, then A is closed.*

Definition 3.4. Let (X, τ) be a topological space and m_X an m-structure on X. A subset A is called an *mlc-set* if $A = U \cap F$, where $U \in m_X$ and F is closed in (X, τ).

Remark 3.3. Let (X, τ) be a topological space and A a subset of X.

(1) if $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\beta(X)$) and A is a *mlc*-set, then A is a locally closed set (resp. an *slc**-set or an *slc*-set, a *plc*-set, an αlc-set, a βlc-set).

(2) every closed set is an *mlc*-set but not conversely by Remark 2.3,

(3) an mg^*-closed set and an *mlc*-set are independent by Remark 2.3,

4. Decompositions of a closed function

Theorem 4.1. *Let (X, τ) be a topological space and m_X a minimal structure on X. Then a subset A of X is closed if and only if it is mg^*-closed and an mlc-set.*

Proof. *Necessity.* Suppose that A is closed in (X, τ). Then, by Definition 3.3, A is mg^*-closed. Furthermore, $A = X \cap A$, where $X \in m_X$ and A is closed and hence A is an *mlc*-set.

Sufficiency. Suppose that A is mg^*-closed and an *mlc*-set. Since A is an *mlc*-set, $A = U \cap F$, where $U \in m_X$ and F is closed in (X, τ). Therefore, we have $A \subset U$ and $A \subset F$. By the hypothesis, we obtain $\mathrm{Cl}(A) \subset U$ and $\mathrm{Cl}(A) \subset F$ and hence $\mathrm{Cl}(A) \subset U \cap F = A$. Thus, $\mathrm{Cl}(A) = A$ and A is closed.

Corollary 4.1. *Let (X, τ) be a topological space. Then, for a subset A of X, the following properties are equivalent:*

(1) A is closed;

(2) A is g-closed and a locally closed set;
(3) A is sg^*-closed and a slc-closed;
(4) A is pg^*-closed and a plc-set;
(5) A is αg^*-closed and an αlc-set;
(6) A is βg^*-closed and a βlc-set.

Proof. This is an immediate consequence of Theorem 4.1.

Definition 4.1. Let (Y, σ) be a topological space and m_Y an m-structure on Y A function $f : (X, \tau) \to (Y, \sigma)$ is said to be mg^*-*closed* (resp. *mlc-closed*) if $f(F)$ is mg^*-closed (resp. an mlc-set) for each closed set F of (X, τ).

Remark 4.1. Let (Y, σ) be a topological space, $m_Y = \sigma$ (resp. $\mathrm{SO}(Y)$, $\mathrm{PO}(Y)$, $\alpha(Y)$, $\beta(Y)$), and $f : (X, \tau) \to (Y, \sigma)$ be a function.

(1) If f is mg^*-closed, then f is g-closed (resp. sg^*-closed, pg^*-closed, αg^*-closed, βg^*-closed);

(2) If f is mlc-closed, then f is LC-closed (resp. *slc*-closed, *plc*-closed, αlc-closed, βlc-closed).

Remark 4.2. Let $f : (X, \tau) \to (Y, \sigma)$ be a function and m_Y an m-structure on Y. Then we have the following properties:

(1) every closed function is mg^*-closed but not conversely by Example 2.1,

(2) every closed function is mlc-closed but not conversely by Example 2.2.

Remark 4.3. The notions of mg^*-closedness and mlc-closedness are independent as shown by the following examples.

Example 4.1. Let $f : (X, \tau) \to (Y, \sigma)$ be the same function as in Example 2.1 and $m_Y = \mathrm{SO}(Y)$. Then f is sg^*-closed but it is not slc-closed. Because $f(\{p\}) = \{b\}$ is an sg^*-closed set which is not slc-closed.

Example 4.2. Let $f : (X, \tau) \to (Y, \sigma)$ be the same function as in Example 2.2. Then f is slc-closed but it is not sg^*-closed. Because $f(\{a\}) = \{a\}$ is an slc-closed set which is not sg^*-closed .

Theorem 4.2. *Let (Y, σ) be a topological space and m_Y an m-structure on X. Then a function $f : (X, \tau) \to (Y, \sigma)$ is closed if and only if it is mg^*-closed and mlc-closed.*

Proof. This is an immediate consequence of Theorem 4.1

Corollary 4.2. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) f is closed;
(2) f is g-closed and LC-closed;
(3) f is sg^*-closed and slc-closed;
(4) f is pg^*-closed and plc-closed;
(5) f is αg^*-closed and αlc-closed;
(6) f is βg^*-closed and a βlc-set.

Proof. This is an immediate consequence of Theorem 4.2.

Definition 4.2. Let (Y, σ) be a topological space with an m-structure m_Y on Y. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *contra m-closed* if $f(F)$ is m_Y-open in (Y, m_Y) for each closed set F of X.

Remark 4.4. If a function $f : (X, \tau) \to (Y, \sigma)$ is contra m-closed and $m_Y = \tau$ (resp. SO(X), PO(X), $\alpha(Y)$, $\beta(Y)$), then f is contra closed [3] (resp. contra semi-closed, contra preclosed, contra α-closed, contra β-closed).

Theorem 4.3. *Let (Y, σ) be a topological space with an m-structure m_Y on Y. If a function $f : (X, \tau) \to (Y, \sigma)$ is mg^*-closed and contra m-closed, then f is closed.*

Proof. Let F be any closed set of (X, τ). Since f is contra m-closed, $f(F)$ is m_Y-open. Since f is mg^*-closed, $f(F)$ is mg^*-closed and hence, by Lemma 3.2, $f(F)$ is closed. Therefore, f is closed.

Corollary 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then f is closed if f satisfies one of the following properties:*

(1) f is g-closed and contra closed;
(2) f is sg^*-closed and contra semi-closed;
(3) f is pg^*-closed and contra preclosed;
(4) f is αg^*-closed and contra α-closed;
(5) f is βg^*-closed and contra β-closed.

5. New forms of decomposition of closed functions

First, we recall the θ-closure and the δ-closure of a subset in a topological space. Let (X, τ) be a topological space and A a subset of X. A point $x \in X$ is called a θ-cluster (resp. δ-cluster) point of A if $\mathrm{Cl}(V) \cap A \neq \emptyset$ (resp. $\mathrm{Int}(\mathrm{Cl}(V)) \cap A \neq \emptyset$) for every open set V containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the *θ-closure* (resp. *δ-closure*) of A and is denoted by $\mathrm{Cl}_\theta(A)$ (resp. $\mathrm{Cl}_\delta(A)$) [23].

Definition 5.1. A subset of a topological space (X, τ) is said to be

(1) *δ-preopen* [20] (resp. *θ-preopen* [15]) if $A \subset \mathrm{Int}(\mathrm{Cl}_\delta(A))$ (resp. $A \subset \mathrm{Int}(\mathrm{Cl}_\theta(A))$),

(2) *δ-β-open* [7] (resp. *θ-β-open* [15]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\delta(A)))$ (resp. $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}_\theta(A)))$).

By $\delta\mathrm{PO}(X)$ (resp. $\delta\beta(X)$, $\theta\mathrm{PO}(X)$, $\theta\beta(X)$), we denote the collection of all δ-preopen (resp. δ-β-open, θ-preopen, θ-β-open) sets of a topological space (X, τ). These four collections are m-structures with property $\mathcal{B}$. In [15], the following diagram is known:

DIAGRAM II

$$\begin{array}{ccccccc}
\alpha\text{-open} & \Rightarrow & \text{preopen} & \Rightarrow & \delta\text{-preopen} & \Rightarrow & \theta\text{-preopen} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{semi-open} & \Rightarrow & \beta\text{-open} & \Rightarrow & \delta\text{-}\beta\text{-open} & \Rightarrow & \theta\text{-}\beta\text{-open}
\end{array}$$

For the new collections of subsets of a topological space (X, τ), we can define many new variations of g-closed sets. For example, in case $m_X = \delta\mathrm{PO}(X)$, $\delta\beta(X)$, $\theta\mathrm{PO}(X)$, $\theta\beta(X)$, we can define new types of g-closed sets as follows:

Definition 5.2. A subset A of a topological space (X, τ) is said to be *δpg^*-closed* (resp. *θpg^*-closed*, *$\delta\beta g^*$-closed*, *$\theta\beta g^*$-closed*) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is δ-preopen (resp. θ-preopen, δ-β-open, θ-β-open) in (X, τ).

By DIAGRAM II and Definitions 5.2, we have the following diagram:

DIAGRAM III

$$\begin{array}{ccccccccc}
g\text{-closed} & \Leftarrow & \alpha g^*\text{-closed} & \Leftarrow & pg^*\text{-closed} & \Leftarrow & \delta pg^*\text{-closed} & \Leftarrow & \theta pg^*\text{-closed} \\
& & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
& & & & sg^*\text{-closed} & \Leftarrow \beta g^*\text{-closed} & \Leftarrow & \delta\beta g^*\text{-closed} & \Leftarrow \\
& & & & \theta\beta g^*\text{-closed} & \Leftarrow \text{closed} & & &
\end{array}$$

Definition 5.3. A subset A of a topological space (X, τ) is called a *δplc-set* (resp. *θplc-set*, *$\delta\beta lc$-set*, *$\theta\beta lc$-set*) if $A = U \cap F$, where U is δ-preopen (resp. θ-preopen, δ-β-open, θ-β-open) in (X, τ) and F is closed in (X, τ).

Corollary 5.1. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) A is closed;

(2) A is δpg^*-closed and a δplc-set;

(3) A is θpg^*-closed and a θplc-set;

(4) A is $\delta\beta g^*$-closed and a $\delta\beta lc$-set;
(5) A is $\theta\beta g^*$-closed and a $\theta\beta lc$-set.

Proof. Let $m_X = \delta\mathrm{PO}(X)$, $\theta\mathrm{PO}(X)$, $\delta\beta(X)$ and $\theta\beta(X)$. Then this is an immediate consequence of Theorem 4.1.

By defining functions similarly to Definition 4.1, we obtain the following decompositions of closed functions:

Corollary 5.2. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*
(1) f is closed;
(2) f is δpg^*-closed and δplc-closed;
(3) f is θpg^*-closed and θplc-closed;
(4) f is $\delta\beta g^*$-closed and $\delta\beta lc$-closed;
(5) f is $\theta\beta g^*$-closed and $\theta\beta lc$-clsoed.

Proof. This is an immediate consequence of Theorem 4.2.

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