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STRONGLY $\mathcal{G}$-β-OPEN SETS AND DECOMPOSITIONS OF CONTINUITY VIA GRILLS

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Abstract. In this paper, we introduce and investigate the notions of strongly $\mathcal{G}$-β-open sets and $\mathcal{G}$-δ-open sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.

1. Introduction

The idea of grills on a topological space was first introduced by Choquet [6]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [4], [5], [14] for details). In [13], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, Hatir and Jafari [7] defined new classes of sets and obtained a new decomposition of continuity in terms of grills. In [1], the present authors defined and investigated the notions of $\mathcal{G}$-α-open sets, $\mathcal{G}$-semi-open sets and $\mathcal{G}$-β-open sets in topological space with a grill.

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By using these sets, we obtained decompositions of continuity. In this paper, we introduce and investigate the notions of strongly $\mathcal{G}$-β-open sets and $\mathcal{G}$-δ-open sets in a topological space with a grill. Furthermore, by using these sets, we obtain new decompositions of continuity.

2. Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $Cl(A)$ and $Int(A)$ denote the closure and the interior of A in (X, τ), respectively. The power set of X will be denoted by $\mathcal{P}(X)$. A subcollection $\mathcal{G}$ (not containing the empty set) of $\mathcal{P}(X)$ is called a grill [6] on X if $\mathcal{G}$ satisfies the following conditions:

(1) $A \in \mathcal{G}$ and $A \subseteq B$ implies that $B \in \mathcal{G}$,
(2) $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any point x of a topological space (X, τ), $\tau(x)$ denotes the collection of all open neighborhoods of x.

Definition 2.1. [13] Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G} \text{ for all } U \in \tau(x)\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ.

Proposition 2.2. [13] *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. Then for all $A, B \subseteq X$:*

(1) $A \subseteq B$ *implies that* $\Phi(A) \subseteq \Phi(B)$,
(2) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,
(3) $\Phi(\Phi(A)) \subseteq \Phi(A) = Cl(\Phi(A)) \subseteq Cl(A)$.

Let $\mathcal{G}$ be a grill on a space X. Then [13] defined a map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ satisfies Kuratowski closure axioms. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X such that $\tau_{\mathcal{G}}$-$Cl(A) = \Psi(A)$ for every $A \subseteq X$. This topology is given by $\tau_{\mathcal{G}} = \{U \subseteq X : \Psi(X - U) = X - U\}$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subseteq \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

3. Strongly $\mathcal{G}$-β-open sets

Definition 3.1. A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be

(1) $\mathcal{G}$-preopen [7] if $A \subseteq Int(\Psi(A))$,
(2) $\mathcal{G}$-semi-open [1] if $A \subseteq \Psi(Int(A))$,
(3) $\mathcal{G}$-α-open [1] if $A \subseteq Int(\Psi(Int(A)))$,
(4) $\mathcal{G}$-β-open [1] if $A \subseteq Cl(Int(\Psi(A)))$,
(5) strongly $\mathcal{G}$-β-open if $A \subseteq \Psi(Int(\Psi(A)))$,
(6) $\mathcal{G}$-γ-open if $A \subseteq \Psi(Int(A)) \cup Int(\Psi(A))$,
(7) almost strongly $\mathcal{G}$-open if f $A \subseteq \Psi(Int(\Phi(A)))$.

The complement of a strongly $\mathcal{G}$-β-open set is said to be strongly $\mathcal{G}$-β-closed.

The family of all $\mathcal{G}$-preopen (resp. $\mathcal{G}$-semi-open, $\mathcal{G}$-α-open, strongly $\mathcal{G}$-β-open) sets in a grill topological space $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}PO(X)$ (resp. $\mathcal{G}SO(X)$, $\mathcal{G}\alpha O(X)$, $S\mathcal{G}\beta O(X)$).

Definition 3.2. A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-δ-open if $Int(\Psi(A)) \subseteq \Psi(Int(A))$. The complement of a $\mathcal{G}$-δ-open set is said to be $\mathcal{G}$-δ-closed.

The family of all $\mathcal{G}$-δ-open subsets of $(X, \tau, \mathcal{G})$ will be denoted by $\mathcal{G}\delta O(X)$.

Definition 3.3. A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-dense-in-itself (resp. $\mathcal{G}$-perfect, $\tau_{\mathcal{G}}$-dense) if and only if $A \subseteq \Phi(A)$ (resp. $A = \Phi(A)$, $\Psi(A) = X$).

Proposition 3.4. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then every almost strongly $\mathcal{G}$-open set is strongly $\mathcal{G}$-β-open.*

Proof. Let A be an almost strongly $\mathcal{G}$-open set. Then $A \subseteq \Psi(Int(\Phi(A))) \subseteq \Psi(Int(\Phi(A) \cup A)) = \Psi(Int(\Psi(A)))$. Hence A is strongly $\mathcal{G}$-β-open. ∎

The converse of Proposition 3.4 need not be true in general as shown in the following example.

Example 3.5. *Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$ and the grill $\mathcal{G} = \{\{b\}, \{a, b\}, \{a, b, c\}, \{c, b, d\}, \{b, c\}, \{b, d\}, \{a, b, d\}, X\}$. Then $A = \{a, b, c\}$ is a strongly $\mathcal{G}$-β-open set which is not almost strongly $\mathcal{G}$-open. For $A = \{a, b, c\}$, $\Phi(A) = \{b, d\}$ and $\Psi(Int(\Phi(A))) = \{\emptyset\}$. Hence $A \nsubseteq \Psi(Int(\Phi(A))) = \{\emptyset\}$ and A*

is not an almost strongly $\mathcal{G}$-open set. On the other hand, since $\Psi(Int(\Psi(A))) = X$, thus A is strongly $\mathcal{G}$-β-open.

Proposition 3.6. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then a subset of X is $\mathcal{G}$-semi-open if and only if it is both $\mathcal{G}$-δ-open and strongly $\mathcal{G}$-β-open.*

Proof. Necessity. This is obvious.
Sufficiency. Let A be $\mathcal{G}$-δ-open and strongly $\mathcal{G}$-β-open, then we have $Int(\Psi(A)) \subseteq \Psi(Int(A))$. Since A is strongly $\mathcal{G}$-β-open, we obtain that $A \subseteq \Psi(Int(\Psi(A))) \subseteq \Psi(\Psi(Int(A))) = \Psi(Int(A))$. Hence A is $\mathcal{G}$-semi-open. ∎

Proposition 3.7. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then a subset of X is $\mathcal{G}$-α-open if and only if it is both $\mathcal{G}$-δ-open and $\mathcal{G}$-preopen.*

Proof. Necessity. This is obvious.
Sufficiency. Let A be a $\mathcal{G}$-δ-open and $\mathcal{G}$-preopen set. Then we have $Int(\Psi(A)) \subseteq \Psi(Int(A))$ and hence $Int(\Psi(A)) \subseteq Int(\Psi(Int(A)))$. Since A is $\mathcal{G}$-preopen, we have $A \subseteq Int(\Psi(A))$. Therefore, we obtain that $A \subseteq Int(\Psi(Int(A)))$ and hence A is a $\mathcal{G}$-α-open set. ∎

Proposition 3.8. *For a subset of a grill topological space $(X, \tau, \mathcal{G})$, the following properties are hold:*

1. *Every $\mathcal{G}$-preopen set is $\mathcal{G}$-γ-open.*
2. *Every $\mathcal{G}$-γ-open set is strongly $\mathcal{G}$-β-open.*
3. *Every strongly $\mathcal{G}$-β-open set is $\mathcal{G}$-β-open.*

Proof. The proof is obvious by Definition 3.1 and $\Psi(A) = A \cup \Phi(A)$. ∎

Proposition 3.9. [1] *For a subset of a grill topological space $(X, \tau, \mathcal{G})$, the following properties hold:*

1. *Every open set is $\mathcal{G}$-α-open.*
2. *Every $\mathcal{G}$-α-open set is $\mathcal{G}$-semi-open.*
3. *Every $\mathcal{G}$-α-open set is $\mathcal{G}$-preopen.*

None of implications in Proposition 3.9 is reversible as shown in [1].

Remark 3.10. By the examples stated below, we obtain the following results:

(1) $\mathcal{G}$-δ-openness and strongly $\mathcal{G}$-β-openness are independent of each other.
(2) $\mathcal{G}$-δ-openness and $\mathcal{G}$-preopenness are independent of each other.

Example 3.11. *Let $X = \{a,b,c,d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\}$ and the grill $\mathcal{G} = \{\{a\},\{b\},\{a,c\},\{a,b\},\{a,d\},\{a,b,c\},\{c,b,d\},\{a,b,d\},\{a,c,d\},\{b,c\},\{b,d\},\{b,c,d\},X\}$. Then $A = \{a,b\}$ is a $\mathcal{G}$-semi-open set which is not $\mathcal{G}$-preopen. For $A = \{a,b\}$, $Int(A) = \{a\}$ and $\Psi(Int(A)) = \{a,b,d\}$. Hence $A \subseteq \Psi(Int(A))$ and hence A is a $\mathcal{G}$-semi-open set. On the other hand, since $Int(\Psi(A)) = \{a\}$, $A = \{a,b\} \nsubseteq Int(\Psi(A)) = \{a\}$ and thus A is not $\mathcal{G}$-preopen.*

Example 3.12. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{b,c\}\}$ and the grill $\mathcal{G} = \{\{a\},\{b\},\{a,b\},\{a,c\},\{b,c\},X\}$. Then $A = \{a,b\}$ is a $\mathcal{G}$-preopen set which is not $\mathcal{G}$-δ-open. For $A = \{a,b\}$, $Int(A) = \{a\}$ and $\Psi(Int(A)) = \{a\}$. Moreover, $\Psi(A) = \{a,b,c\}$ and $Int(\Psi(A)) = \{a,b,c\}$. Hence $Int(\Psi(A)) \nsubseteq \Psi(Int(A))$ and A is not a $\mathcal{G}$-δ-open set. On the other hand, since $Int(\Psi(A)) = \{a,b,c\}$, $A = \{a,b\} \subseteq Int(\Psi(A)) = \{a,b,c\}$ and thus A is $\mathcal{G}$-preopen.*

Example 3.13. *Let $\mathbb{R}$ be the set of real numbers with the usual topology, and let $A = [1,2) \cap \mathbb{Q}$, where $\mathbb{Q}$ stands for the set of rational numbers with the grill $\mathcal{G} = \mathcal{P}(\mathbb{R}) - \{\emptyset\}$. Then A is a strongly $\mathcal{G}$-β-open set which is not $\mathcal{G}$-γ-open. For $A = [1,2) \cap \mathbb{Q}$, $Int(A) = \emptyset$ and $\Psi(Int(A)) = \emptyset$. Moreover, $\Psi(A) = [1,2]$ and $Int(\Psi(A)) = (1,2)$. Hence $A \nsubseteq Int(\Psi(A)) \cup \Psi(Int(A))$ and A is not $\mathcal{G}$-γ-open. On the other hand, since $\Psi(Int(\Psi(A))) = [1,2]$, and $A \subseteq \Psi(Int(\Psi(A))) = [1,2]$ and thus A is strongly $\mathcal{G}$-β-open.*

Example 3.14. *In Example 3.5, $A = \{a,c,d\}$ is a $\mathcal{G}$-δ-open set and a $\mathcal{G}$-β-open set which is not strongly $\mathcal{G}$-β-open. For $A = \{a,c,d\}$, $Int(A) = \{a,c\}$ and $\Psi(Int(A)) = \{a,c\}$. Moreover, $\Psi(A) = \{a,c,d\}$, $Int(\Psi(A)) = \{a,c\}$ and $Cl(Int(\Psi(A)) = X$. Hence $Int(\Psi(A)) \subseteq \Psi(Int(A))$ and $A \subseteq Cl(Int(\Psi(A)))$ so A is a $\mathcal{G}$-δ-open set and a $\mathcal{G}$-β-open set. On the other hand, since $\Psi(Int(\Psi(A))) = \{a,c\}$, $A = \{a,c,d\} \nsubseteq \Psi(Int(\Psi(A))) = \{a,c\}$ and thus A is not strongly $\mathcal{G}$-β-open (This implies that $\mathcal{G}$-δ-open set may not be $\mathcal{G}$-semi-open).*

Proposition 3.15. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a subset of X the following implications hold:*

$$\begin{array}{ccccccc} open \to \mathcal{G}-\alpha-open & \to & \mathcal{G}-semi-open & \to & \mathcal{G}-\delta-open & & \\ \downarrow & & \downarrow & & & & \\ \to \mathcal{G}-preopen & \to & \mathcal{G}-\gamma-open & \to & strongly\mathcal{G}\beta-open & \to & \mathcal{G}-\beta-open \\ & & & & & & \uparrow \\ & & & & & & almoststrongly\mathcal{G}-open \end{array}$$

Remark 3.16. None of implications in Proposition 3.15 is reversible as shown by the above examples.

Proposition 3.17. *Let A and B be subsets of a grill topological space $(X, \tau, \mathcal{G})$. If $A \subseteq B \subseteq \Psi(A)$ and $A \in \mathcal{G}\delta O(X)$, then $B \in \mathcal{G}\delta O(X)$.*

Proof. Suppose that $A \subseteq B \subseteq \Psi(A)$ and $A \in \mathcal{G}\delta O(X)$. Then, since $A \in \mathcal{G}\delta O(X)$, we have $Int(\Psi(A)) \subseteq \Psi(Int(A))$. Since $A \subseteq B$, $\Psi(Int(A)) \subseteq \Psi(Int(B))$ and $Int(\Psi(A)) \subseteq \Psi(Int(A)) \subseteq \Psi(Int(B))$. Since $B \subseteq \Psi(A)$, we have $\Psi(B) \subseteq \Psi(\Psi(A)) = \Psi(A)$ and $Int(\Psi(B)) \subseteq Int(\Psi(A))$. Therefore, we obtain that $Int(\Psi(B)) \subseteq \Psi(Int(B))$. This shows that B is a $\mathcal{G}$-δ-open set. ∎

Corollary 3.18. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If a subset A of X is $\mathcal{G}$-δ-open and $\tau_{\mathcal{G}}$-dense, then every subset of X containing A is $\mathcal{G}$-δ-open.*

Proof. The proof is obvious by Proposition 3.17. ∎

Lemma 3.19. [13] *Let (X, τ) be a topological space and $\mathcal{G}$ be a grill on X. If $U \in \tau$, then $U \cap \Phi(A) = U \cap \Phi(U \cap A)$ for any $A \subseteq X$.*

Lemma 3.20. *Let A be a subset of a grill topological space $(X, \tau, \mathcal{G})$. If $U \in \tau$, then $U \cap \Psi(A) \subseteq \Psi(U \cap A)$.*

Proof. Since $U \in \tau$, by Lemma 3.19 we obtain $U \cap \Psi(A) = U \cap (A \cup \Phi(A) = (U \cap A) \cup (U \cap \Phi(A)) \subseteq (U \cap A) \cup \Phi(U \cap A) = \Psi(U \cap A)$. ∎

Theorem 3.21. *A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is strongly $\mathcal{G}$-β-closed if and only if $\tau_{\mathcal{G}}$-$Int(Cl(\tau_{\mathcal{G}}$-$Int(A))) \subseteq A$.*

Proof. Let A be a strongly $\mathcal{G}$-β-closed set of $(X, \tau, \mathcal{G})$. Then $X - A$ is strongly $\mathcal{G}$-β-open and hence $X - A \subseteq \Psi(Int(\Psi(X - A))) = X - \tau_{\mathcal{G}}$-$Int(Cl(\tau_{\mathcal{G}}$-$Int(A)))$. Therefore, we have $\tau_{\mathcal{G}}$-$Int(Cl(\tau_{\mathcal{G}}$-$Int(A))) \subseteq A$.

Conversely, let $\tau_{\mathcal{G}}$-$Int(Cl(\tau_{\mathcal{G}}$-$Int(A))) \subseteq A$. Then, $X - A \subseteq \Psi(Int(\Psi(X - A)))$ and hence $X - A$ is strongly $\mathcal{G}$-β-open. Therefore, A is strongly $\mathcal{G}$-β-closed. ∎

Proposition 3.22. *If a subset A of a grill topological space $(X, \tau, \mathcal{G})$ is strongly $\mathcal{G}$-β-closed, then $Int(\Psi(Int(A))) \subseteq A$.*

Proof. Let A be any strongly $\mathcal{G}$-β-closed set of a grill topological space $(X, \tau, \mathcal{G})$. Since $\tau_{\mathcal{G}}$ is finer than τ, we have $Int(\Psi(Int(A))) \subseteq \tau_{\mathcal{G}} - Int(\Psi(\tau_{\mathcal{G}} - Int(A))) \subseteq \tau_{\mathcal{G}} - (Cl(\tau_{\mathcal{G}} - Int(A)))$. Therefore, by Theorem 3.21, we obtain $Int(\Psi(Int(A))) \subseteq A$. ∎

Lemma 3.23. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $\{A_\alpha : \alpha \in \Delta\}$ a family of strongly $\mathcal{G}$-β-open subsets of X. Then $\cup\{A_\alpha : \alpha \in \Delta\}$ is strongly $\mathcal{G}$-β-open.*

Proof. Since $\{A_\alpha : \alpha \in \Delta\} \subseteq S\mathcal{G}\beta O(X)$, then $A_\alpha \subseteq \Psi(Int(\Psi(A_\alpha)))$ for each $\alpha \in \Delta$. Then we have

$$\begin{aligned}\cup_{\alpha\in\Delta} A_\alpha &\subseteq \cup_{\alpha\in\Delta}\ \Psi(Int(\Psi(A_\alpha)))\\ &\subseteq \Psi(\cup_{\alpha\in\Delta}(Int(\Psi(A_\alpha))))\\ &\subseteq \Psi(Int(\cup_{\alpha\in\Delta}(\Psi(A_\alpha))))\\ &\subseteq \Psi(Int(\Psi(\cup_{\alpha\in\Delta} A_\alpha))).\end{aligned}$$

This shows that $\cup_{\alpha\in\Delta} A_\alpha \in S\mathcal{G}\beta O(X)$. ∎

Theorem 3.24. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If A is strongly $\mathcal{G}$-β-open and B is $\mathcal{G}$-α-open, then $A \cap B$ is strongly $\mathcal{G}$-β-open.*

Proof. Since A is strongly $\mathcal{G}$-β-open, we have $A \subseteq \Psi(Int(\Psi(A)))$. Since B is $\mathcal{G}$-α-open, $B \subseteq Int(\Psi(Int(B)))$ and hence by using Lemma 3.20, we have have

$$\begin{aligned}B \cap A &\subseteq [Int(\Psi(Int(B))) \cap \Psi(Int(\Psi(A)))]\\ &\subseteq \Psi[Int(\Psi(Int(B))) \cap Int(\Psi(A))]\\ &\subseteq \Psi(Int[\Psi(Int(B)) \cap Int(\Psi(A))])\\ &\subseteq \Psi(Int(\Psi[Int(B) \cap Int(\Psi(A))]))\\ &\subseteq \Psi(Int(\Psi[Int(B) \cap \Psi(A)]))\\ &\subseteq \Psi(Int(\Psi(\Psi[Int(B) \cap A])))\\ &\subseteq \Psi(Int(\Psi[B \cap A])).\end{aligned}$$

Therefore, $B \cap A$ is strongly $\mathcal{G}$-β-open. ∎

By Proposition 3.15 and Theorem 3.24 we have.

Corollary 3.25. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If A is strongly $\mathcal{G}$-β-open and B is open, then $A \cap B$ is strongly $\mathcal{G}$-β-open.*

Lemma 3.26. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. Then the following properties hold:*

(1) *If $\{A_\alpha : \alpha \in \Delta\}$ is a family of almost strongly $\mathcal{G}$-open subsets of X, then $\cup\{A_\alpha : \alpha \in \Delta\}$ is almost strongly $\mathcal{G}$-open.*

(2) *If A is almost strongly $\mathcal{G}$-open and B is $\mathcal{G}$-α-open, then $A \cap B$ is almost strongly $\mathcal{G}$-open.*

Proof. (1) For each $\alpha \in \Delta$, we have $A_\alpha \subseteq \Psi(Int(\Phi(A_\alpha))) \subseteq \Psi(Int(\Phi(\cup A_\alpha)))$ and hence $\cup A_\alpha \subseteq \Psi(Int(\Phi(\cup A_\alpha)))$. This shows that $\cup A_\alpha$ is almost strongly $\mathcal{G}$-open.
(2) Let A be almost strongly $\mathcal{G}$-open and B be $\mathcal{G}$-α-open. Then, by Lemma 3.19 and Proposition 2.2 we have

$$\begin{aligned}
B \cap A \subseteq & [Int(\Psi(Int(B))) \cap \Psi(Int(\Phi(A)))] \\
\subseteq & \Psi[Int(\Psi(Int(B))) \cap Int(\Phi(A))] \\
= & \Psi(Int[\Psi(Int(B)) \cap Int(\Phi(A))]) \\
\subseteq & \Psi(Int(\Psi[Int(B) \cap \Phi(A)])) \\
\subseteq & \Psi(Int(\Psi[\Phi(Int(B) \cap A)])) \\
\subseteq & \Psi(Int(\Psi[\Phi(B \cap A)])) \\
= & \Psi(Int[\Phi(\Phi(B \cap A)) \cup \Phi(B \cap A)]) \\
= & \Psi(Int[\Phi(B \cap A)]).
\end{aligned}$$

This shows that $B \cap A$ is almost strongly $\mathcal{G}$-open. ∎

Proposition 3.27. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $A \subseteq W \subseteq \Psi(A)$, then W is strongly $\mathcal{G}$-β-open if and only if A is strongly $\mathcal{G}$-β-open.*

Proof. Suppose $A \subseteq W \subseteq \Psi(A)$ and W is strongly $\mathcal{G}$-β-open. Then $\Psi(A) = \Psi(W)$. Now $A \subseteq W \subseteq \Psi(Int(\Psi(W))) = \Psi(Int(\Psi(A)))$ and so A is strongly $\mathcal{G}$-β-open. Conversely, suppose $A \subseteq W \subseteq \Psi(A)$ and A is strongly $\mathcal{G}$-β-open. Now $W \subseteq \Psi(A) \subseteq \Psi(\Psi(Int(\Psi(A)))) = \Psi(Int(\Psi(A))) = \Psi(Int(\Psi(W)))$ and hence W is strongly $\mathcal{G}$-β-open. ∎

Corollary 3.28. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. If $A \subseteq X$ is strongly $\mathcal{G}$-β-open, then $\Psi(A)$ and $\Psi(Int(\Psi(A)))$ are strongly $\mathcal{G}$-β-open.*

Proposition 3.29. *If A is a $\mathcal{G}$-δ-open set, then there exists two disjoint subsets B and C with $B \in \mathcal{G}\alpha O(X)$ and $Int(\Psi(C)) = \emptyset$ such that $A = B \cup C$.*

Proof. Suppose that $A \in \mathcal{G}\delta O(X)$. Then we have $Int(\Psi(A)) \subseteq \Psi(Int(A))$ and $Int(\Psi(A)) \subseteq Int(\Psi(Int(A)))$. Now we have $A = [Int(\Psi(A)) \cap A] \cup [A \setminus Int(\Psi(A))]$. Now we set $B = Int(\Psi(A)) \cap A$ and $C = A \setminus Int(\Psi(A))$. We first show that $B \in \mathcal{G}\alpha O(X)$, that is, $B \subseteq Int(\Psi(Int(B)))$. Now we have

$$\begin{aligned} Int(\Psi(Int(B))) &= Int(\Psi(Int[Int(\Psi(A)) \cap A])) \\ &= Int(\Psi[Int(\Psi(A)) \cap Int(A)]) \\ &= Int(\Psi[Int(A)]). \end{aligned}$$

Since $A \in \mathcal{G}\delta O(X)$, $B \subseteq Int(\Psi(A)) \subseteq Int(\Psi(Int(A))) = Int(\Psi(Int(B)))$ and thus $B \in \mathcal{G}\alpha O(X)$. Next we show that $Int(\Psi(C)) = \emptyset$. Since $\tau \subseteq \tau_{\mathcal{G}}$, $\Psi(S) \subseteq Cl(S)$ for any subset S of X. Therefore, we have

$$\begin{aligned} Int(\Psi(C)) &= Int(\Psi[A \cap (X \setminus Int(\Psi(A)))]) \\ &\subseteq Int(\Psi(A)) \cap Int(\Psi(X \setminus Int(\Psi(A)))) \\ &\subseteq Int(\Psi(A)) \cap Int(Cl(X \setminus Int(\Psi(A)))) \\ &\subseteq Int(\Psi(A)) \cap (X \setminus Int(\Psi(A))) = \emptyset. \end{aligned}$$

It is obvious that $B \cap C = [Int(\Psi(A)) \cap A] \cap [A \setminus Int(\Psi(A))] = \emptyset$. ∎

Proposition 3.30. *Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subseteq X$. Then the following statements are equivalent:*

(1) *A is almost strongly $\mathcal{G}$-open;*
(2) *A is strongly $\mathcal{G}$-β-open and $\mathcal{G}$-dense-in-itself.*

Proof. (1)$\Rightarrow$ (2): Every almost strongly $\mathcal{G}$-open set is strongly $\mathcal{G}$-β-open from Proposition 3.4. On the other hand, $A \subseteq \Psi(Int(\Phi(A))) = Int(\Phi(A)) \cup \Phi(Int(\Phi(A))) \subseteq Int(\Phi(A)) \cup \Phi(\Phi(A)) \subseteq \Phi(A)$. This shows that A is $\mathcal{G}$-dense-in-itself.
(2)$\Rightarrow$ (1): By the assumption, $A \subseteq \Psi(Int(\Psi(A))) = \Psi(Int(\Phi(A) \cup A)) \subseteq \Psi(Int(\Phi(A)))$. This shows that A is almost strongly $\mathcal{G}$-open. ∎

Definition 3.31. A subset A of a grill topological space $(X, \tau, \mathcal{G})$ is called a $D_{\mathcal{G}}$-set if $A = U \cap V$, where $U \in \tau$ and $Int(V) = \Psi(Int(\Psi(V)))$.

Theorem 3.32. *Let $(X, \tau, \mathcal{G})$ be a grill topological space. For a subset A of X, the following statements are equivalent:*

(1) *A is open;*
(2) *A is strongly $\mathcal{G}$-β-open and a $D_{\mathcal{G}}$-set.*

Proof. (1)$\Rightarrow$ (2): The proof is obvious by Proposition 3.15 and Definition 3.31.
(2)$\Rightarrow$ (1): Let A be strongly $\mathcal{G}$-β-open and a $D_{\mathcal{G}}$-set. Then $A = U \cap V$, where $U \in \tau$ and $Int(V) = \Psi(Int(\Psi(V)))$. Therefore, we have

$$\begin{aligned} A &\subseteq \Psi(Int(\Psi(A))) = \Psi(Int(\Psi(U \cap V))) \\ &\subseteq \Psi(Int(\Psi(U) \cap \Psi(V))) \\ &= \Psi[Int(\Psi(U)) \cap Int(\Psi(V))] \\ &\subseteq \Psi[Int(\Psi(U))] \cap \Psi[Int(\Psi(V))] \\ &\subseteq \Psi[Int(\Psi(U))] \cap Int(V). \end{aligned}$$

Hence, $A = U \cap A \subseteq U \cap \Psi(Int(\Psi(U))) \cap Int(V) = U \cap Int(V) = Int(U \cap V) = Int(A)$. Thus A is open. ∎

By the examples stated below, we obtain that strongly $\mathcal{G}$-β-open sets and $D_{\mathcal{G}}$-sets are independent of each other.

Example 3.33. *In Example 3.12, $A = \{a, b\}$ is a strongly $\mathcal{G}$-β-open set but it is not a $D_{\mathcal{G}}$-set.*

Example 3.34. *In Example 3.14, $A = \{a, c, d\}$ is a $D_{\mathcal{G}}$-set but it is not strongly $\mathcal{G}$-β-open.*

4. Decompositions of continuity

Definition 4.1. A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be $\mathcal{G}$-precontinuous [7] (resp. $\mathcal{G}$-semi-continuous [1], $\mathcal{G}$-α-continuous [1], strongly $\mathcal{G}$-β-continuous, $\mathcal{G}$-δ-continuous, $D_{\mathcal{G}}$-continuous) if for every $V \in \sigma$, $f^{-1}(V)$ is $\mathcal{G}$-preopen (resp. $\mathcal{G}$-semi-open, $\mathcal{G}$-α-open, strongly $\mathcal{G}$-β-open, $\mathcal{G}$-δ-open, a $D_{\mathcal{G}}$-set) in $(X, \tau, \mathcal{G})$.

Theorem 4.2. *For a function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is strongly $\mathcal{G}$-β-continuous;*

(2) *The inverse image of each closed set in* (Y,σ) *is strongly* $\mathcal{G}$-β-*closed;*
(3) *For each* $x \in X$ *and* $V \in \sigma$ *containing* $f(x)$*, there exists* $U \in S\mathcal{G}\beta O(X)$ *containing* x *such that* $f(U) \subseteq V$.

Proof. The proof is obvious form Lemma 3.23 and is thus omitted. ∎

Theorem 4.3. *A function* $f : (X,\tau,\mathcal{G}) \to (Y,\sigma)$ *is strongly* $\mathcal{G}$-β-*continuous if and only if its graph function* g *is strongly* $\mathcal{G}$-β-*continuous.*

Proof. Necessity. Let f be strongly $\mathcal{G}$-β-continuous, $x \in X$ and H an open set in $X \times Y$ containing $g(x)$. Then, there exist $U \in \tau$ and $V \in \sigma$ such that $g(x) = (x, f(x)) \in U \times V \subseteq H$. By Theorem 4.2, there exists $W \in S\mathcal{G}\beta O(X)$ containing x such that $f(W) \subseteq V$. We have $x \in (U \cap W) \in S\mathcal{G}\beta O(X)$ by using Corollary 3.25. Hence $((U \cap W) \times V) \subseteq U \times V \subseteq H$ and therefore $g(U \cap W) \subseteq H$. It follows from Theorem 4.2 that $g : (X,\tau,\mathcal{G}) \to (X \times Y, \tau \times \sigma)$ is strongly $\mathcal{G}$-β-continuous.
Sufficiency. Let $x \in X$ and $V \in \sigma$ containing $f(x)$, then $g(x) \in X \times V$. Since g is strongly $\mathcal{G}$-β-continuous, there exists $W \in S\mathcal{G}\beta O(X)$ containing x such that $g(W) \subseteq X \times V$. Therefore, we obtain $f(W) \subseteq V$. Hence by Theorem 4.2 f is strongly $\mathcal{G}$-β-continuous. ∎

Theorem 4.4. *Let* $(X,\tau,\mathcal{G})$ *be a grill topological space. For a function* $f : (X,\tau,\mathcal{G}) \to (Y,\sigma)$*, the following conditions are equivalent:*
(1) f *is* $\mathcal{G}$-*semi-continuous.*
(2) f *is strongly* $\mathcal{G}$-β-*continuous and* $\mathcal{G}$-δ-*continuous.*

Proof. This is an immediate consequence of Proposition 3.6. ∎

Theorem 4.5. *Let* $(X,\tau,\mathcal{G})$ *be a grill topological space. For a function* $f : (X,\tau,\mathcal{G}) \to (Y,\sigma)$*, the following conditions are equivalent:*
(1) f *is* $\mathcal{G}$-α-*continuous;*
(2) f *is* $\mathcal{G}$-*precontinuous and* $\mathcal{G}$-*semi-continuous;*
(3) f *is* $\mathcal{G}$-*precontinuous and* $\mathcal{G}$-δ-*continuous.*

Proof. The proof is obvious by Propositions 3.6 and 3.7. ∎

Theorem 4.6. *Let* $(X,\tau,\mathcal{G})$ *be a grill topological space. For a function* $f : (X,\tau,\mathcal{G}) \to (Y,\sigma)$*, the following conditions are equivalent:*
(1) f *is continuous;*

(2) *f is strongly $\mathcal{G}$-β-continuous and $D_{\mathcal{G}}$-continuous.*

Proof. The proof is obvious by Theorem 3.32. ∎

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