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COMMON FIXED POINT THEOREMS FOR TWO PAIRS OF WEAKLY COMPATIBLE MAPPINGS IN MENGER SPACES AND FUZZY METRIC SPACES

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Abstract. In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings satisfying a contraction type condition in Menger spaces. As application to our result, we obtain the corresponding common fixed point theorem in fuzzy metric spaces.

1. Introduction

There have been a number of generalizations of metric spaces. One such generalization is Menger space initiated by Menger [14]. The idea thus appears that, instead of a single positive number, we should associate a distribution function with the point pairs. Thus the concept of a probabilistic metric space corresponds to the situations when we do not know the distance between the points, i.e. the distance between the points is inexact. Rather than a single real number, we know only probabilities of possible values of this distance. The study of this space was expanded rapidly with the pioneering works of Schweizer and Sklar [25] and some of their coworkers. Such a probabilistic generalization of a metric space appears to be well adapted for the investigation of physical quantities and physiological threshold. It is also of fundamental importance in probabilistic functional analysis, nonlinear analysis and applications [2, 11, 24].

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In 1972, Sehgal and Bharucha-Reid [26] initiated the study of contraction mappings on probabilistic metric spaces. Several interesting and elegant results have been obtained by various authors in this direction. For example, Jungck [8] obtained a common fixed point theorem for a pair of commuting mappings. Sessa [27] formulated the notion of weak commutativity and obtained common fixed point theorems concerning them. Jungck [9] introduced the concept of compatible maps. This condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [10]. The notion of R-weakly commuting mappings was introduced by Pant [19]. For detailed description of these concepts, we refer to Singh and Tomar [30]. The concept of weakly compatible mappings is most general as every commuting pair is R-weakly commuting, each pair of R-weakly commuting mappings is compatible and each pair of compatible mappings is weakly compatible but the reverse is not true. In 2005, Singh and Jain [29] extended the notion of weakly compatible mappings to Menger spaces and proved common fixed point theorems (see [20, 21]).

In the present paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings in Menger space. We also present the corresponding common fixed point theorems in fuzzy metric spaces.

2. Preliminaries

Definition 2.1 [25] A triangular norm $*$ (shortly t-norm) is a binary operation on the unit interval $[0, 1]$ such that for all $a, b, c, d \in [0, 1]$ the following conditions are satisfied:

(1) $a * 1 = a$,
(2) $a * b = b * a$,
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$,
(4) $a * (b * c) = (a * b) * c$.

Examples of t-norm are $a * b = \min\{a, b\}$,$a * b = ab$ and $a * b = \max\{a + b - 1, 0\}$.

Definition 2.2 [25] A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$. We shall denote the set of all distribution functions on $[-\infty, \infty]$ by $\Im$ while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y}$.

Definition 2.3 [25] The ordered pair $(X, \mathcal{F})$ is called a probabilistic metric space (shortly PM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$

(1) $F_{x,y}(t) = H(t)$ if and only if $x = y$,
(2) $F_{x,y}(t) = F_{y,x}(t)$,
(3) $F_{x,z}(t) = 1, F_{z,y}(s) = 1 \Rightarrow F_{x,y}(t + s) = 1$.

The ordered triple $(X, \mathcal{F}, *)$ is called a Menger space if $(X, \mathcal{F})$ is a PM-space, $*$ is a t-norm and the following inequality holds:

$$F_{x,y}(t + s) \geq F_{x,z}(t) * F_{z,y}(s),$$

for all $x, y, z \in X$ and $t, s > 0$. Every metric space (X, d) can be realized as a PM-space by taking $\mathcal{F} : X \times X \to \Im$ defined by $F_{x,y}(t) = H\left(t - d(x, y)\right)$ for all $x, y \in X$.

Definition 2.4 [25] Let $(X, \mathcal{F}, *)$ be a Menger space and $*$ be a continuous t-norm.

(1) A sequence $\{x_n\}$ in X is said to be converge to a point x in X if and only if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $\mathbb{N}$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ for all $n \geq \mathbb{N}$.
(2) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists an integer $\mathbb{N}$ such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ for all $n, m \geq \mathbb{N}$.
(3) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.5 [17] Two self mappings A and B of a Menger space $(X, \mathcal{F}, *)$ are said to be compatible if $F_{ABx_n,BAx_n}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to x$ for some x in X as $n \to \infty$.

Definition 2.6 [29] Two self mappings A and B of a Menger space $(X, \mathcal{F}, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$, then $ABx = BAx$.

Remark 2.1 [29] If self mappings A and B of a Menger space $(X, \mathcal{F}, *)$ are compatible then they are weakly compatible.

The following is an example of pair of self mappings in a Menger space which are weakly compatible but not compatible.

Example 2.1 Let (X, d) be a metric space where $X = [0, 1]$ and $(X, \mathcal{F}, *)$ be the induced Menger space with $F_{x,y}(t) = \frac{t}{t+d(x,y)}$, for all $t > 0$. Define self mappings A and B by

$$A(x) = \begin{cases} 1 - x, & \text{if } 0 \leq x < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \quad B(x) = \begin{cases} x, & \text{if } 0 \leq x < \frac{1}{2}; \\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Taking sequence $\{x_n\} = \left\{\frac{1}{2} - \frac{1}{n}\right\}_{n\in\mathbb{N}}$, we get $Ax_n = \frac{1}{2} + \frac{1}{n}$, $Bx_n = \frac{1}{2} - \frac{1}{n}$. Thus $Ax_n \to \frac{1}{2}$, $Bx_n \to \frac{1}{2}$. Hence $x = \frac{1}{2}$. Further, $ABx_n = \frac{1}{2} + \frac{1}{n}$, $BAx_n = 1$. Now $\lim_{n\to\infty} F_{ABx_n,BAx_n}(t) = \lim_{n\to\infty} F_{\frac{1}{2}+\frac{1}{n},1}(t) = \frac{t}{t+\frac{1}{2}} = \frac{2t}{2t+1} < 1$, for all $t > 0$. Hence the pair (A, B) is not compatible.

Coincidence points of A and B are in $[\frac{1}{2}, 1]$. Now for any $x \in [\frac{1}{2}, 1]$, $Ax = Bx = 1$ and $AB(x) = A(1) = 1 = B(1) = BA(x)$. Thus the pair (A, B) is weakly compatible.

Lemma 2.1 [18, 28] Let $(X, \mathcal{F}, *)$ be a Menger probabilistic metric space and define $E_{\lambda,F} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by

$$E_{\lambda,F}(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\}$$

for each $\lambda \in (0, 1)$ and $x, y \in X$. Then we have

(1) For any $\mu \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$E_{\mu,F}(x_1, x_n) \leq E_{\lambda,F}(x_1, x_2) + \ldots + E_{\lambda,F}(x_{n-1}, x_n),$$

for any $x_1, \ldots, x_n \in X$.

(2) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent with respect to Menger probabilistic metric $\mathcal{F}$ if and only if $E_{\lambda,F}(x_n, x) \to 0$. Also the sequence $\{x_n\}$ is a Cauchy sequence with respect to Menger probabilistic metric $\mathcal{F}$ if and only if it is a Cauchy sequence with $E_{\lambda,F}$.

Lemma 2.2 [16] Let $(X, \mathcal{F}, *)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t)$$

for all $t > 0$ with fixed $x, y \in X$ then $x = y$.

3. Main result

Theorem 3.1 Let A, L, M and S be self mappings of a complete Menger space $(X, \mathcal{F}, *)$ with continuous t-norm $* = \min$ and satisfy the following conditions:

(1) $L(X) \subseteq S(X)$, $M(X) \subseteq A(X)$,
(2) one of $S(X)$ and $A(X)$ is a closed subset of X,

(3) $$[1+aF_{Ax,Sy}(kt)]*F_{Lx,My}(kt) \geq a\left\{\begin{array}{c} F_{Lx,Ax}(kt)*F_{My,Sy}(kt)* \\ F_{Lx,Sy}(2kt)*F_{My,Ax}(2kt) \end{array}\right\} + \left\{\begin{array}{c} F_{Ax,Sy}(t)*F_{Lx,Ax}(t)*F_{My,Sy}(t)* \\ F_{Lx,Sy}(2t)*F_{My,Ax}(2t) \end{array}\right\}$$

for all $t>0$, $x,y\in X$, $a\geq 0$ and $k\in(0,1)$,

(4) the pairs (L,A) and (M,S) are weakly compatible.

Then A,L,M and S have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $L(X)\subseteq S(X)$ one can find a point x_1 in X with $Lx_0=Sx_1=y_0$. Again, as $M(X)\subseteq A(X)$ one can also choose a point $x_2\in X$ with $Mx_1=Ax_2=y_2$. Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Lx_{2n}=Sx_{2n+1}=y_{2n}$ and $Mx_{2n+1}=Ax_{2n+2}=y_{2n+1}$ for $n=0,1,2,\ldots$.

Putting $x=x_{2n}$ and $y=x_{2n+1}$ in (3), we get

$$[1+aF_{Ax_{2n},Sx_{2n+1}}(kt)]*F_{Lx_{2n},Mx_{2n+1}}(kt) \geq a\left\{\begin{array}{c} F_{Lx_{2n},Ax_{2n}}(kt)*F_{Mx_{2n+1},Sx_{2n+1}}(kt)* \\ F_{Lx_{2n},Sx_{2n+1}}(2kt)*F_{Mx_{2n+1},Ax_{2n}}(2kt) \end{array}\right\} + \left\{\begin{array}{c} F_{Ax_{2n},Sx_{2n+1}}(t)*F_{Lx_{2n},Ax_{2n}}(t)* \\ F_{Mx_{2n+1},Sx_{2n+1}}(t)*F_{Lx_{2n},Sx_{2n+1}}(2t)* \\ F_{Mx_{2n+1},Ax_{2n}}(2t) \end{array}\right\},$$

$$[1+aF_{y_{2n-1},y_{2n}}(kt)]*F_{y_{2n},y_{2n+1}}(kt) \geq a\left\{\begin{array}{c} F_{y_{2n},y_{2n-1}}(kt)*F_{y_{2n+1},y_{2n}}(kt)* \\ F_{y_{2n},y_{2n}}(2kt)*F_{y_{2n-1},y_{2n+1}}(2kt) \end{array}\right\} + \left\{\begin{array}{c} F_{y_{2n-1},y_{2n}}(t)*F_{y_{2n},y_{2n-1}}(t)* \\ F_{y_{2n+1},y_{2n}}(t)*F_{y_{2n},y_{2n}}(2t)* \\ F_{y_{2n-1},y_{2n+1}}(2t) \end{array}\right\}.$$

From Definition 2.3, we have

$$F_{y_{2n-1},y_{2n}}(kt)*F_{y_{2n},y_{2n-1}}(kt)\leq F_{y_{2n-1},y_{2n+1}}(2kt)$$

and

$$F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n-1}}(t) \leq F_{y_{2n-1},y_{2n+1}}(2t),$$

hence

$$[1 + aF_{y_{2n-1},y_{2n}}(kt)] * F_{y_{2n},y_{2n+1}}(kt)$$
$$\geq a \left\{ \begin{array}{c} F_{y_{2n},y_{2n-1}}(kt) * F_{y_{2n+1},y_{2n}}(kt) * \\ 1 * F_{y_{2n-1},y_{2n}}(kt) * F_{y_{2n},y_{2n+1}}(kt) \end{array} \right\} + \left\{ \begin{array}{c} F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t) * \\ 1 * F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n+1}}(t) \\ \end{array} \right\},$$

$F_{y_{2n},y_{2n+1}}(kt) + a[F_{y_{2n-1},y_{2n}}(kt) * F_{y_{2n},y_{2n+1}}(kt)]$
$\geq a\left\{F_{y_{2n},y_{2n-1}}(kt) * F_{y_{2n+1},y_{2n}}(kt)\right\} + F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t),$

$$F_{y_{2n},y_{2n+1}}(kt) \geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n+1}}(t).$$

Hence

$$F_{y_{2n},y_{2n+1}}(kt) \geq \min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t)\}.$$

Similarly,

$$F_{y_{2n+1},y_{2n+2}}(kt) \geq \min\{F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n+2}}(t)\}.$$

Therefore, for all n we have

$$F_{y_n,y_{n+1}}(kt) \geq \min\{F_{y_{n-1},y_n(t),}F_{y_n,y_{n+1}}(t)\}.$$

Consequently,

$$F_{y_n,y_{n+1}}(t) \geq \min\{F_{y_{n-1},y_n}(k^{-1}t), F_{y_n,y_{n+1}}(k^{-1}t)\}.$$

By repeated application of above inequality, we get for each $m \in \{1, 2, 3, \ldots\}$.

$$\begin{aligned} F_{y_n,y_{n+1}}(t) &\geq \min\left\{F_{y_{n-1},y_n}(k^{-1}t), F_{y_{n-1},y_n}(k^{-2}t), F_{y_n,y_{n+1}}(k^{-2}t)\right\} \\ &= \min\left\{F_{y_{n-1},y_n}(k^{-1}t), F_{y_n,y_{n+1}}(k^{-2}t)\right\} \\ &\geq \ldots \geq \min\left\{F_{y_{n-1},y_n}(k^{-1}t), F_{y_n,y_{n+1}}(k^{-m}t)\right\}, \end{aligned}$$

and so for each $\lambda \in (0, 1)$ we have

$$
\begin{aligned}
E_{\lambda,F}(y_n, y_{n+1}) &= \inf\left\{t > 0 : F_{y_n,y_{n+1}}(t) > 1-\lambda\right\} \\
&\leq \inf\left\{t > 0 : \min\{F_{y_{n-1},y_n}(k^{-1}t), F_{y_n,y_{n+1}}(k^{-m}t)\} > 1-\lambda\right\} \\
&\leq \max\left\{\begin{array}{l} \inf\{t > 0 : F_{y_{n-1},y_n}(k^{-1}t) > 1-\lambda\}, \\ \inf\{t > 0 : F_{y_n,y_{n+1}}(k^{-m}t) > 1-\lambda\} \end{array}\right\} \\
&\leq \max\left\{kE_{\lambda,F}(y_{n-1}, y_n), k^m E_{\lambda,F}(y_n, y_{n+1})\right\}.
\end{aligned}
$$

Since, $k^m E_{\lambda,F}(y_n, y_{n+1}) \to 0$ as $m \to \infty$, it follows that

$E_{\lambda,F}(y_n, y_{n+1}) \leq kE_{\lambda,F}(y_{n-1}, y_n) \leq k^n E_{\lambda,F}(y_0, y_1)$, for every $\lambda \in (0,1)$.

Now, we show that $\{y_n\}$ is a Cauchy sequence. For every $\mu \in (0,1)$, there exists $\gamma \in (0,1)$ such that, for $m \geq n$,

$$
\begin{aligned}
E_{\mu,F}(y_n, y_m) &\leq E_{\gamma,F}(y_{m-1}, y_m) + E_{\gamma,F}(y_{m-2}, y_{m-1}) + \ldots \\
&\quad \ldots + E_{\gamma,F}(y_n, y_{n+1}) \\
&\leq E_{\gamma,F}(y_0, y_1) \sum_{i=n}^{m-1} k^i \to 0,
\end{aligned}
$$

as $m, n \to \infty$. Hence by Lemma 2.1, $\{y_n\}$ is a Cauchy sequence in X, which is complete. Therefore $\{y_n\}$ converges to $z \in X$. That is

$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Lx_{2n} = \lim_{n\to\infty} Mx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Ax_{2n} = z$.

Suppose that $S(X)$ is a closed subset of X then for some $v \in X$ we have $z = Sv \in S(X)$. Putting $x = x_{2n}$ and $y = v$ in (3), we have

$$
\begin{aligned}
&[1 + aF_{Ax_{2n},Sv}(kt)] * F_{Lx_{2n},Mv}(kt) \geq a\left\{\begin{array}{l} F_{Lx_{2n},Ax_{2n}}(kt) * F_{Mv,Sv}(kt) * \\ F_{Lx_{2n},Sv}(2kt) * F_{Mv,Ax_{2n}}(2kt) \end{array}\right\} \\
&\qquad + \left\{\begin{array}{c} F_{Ax_{2n},Sv}(t) * F_{Lx_{2n},Ax_{2n}}(t) * F_{Mv,Sv}(t) * \\ F_{Lx_{2n},Sv}(2t) * F_{Mv,Ax_{2n}}(2t) \end{array}\right\},
\end{aligned}
$$

passing to limit as $n \to \infty$, we get

$$
[1 + aF_{z,z}(kt)] * F_{z,Mv}(kt) \geq a\left\{\begin{array}{l} F_{z,z}(kt) * F_{Mv,z}(kt) * \\ F_{z,z}(2kt) * F_{Mv,z}(2kt) \end{array}\right\}
$$

$$+\left\{\begin{array}{c} F_{z,z}(t) * F_{z,z}(t) * F_{Mv,z}(t) \\ *F_{z,z}(2t) * F_{Mv,z}(2t) \end{array}\right\},$$

since $F_{Mv,z}(kt) \leq F_{Mv,z}(2kt)$ and $F_{Mv,z}(t) \leq F_{Mv,z}(2t)$, hence

$$(1+a) * F_{z,Mv}(kt) \geq aF_{Mv,z}(kt) + F_{Mv,z}(t).$$

Since $a \geq 0$, it follows that $(1+a) * F_{z,Mv}(kt) \leq F_{z,Mv}(kt) + aF_{z,Mv}(kt)$. The above inequality implies

$$F_{z,Mv}(kt) + aF_{z,Mv}(kt) \geq aF_{Mv,z}(kt) + F_{Mv,z}(t),$$

$$F_{z,Mv}(kt) \geq F_{z,Mv}(t).$$

Thus, by Lemma 2.2, $z = Mv$. Therefore, $M(v) = S(v) = z$. From weak compatibility of the pair (M, S), we have $Mz = MSv = SMv = Sz$. Now we put $x = x_{2n}$ and $y = z$ in (3) and letting $n \to \infty$, we obtain

$$[1 + aF_{z,Sz}(kt)] * F_{z,Sz}(kt) \geq aF_{Sz,z}(2kt) + F_{Sz,z}(t) * F_{Sz,z}(2t),$$

hence

$$[1 + aF_{z,Sz}(kt)] * F_{z,Sz}(kt) \geq aF_{Sz,z}(kt) + F_{Sz,z}(t).$$

Since $a \geq 0$, we get $[1 + aF_{z,Sz}(kt)] * F_{z,Sz}(kt) \leq F_{z,Sz}(kt) + aF_{z,Sz}(kt)$. This implies

$$F_{z,Sz}(kt) + aF_{z,Sz}(kt) \geq aF_{Sz,z}(kt) + F_{Sz,z}(t),$$

$$F_{z,Sz}(kt) \geq F_{z,Sz}(t).$$

Thus, by Lemma 2.2, $z = Sz$. Therefore, $z = Mz = Sz$. Since $M(X) \subseteq A(X)$, there exists $w \in X$ such that $Aw = Mz = Sz = z$. Putting $x = w$ and $y = z$ in (3), we have

$$[1+aF_{z,z}(kt)]*F_{Lw,z}(kt) \geq a\{F_{Lw,z}(kt)*F_{Lw,z}(2kt)\}+F_{Lw,z}(t)*F_{Lw,z}(2t),$$

i.e.

$$(1+a) * F_{Lw,z}(kt) \geq aF_{Lw,z}(kt) + F_{Lw,z}(t).$$

Since $a \geq 0$, we have $(1+a) * F_{Lw,z}(kt) \leq F_{Lw,z}(kt) + aF_{Lw,z}(kt)$. This implies

$$F_{Lw,z}(kt) + aF_{Lw,z}(kt) \geq aF_{Lw,z}(kt) + F_{Lw,z}(t),$$

$$F_{Lw,z}(kt) \geq F_{Lw,z}(t).$$

Thus, by Lemma 2.2, $z = Lw$. Therefore, $Lw = Aw = z = Mz = Sz$. Since $Lw = Aw$ and the pair (L, A) is weakly compatible, we obtain $Lz = LAw = ALw = Az$. Putting $x = z$ and $y = x_{2n+1}$ in (3) and taking limit as $n \to \infty$, we get

$$[1 + aF_{Az,z}(kt)] * F_{Az,z}(kt) \geq aF_{Az,z}(2kt) + F_{Az,z}(t) * F_{Az,z}(2t),$$

i.e.

$$[1 + aF_{Az,z}(kt)] * F_{Az,z}(kt) \geq aF_{Az,z}(kt) + F_{Az,z}(t).$$

Since $a \geq 0$, it follows that $[1 + aF_{Az,z}(kt)] * F_{Az,z}(kt) \leq F_{Az,z}(kt) + aF_{Az,z}(kt)$. This implies

$$F_{Az,z}(kt) + aF_{Az,z}(kt) \geq aF_{Az,z}(kt) + F_{Az,z}(t),$$

$$F_{Az,z}(kt) \geq F_{Az,z}(t).$$

Thus, by Lemma 2.2, $z = Az$. Therefore, $z = Lz = Az$. Now we combine all the results, we have $z = Lz = Az = Mz = Sz$, i.e. z is the common fixed point of self mappings A, L, M and S.

Uniqueness: Let u be a common fixed point of self mappings A, L, M and S. We show that $u = z$. Putting $x = z$ and $y = u$ in (3), we obtain

$$[1 + aF_{z,u}(kt)] * F_{z,u}(kt) \geq aF_{z,u}(2kt) + F_{z,u}(t) * F_{z,u}(2t),$$

i.e.

$$[1 + aF_{z,u}(kt)] * F_{z,u}(kt) \geq aF_{z,u}(kt) + F_{z,u}(t).$$

Since $a \geq 0$, we get $[1 + aF_{z,u}(kt)] * F_{z,u}(kt) \leq F_{z,u}(kt) + aF_{z,u}(kt)$. This implies

$$F_{z,u}(kt) + aF_{z,u}(kt) \geq aF_{z,u}(kt) + F_{z,u}(t),$$

hence

$$F_{z,u}(kt) \geq F_{z,u}(t).$$

Thus, by Lemma 2.2, $z = u$ and so the uniqueness of the common fixed point.

The proof is similar when $A(X)$ is assumed to be a closed subset of X.

From Theorem 3.1 with $a = 0$, we obtain the following interesting result:

Corollary 3.1 Let A, L, M and S be self mappings of a complete Menger space $(X, \mathcal{F}, *)$ with continuous t-norm $* = \min$ satisfying the conditions (1), (2) and (4) of Theorems 3.1 such that

$$F_{Lx,My}(kt) \geq \left\{ \begin{array}{c} F_{Ax,Sy}(t) * F_{Lx,Ax}(t) * F_{My,Sy}(t) * \\ F_{Lx,Sy}(2t) * F_{My,Ax}(2t) \end{array} \right\}$$

holds for all $t > 0$, $x, y \in X$ and $k \in (0, 1)$. Then A, L, M and S have a unique common fixed point in X.

The following example illustrates Theorem 3.1 and Corollary 3.1.

Example 3.2 Let $X = [0, 15]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0. \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, *)$ is a complete Menger space, where $*$ is defined as $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define A, L, M and $S : X \to X$ by

$$A(x) = \begin{cases} 0, & \text{if } x = 0; \\ 10 - x, & \text{if } 0 < x \leq 10; \\ x - 7, & \text{if } 10 < x \leq 15. \end{cases}$$

$$S(x) = \begin{cases} 0, & \text{if } x = 0; \\ 10 - x, & \text{if } 0 < x \leq 10; \\ x - 3, & \text{if } 10 < x \leq 15. \end{cases}$$

$$L(x) = \begin{cases} 0, & \text{if } x = 0; \\ 3, & \text{if } 0 < x \leq 15. \end{cases}$$

$$M(x) = \begin{cases} 0, & \text{if } x = 0; \\ 7, & \text{if } 0 < x \leq 15. \end{cases}$$

Then A, L, M and S satisfy all the conditions of Theorem 3.1 and Corollary 3.1 for some fixed $k \in (0, 1)$ and have a unique common fixed point $0 \in X$. The mappings L and A commute at coincidence point $0 \in X$. So L and A are weakly compatible mappings. Similarly, M and S are weakly compatible mappings. To see the pairs (L, A) and

(M, S) are not compatible, let us consider a sequence $\{x_n\}$ defined as $x_n = 10 + \frac{1}{n}, n \geq 1$, then $x_n \to 10$ as $n \to \infty$. Then $\lim_{n\to\infty} Lx_n = 3, \lim_{n\to\infty} Ax_n = 3$ but $\lim_{n\to\infty} F_{LAx_n,ALx_n}(t) = \frac{t}{t+|3-7|} \neq 1$. Thus the pair (L, A) is not compatible.

Also $\lim_{n\to\infty} Mx_n = 7, \lim_{n\to\infty} Sx_n = 7$ but $\lim_{n\to\infty} F_{MSx_n,SMx_n}(t) = \frac{t}{t+|7-3|} \neq 1$, it implies the pair (M, S) is not compatible. It may be noted that all the mappings involved in this example are discontinuous even at the common fixed point $x = 0$.

On taking $A = S$ and $L = M$ in Theorem 3.1 and Corollary 3.1, we get the following results:

Corollary 3.2 Let A and L be self mappings of a complete Menger space $(X, \mathcal{F}, *)$ with continuous t-norm $* = \min$ and satisfy the following conditions:

(1) $L(X) \subseteq A(X)$,
(2) $A(X)$ is a closed subset of X,
(3) $[1 + aF_{Ax,Ay}(kt)] * F_{Lx,Ly}(kt)$

$$\geq a \left\{ \begin{array}{c} F_{Lx,Ax}(kt) * F_{Ly,Ay}(kt)* \\ F_{Lx,Ay}(2kt) * F_{Ly,Ax}(2kt) \end{array} \right\} + \left\{ \begin{array}{c} F_{Ax,Ay}(t) * F_{Lx,Ax}(t) * F_{Ly,Ay}(t)* \\ F_{Lx,Ay}(2t) * F_{Ly,Ax}(2t) \\ \end{array} \right\}$$

for all $t > 0$, $x, y \in X$, $a \geq 0$ and $k \in (0, 1)$,

(4) the pair (L, A) is weakly compatible.

Then A and L have a unique common fixed point in X.

Corollary 3.3 Let A and L be self mappings of a complete Menger space $(X, \mathcal{F}, *)$ with continuous t-norm $* = \min$ satisfying the conditions (1), (2) and (4) of Corollary 3.2 such that

$$F_{Lx,Ly}(kt) \geq \left\{ \begin{array}{c} F_{Ax,Ay}(t) * F_{Lx,Ax}(t) * F_{Ly,Ay}(t)* \\ F_{Lx,Ay}(2t) * F_{Ly,Ax}(2t) \\ \end{array} \right\}$$

holds for all $t > 0$, $x, y \in X$ and $k \in (0, 1)$. Then A and L have a unique common fixed point in X.

4. Application to fuzzy metric spaces

Fixed point theory in fuzzy metric spaces for different contractive-type mappings is closely related to that in probabilistic metric spaces (refer [1, Chapters VIII, IX], [6, Chapters 3-5], [15], [26]). Various mathematicians; for example, Hadžić and Pap [7], Razani and Shirdaryazdi [23], Razani and Kouladgar [22] and Liu and Li [13] have studied the applications of fixed point theorems in PM-spaces to fuzzy metric spaces.

First we recall some definitions, lemma and remark in fuzzy metric spaces from [3, 4, 5, 12, 17].

Definition 4.1 The 3-tuple $(X, \mathrm{M}, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$

(1) $\mathrm{M}(x, y, 0) = 0$,
(2) $\mathrm{M}(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(3) $\mathrm{M}(x, y, t) = \mathrm{M}(y, x, t)$,
(4) $\mathrm{M}(x, z, t + s) \geq \mathrm{M}(x, y, t) * \mathrm{M}(y, z, s)$,
(5) $\mathrm{M}(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

In the following example (see [5]), we know that every metric induces a fuzzy metric:

Example 4.1 Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) for all $x, y \in X$ and $t > 0$,

$$\mathrm{M}(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, \mathrm{M}, *)$ is a fuzzy metric space and the fuzzy metric M induced by the metric d is often referred to as the standard fuzzy metric.

Lemma 4.1 Let $(X, \mathrm{M}, *)$ be a fuzzy metric space. Then $\mathrm{M}(x, y, t)$ is non-decreasing with respect to t for all $x, y \in X$.

Definition 4.2 Let $(X, \mathrm{M}, *)$ be a fuzzy metric space. Then

(1) a sequence $\{x_n\}$ in X is said to be converge to a point x in X if and only if $\lim_{n\to\infty} \mathrm{M}(x_n, x, t) = 1$ for all $t > 0$.
(2) a sequence $\{x_n\}$ in X is said to be Cauchy if and only if for each $\varepsilon \in (0, 1)$ and $t > 0$, there exists an integer $\mathbb{N}$ such that $\mathrm{M}(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq \mathbb{N}$.
(3) a fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 4.3 Let A and B be mappings from fuzzy metric space $(X, \mathrm{M}, *)$ into itself. The mappings A and B are said to be compatible if $\lim_{n\to\infty} \mathrm{M}(ABx_n, BAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some x in X.

Definition 4.4 Two self mappings A and B of a fuzzy metric space $(X, \mathrm{M}, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Remark 4.1 If self mappings A and B of a fuzzy metric space $(X, \mathrm{M}, *)$ are compatible then they are weakly compatible.

As as application, we present the fuzzy version of Theorem 3.1.

Theorem 4.1 Let A, L, M and S be self mappings of a complete fuzzy metric space $(X, \mathrm{M}, *)$ with continuous t-norm $* = \min$ and satisfy the following conditions:

(1) $L(X) \subseteq S(X), M(X) \subseteq A(X)$,

(2) one of $S(X)$ and $A(X)$ is a closed subset of X,

(3) $$[1 + a\mathrm{M}(Ax, Sy, kt)] * \mathrm{M}(Lx, My, kt) \geq a \left\{ \begin{array}{c} \mathrm{M}(Lx, Ax, kt) * \mathrm{M}(My, Sy, kt) \\ *\mathrm{M}(Lx, Sy, 2kt) * \mathrm{M}(My, Ax, 2kt) \end{array} \right\} + \left\{ \begin{array}{c} \mathrm{M}(Ax, Sy, t) * \mathrm{M}(Lx, Ax, t) * \mathrm{M}(My, Sy, t) \\ *\mathrm{M}(Lx, Sy, 2t) * \mathrm{M}(My, Ax, 2t) \\ \end{array} \right\}$$

for all $t > 0$, $x, y \in X$, $a \geq 0$ and $k \in (0, 1)$,

(4) the pairs (L, A) and (M, S) are weakly compatible.

Then A, L, M and S have a unique common fixed point in X.

Proof. For every fuzzy metric M we define $F_{x,y}(t) = \mathrm{M}(x, y, t)$, where $(x, y, t) \in X \times X \times [0, \infty)$.

By the axioms of the fuzzy metric space in the sense of George and Veeramani, $(X, \mathcal{F}, *)$ is a Menger space.

From Theorem 4.1 with $a = 0$, we have the following interesting result:

Corollary 4.1 Let A, L, M and S be self mappings of a complete fuzzy metric space $(X, \mathrm{M}, *)$ with continuous t-norm $* = \min$ satisfying the conditions (1), (2) and (4) of Theorem 4.1 such that

$$\mathrm{M}(Lx, My, kt) \geq \left\{ \begin{array}{c} \mathrm{M}(Ax, Sy, t) * \mathrm{M}(Lx, Ax, t) * \mathrm{M}(My, Sy, t) \\ *\mathrm{M}(Lx, Sy, 2t) * \mathrm{M}(My, Ax, 2t) \\ \end{array} \right\}$$

holds for all $t > 0$, $x, y \in X$ and $k \in (0, 1)$. Then A, L, M and S have a unique common fixed point in X.

On taking $A = S$ and $L = M$ in Theorem 4.1 and Corollary 4.1, we obtain the following results:

Corollary 4.2 Let A and L be self mappings of a complete fuzzy metric space $(X, \mathrm{M}, *)$ with continuous t-norm $* = \min$ and satisfy the following conditions:

(1) $L(X) \subseteq A(X)$,

(2) $A(X)$ is a closed subset of X,

(3) $$[1 + a\mathrm{M}(Ax, Ay, kt)] * \mathrm{M}(Lx, Ly, kt) \geq a \left\{ \begin{array}{c} \mathrm{M}(Lx, Ax, kt) * \mathrm{M}(Ly, Ay, kt) * \\ \mathrm{M}(Lx, Ay, 2kt) * \mathrm{M}(Ly, Ax, 2kt) \end{array} \right\} + \left\{ \begin{array}{c} \mathrm{M}(Ax, Ay, t) * \mathrm{M}(Lx, Ax, t) * \mathrm{M}(Ly, Ay, t) \\ * \mathrm{M}(Lx, Ay, 2t) * \mathrm{M}(Ly, Ax, 2t) \end{array} \right\},$$

holds for all $t > 0$, $x, y \in X$, $a \geq 0$ and $k \in (0, 1)$,

(4) the pair (L, A) is weakly compatible.

Then A and L have a unique common fixed point in X.

Corollary 4.3 Let A and L be self mappings of a complete fuzzy metric space $(X, \mathrm{M}, *)$ with continuous t-norm $* = \min$ satisfying the conditions (1), (2) and (4) of Corollary 4.2 such that

$$\mathrm{M}(Lx, Ly, kt) \geq \left\{ \begin{array}{c} \mathrm{M}(Ax, Ay, t) * \mathrm{M}(Lx, Ax, t) * \mathrm{M}(Ly, Ay, t) \\ * \mathrm{M}(Lx, Ay, 2t) * \mathrm{M}(Ly, Ax, 2t) \end{array} \right\}$$

holds for all $t > 0$, $x, y \in X$ and $k \in (0, 1)$. Then A and L have a unique common fixed point in X.

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