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WELL-POSEDNESS AND PERIODIC POINT PROPERTY OF MAPPINGS SATISFYING A RATIONAL INEQUALITY IN AN ORDERED COMPLEX VALUED METRIC SPACE

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Abstract. Azam, Fisher and Khan [A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric spaces, *Numerical Functional Analysis and Optimization,* 32(3)(2011), 243-253] introduced a notion of complex valued metric space and obtained common fixed point result for mappings in the context of complex valued metric spaces. In this paper, employing the concept of weakly increasing mappings, the existence of common fixed points is obtained in an ordered complex valued metric space. We apply our results to study well-posedness of a common fixed point problem for two rational type contractive mappings and a periodic point property of mapping involved therein.

1. Introduction and preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a broad set of applications. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer on the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful tools in the study of nonlinear equations. There are many

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generalizations of the Banach's contraction mapping principle in the literature. These generalization were made either by using the contractive condition or by imposing some additional conditions on an ambient space. There have been a number of generalizations of metric spaces such as, rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces (see [1, 8, 11, 16, 21]). Recently, A. Azam, B. Fisher and M. Khan [6] obtained the generalization of Banach's contraction principal introducing the concept of complex valued metric space. Common fixed point problem for two maps under several variants of non-commutativity has been studied by many authors.

The existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [18], and further studied by Nieto and Lopez [14].

Several authors have studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions in the framework of partially ordered metric spaces (e.g. [2, 3, 4, 5, 7, 10, 17, 20]). The purpose of this paper is to study common fixed points of two mappings satisfying a rational inequality, without exploiting any type of commutativity condition in the framework of a complex valued metric space. The results presented in this paper substantially extend and strengthen the results given in [6].

Consistent with Azam, Fisher and Khan [6], the following definitions and results will be needed in the sequel.

Let $\mathbb{C}$ be the set of complex numbers and let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if Re $(z_1) \leq \text{Re}(z_2)$, $\text{Im}(z_1) \leq \text{Im}(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(1) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$,
(2) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$,
(3) $\text{Re}(z_1) < \text{Re}(z_2)$, $\text{Im}(z_1) < \text{Im}(z_2)$,
(4) $\text{Re}(z_1) = \text{Re}(z_2)$, $\text{Im}(z_1) = \text{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if one of (1), (2) and (3) is satisfied and we will write $z_1 \prec z_2$ if only (3) is satisfied.

Some elementary properties of the partial order $\precsim$ on $\mathbb{C}$ are the following:

(i) If $0 \precsim z_1 \precnsim z_2$, then $|z_1| < |z_2|$.
(ii) $z_1 \precsim z_2$ is equivalent to $z_1 - z_2 \precsim 0$.
(iii) If $z_1 \precsim z_2$ and $r \geq 0$ is a real number, then $rz_1 \precsim rz_2$.

(iv) If $0 \le z_1$ and $0 \le z_2$ with $z_1 + z_2 \ne 0$, then $\dfrac{z_1^2}{z_1+z_2} \le z_1$.

Note that $\dfrac{z_1^2}{z_1+z_2} \le z_1$ is equivalent to $0 \le \dfrac{z_1 z_2}{z_1+z_2}$. Indeed,

$$\frac{z_1^2}{z_1+z_2} \le z_1 \Leftrightarrow 0 \le z_1 - \frac{z_1^2}{z_1+z_2} \Leftrightarrow 0 \le \frac{z_1^2 + z_1 z_2 - z_1^2}{z_1+z_2} \Leftrightarrow 0 \le \frac{z_1 z_2}{z_1+z_2}.$$

(v) $0 \le z_1$ and $0 \le z_2$ do not imply $0 \le z_1 z_2$.

(vi) $0 \le z_1$ does not imply $0 \le \dfrac{1}{z_1}$. Moreover, if $0 < z_1$ and $0 \le \dfrac{1}{z_1}$, then $\mathrm{Im}(z_1) = 0$.

Definition 1.1 Let X be a non-empty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

(a) $0 \le d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(b) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(c) $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a *complex valued metric* on X and (X,d) is called a *complex valued metric space.*

A point $x \in X$ is called an *interior point* of a set $A \subseteq X$ whenever there exists $0 < r \in \mathbb{C}$ such that $B(x,r) = \{y \in X : d(x,y) < r\} \subseteq A$. A subset A in X is called *open* whenever each point of A is an interior point of A. The family $F = \{B(x,r) : x \in X,\ 0 < r\}$ is a sub-basis for a Hausdorff topology τ on X.

A point $x \in X$ is called a *limit point* of A whenever for every $0 < r \in \mathbb{C}$, $B(x,r) \cap (A \backslash x) \ne \phi$. A subset $B \subseteq X$ is called *closed* whenever each limit point of B belongs to B.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 < c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) < c$, then x is called the *limit* of $\{x_n\}$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in \mathbb{C}$, with $0 < c$, there is an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) < c$, then $\{x_n\}$ is called a *Cauchy sequence* in (X,d). If every Cauchy sequence is convergent in (X,d), then (X,d) is called a *complete complex valued metric space.*

Lemma 1.2. [6] Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.

Lemma 1.3. Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $\sup\limits_{m\ge 1} |d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.

Definition 1.4. Let X be a nonempty set. A binary relation "$\preceq$" on X is called a *partial order* on X if for x, y and $z \in X$, the following axioms are satisfied:

(1) $x \preceq x$ (reflexivity)
(2) $x \preceq y$ and $y \preceq x$ imply $x = y$ (anti-symmetry)
(3) $x \preceq y$ and $y \preceq z$ imply $x \preceq z$ (transitivity).

The pair $(X, \preceq)$ is called a *partially ordered set.*

The following definition is due to Altun([3]).

Definition 1.5. [3] Let $(X, \preceq)$ be a partially ordered set. A pair (f, g) of self-maps of X is said to be *weakly increasing* if $fx \preceq gfx$ and $gx \preceq fgx$ for all $x \in X$. If $f = g$, then we have $fx \preceq f^2x$ for all x in X and in this case, we say that f is a *weakly increasing map.*

Example 1.6. Let $X = [0, \infty)$ be endowed with usual ordering. Let $f : X \to X$ be defined by

$$fx = \begin{cases} x^{1/3}, \text{ if } 0 \leq x \leq 1, \\ x, \text{ if } 1 < x \leq 2, \\ 0, \text{ if } 2 < x < \infty. \end{cases}$$

Note that if $x \in [0, 1]$, then $fx = x^{\frac{1}{3}} \leq x^{\frac{1}{9}} = f^2x$. Also when $x \in (1, 2]$, then $fx = x \leq f^2x$ and if $x \in (2, \infty)$, then $fx = 0 = f^2x$. Thus $fx \leq f^2x$ for all x in X an so f is a weakly increasing map. Note that f not increasing since $2 < 3$ and $f(2) = 2 \nleq 0 = f(3)$.

A point x in X said to be a *fixed point* of a self-map f on X if $fx = x$. A fixed point problem is to find some x in X such that $fx = x$ and we denote it by $FP(f, X)$. A point $x \in X$ is called a *common fixed point of the pair* (f, g) if $x = fx = gx$, where f and g are two self-maps on X. A common fixed point problem is to find some x in X such that $x = fx = gx$, and we denote it by $CFP(f, g, X)$. A nonempty subset W of a partially ordered set X is said to be *totally ordered* if every two elements of W are comparable.

2. Main results

We begin with a common fixed point theorem for weakly increasing maps on an ordered complex valued metric space.

Theorem 2.1. Let $(X, \preceq)$ be a partially ordered set such that there exists a complete complex valued metric d on X and let S and T be weakly increasing self-maps on X. Also, for every comparable $x, y \in X$, we have either

$$d(Sx,Ty) \leq \frac{a[\{d(x,Ty)\}^2 + \{d(y,Sx)\}^2]}{d(x,Ty)+d(y,Sx)} + \tag{2.1}$$
$$+b[d(x,Sx)+d(y,Ty)] + c[d(x,Ty)+d(y,Sx)] + ed(x,y),$$

if $d(x,Ty)+d(y,Sx) \neq 0$, $a,b,c,e \geq 0$ and $2a+2b+2c+e<1$, or

$$d(Sx,Ty) = 0 \text{ if } d(x,Ty)+d(y,Sx) = 0. \tag{2.2}$$

If S or T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \preceq z$ for all $n \in \mathbb{N}$, then S and T have a common fixed point. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have one and only one common fixed point.

Proof. First we shall show that if S or T has a fixed point, then it is a common fixed point of S and T. Let u be a fixed point of S. Then from (2.1) with $x=y=u$, we get

$$\begin{aligned} d(u,Tu) &= d(Su,Tu) \\ &\leq \frac{a[\{d(u,Tu)\}^2 + \{d(u,Su)\}^2]}{d(u,Tu)+d(u,Su)} + \\ &+ b[d(u,Su)+d(u,Tu)] + c[d(u,Tu)+d(u,Su)] + ed(u,u) \\ &= \frac{a[\{d(u,Tu)\}^2 + \{d(u,u)\}^2]}{d(u,Tu)+d(u,u)} + \\ &+ b[d(u,u)+d(u,Tu)] + c[d(u,Tu)+d(u,u)] \\ &= (a+b+c)d(u,Tu). \end{aligned}$$

Hence

$$|d(u,Tu)| \leq (a+b+c)\,|d(u,Tu)|\,,$$

where $a+b+c<1$ implies $d(u,Tu)=0$ and so u is a common fixed point of S and T.

Similarly, if u is a fixed point of T, then it is also fixed point of S.

Now let x_0 be an arbitrary point of X. If $Sx_0 = x_0$, then the proof is finished. Assume that $Sx_0 \neq x_0$. Define a sequence $\{x_n\}$ in X as follows:

$$\begin{aligned} x_1 &= Sx_0 \preceq TSx_0 = Tx_1 = x_2, \\ x_2 &= Tx_1 \preceq STx_1 = Sx_2 = x_3. \end{aligned}$$

Continuing this process, we have

$$x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots .$$

Assume that $d(x_{2n}, x_{2n+1}) > 0$, for every $n \in \mathbb{N}$. If not, then $x_{2n} = x_{2n+1}$ for some n. For all those n, $x_{2n} = x_{2n+1} = Sx_{2n}$ and the proof is finished. Now, since x_{2n} and x_{2n+1} are comparable, then taking $d(x_{2n}, x_{2n+1}) > 0$ for $n = 0, 1, 2, 3, ...$, consider

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}) = d(Sx_{2n}, Tx_{2n+1}) + \\
&\leq \frac{a[\{d(x_{2n}, Tx_{2n+1})\}^2 + \{d(x_{2n+1}, Sx_{2n})\}^2]}{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})} \\
&+b[d(x_{2n}, Sx_{2n}) + d(x_{2n+1}, Tx_{2n+1})] + \\
&+c[d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Sx_{2n})] + ed(x_{2n}, x_{2n+1}) \\
&= \frac{a[\{d(x_{2n}, x_{2n+2})\}^2 + \{d(x_{2n+1}, x_{2n+1})\}^2]}{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})} + \\
&+b[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] \\
&+c[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})] + ed(x_{2n}, x_{2n+1}) \\
&= ad(x_{2n}, x_{2n+2}) + b[d(x_{2n}, x_{2n+1}) + \\
&+d(x_{2n+1}, x_{2n+2})] + cd(x_{2n}, x_{2n+2}) + ed(x_{2n}, x_{2n+1}) \\
&\leq (a + b + c + e)d(x_{2n}, x_{2n+1}) + \\
&+(a + b + c)d(x_{2n+1}, x_{2n+2}),
\end{aligned}$$

which implies that

$$d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1})$$

for all $n \geq 0$, where

$$0 \leq h = \frac{(a + b + c + e)}{1 - (a + b + c)} < 1.$$

Similarly, we have $d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-1}, x_{2n})$ for all $n \geq 0$. Hence for all $n \geq 0$

$$d(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1})$$

and consequently

$$d(x_{n+1}, x_{n+2}) \leq hd(x_n, x_{n+1}) \leq \ldots \leq h^{n+1}d(x_0, x_1)$$

for all $n \geq 0$. Now for $m > n$, we have

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m+1}, x_m) \\
&\leq h^n d(x_0, x_1) + h^{n+1}d(x_0, x_1) + \ldots + h^{m-1}d(x_0, x_1) \\
&\leq \frac{h^n}{1 - h} d(x_0, x_1).
\end{aligned}$$

Therefore,

$$|d(x_n, x_m)| \leq \frac{h^n}{1 - h} |d(x_0, x_1)|,$$

and so $\sup\limits_{m\geq 1} |d(x_n, x_m)| \to 0$ as $n \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, the sequence $\{x_n\}$ converges to a point u in X.

If S or T is continuous, then it is clear that $Su = u = Tu$.

If neither S, nor T is continuous, $x_n \preceq u$ for all $n \in \mathbb{N}$. We claim that u is a fixed point of S. If not, then $d(u, Su) = z > 0$. From (2.1), we obtain

$$\begin{aligned}
z &\leq d(u, x_{n+2}) + d(x_{n+2}, Su) \\
&= d(u, x_{n+2}) + d(Su, Tx_{n+1}) \\
&\leq d(u, x_{n+2}) + \frac{a[\{d(u, Tx_{n+1})\}^2 + \{d(x_{n+1}, Su)\}^2]}{d(u, Tx_{n+1}) + d(x_{n+1}, Su)} \\
&\quad + b[d(u, Su) + d(x_{n+1}, Tx_{n+1})] \\
&\quad + c[d(u, Tx_{n+1}) + d(x_{n+1}, Su)] + ed(u, x_{n+1}) \\
&= d(u, x_{n+2}) + \frac{a[\{d(u, x_{n+2})\}^2 + \{d(x_{n+1}, Su)\}^2]}{d(u, x_{n+2}) + d(x_{n+1}, Su)} \\
&\quad + b[d(u, Su) + d(x_{n+1}, x_{n+2})] \\
&\quad + c[d(u, x_{n+2}) + d(x_{n+1}, Su)] + ed(u, x_{n+1}),
\end{aligned}$$

and so

$$\begin{aligned}
&|z| \leq |d(u, x_{n+2})| + \tfrac{a[|d(u,x_{n+2})|^2 + |d(x_{n+1},Su)|^2]}{|d(u,x_{n+2}) + d(x_{n+1},Su)|} + \\
&+ b[|d(u, Su)| + |d(x_{n+1}, x_{n+2})|] + \\
&+ c[|d(u, x_{n+2})| + |d(x_{n+1}, Su)|] + e\,|d(u, x_{n+1})|,
\end{aligned}$$

which on taking limit as $n \to \infty$ gives

$$|z| \leq (a + b + c)\,|z|,$$

a contradiction, and so $u = Su$. Therefore $Su = Tu = u$.

Now suppose that set of common fixed points of S and T is totally ordered. We prove that common fixed point of S and T is unique. Assume on contrary that u and v are distinct common fixed points of S and T. By supposition, we can replace x by u and y by v in (2.1) to obtain

$$
\begin{aligned}
d(u,v) &= d(Su,Tv) \\
&\leq \frac{a[\{d(u,Tv)\}^2+\{d(v,Su)\}^2]}{d(u,Tv)+d(v,Su)} + \\
&+ b[d(u,Su)+d(v,Tv)] + \\
&+ c[d(u,Tv)+d(v,Su)] + ed(u,v) \\
&= \frac{a[\{d(u,v)\}^2+\{d(v,u)\}^2]}{d(u,v)+d(v,u)} + b[d(u,u)+d(v,v)] \\
&+ c[d(u,v)+d(v,u)] + ed(u,v) \\
&= (a+2c+e)d(u,v),
\end{aligned}
$$

which implies that

$$|d(u,v)| \leq (a+2c+e)\,|d(u,v)|\,,$$

a contradiction. Hence $u = v$.

Conversely, if S and T have only one common fixed point then the set of common fixed point of S and T being singleton is totally ordered. □

Example 2.2. Let $X = [0,1]$ be endowed with order $x \preceq y$ if and only if $y \leq x$. Then $\preceq$ is a partial order in X. Let $d(x,y) = |x-y|\, e^{i\theta}$ where $\theta \in [0, \frac{\pi}{2}]$. We define $S, T : X \to X$ by

$$Sx = \frac{x}{12} \text{ for } x \in X \textit{ and } Tx = \begin{cases} \dfrac{x}{6}, \text{ for } x \in [0, \dfrac{1}{2}), \\ \dfrac{x}{4}, \text{ for } x \in [\dfrac{1}{2}, 1]. \end{cases}$$

Note that T is discontinuous map and that $ST(\frac{1}{2}) = \frac{1}{96} \neq \frac{1}{144} = TS(\frac{1}{2})$, which shows that S and T do not commute. Now

(I) for $x, y \in [0, \frac{1}{2})$, we have $Sx = \dfrac{x}{12}$, $Ty = \dfrac{y}{6}$. If $y \preceq \dfrac{x}{2}$, then

$$\begin{aligned}
&d(Sx,Ty) = \frac{1}{6}\left(y - \frac{x}{2}\right)e^{i\theta} \leq \frac{1}{6}ye^{i\theta} \leq \frac{7}{20}\left(\frac{11x}{12} + \frac{5y}{6}\right)e^{i\theta} \\
&\leq \frac{1}{20}\frac{[\left|x - \frac{y}{6}\right|^2 + \left|y - \frac{x}{12}\right|^2]e^{i\theta}}{\left|x - \frac{y}{6}\right| + \left|y - \frac{x}{12}\right|} + \frac{7}{20}\left(\frac{11x}{12} + \frac{5y}{6}\right)e^{i\theta} + \\
&+ \frac{1}{20}\left[\left|x - \frac{y}{6}\right| + \left|y - \frac{x}{12}\right|\right]e^{i\theta} + \frac{1}{20}\left|x - y\right|e^{i\theta} + \\
&= \frac{a[\{d(x,Ty)\}^2 + \{d(y,Sx)\}^2]}{d(x,Ty) + d(y,Sx)} + \\
&+ b[d(x,Sx) + d(y,Ty)] + \\
&+ c[d(x,Ty) + d(y,Sx)] + ed(x,y),
\end{aligned}$$

and if $y \succ \dfrac{x}{2}$, then

$$\begin{aligned}
d(Sx,Ty) &= \tfrac{1}{6}\left(\tfrac{x}{2} - y\right)e^{i\theta} \leq \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{5y}{6}\right)e^{i\theta} \\
&\leq \tfrac{1}{20}\frac{\left[\left|x - \frac{y}{6}\right|^2 + \left|y - \frac{x}{12}\right|^2\right]e^{i\theta}}{\left|x - \frac{y}{6}\right| + \left|y - \frac{x}{12}\right|} + \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{5y}{6}\right)e^{i\theta} \\
&+ \tfrac{1}{20}\left[\left|x - \tfrac{y}{6}\right| + \left|y - \tfrac{x}{12}\right|\right]e^{i\theta} + \tfrac{1}{20}(y - x)e^{i\theta} \\
&= \tfrac{a[\{d(x,Ty)\}^2 + \{d(y,Sx)\}^2]}{d(x,Ty) + d(y,Sx)} + \\
&+ b[d(x,Sx) + d(y,Ty)] + \\
&+ c[d(x,Ty) + d(y,Sx)] + ed(x,y).
\end{aligned}$$

(II) For $x \in [0, \frac{1}{2})$ and $y \in [\frac{1}{2}, 1)$, we have $Sx = \dfrac{x}{12}$, $Ty = \dfrac{y}{4}$ and

$$\begin{aligned}
&d(Sx,Ty) = \left(\tfrac{y}{4} - \tfrac{x}{12}\right)e^{i\theta} = \tfrac{1}{4}\left(y - \tfrac{x}{3}\right)e^{i\theta} \\
&\leq \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{3y}{4}\right)e^{i\theta} \\
&\leq \tfrac{1}{20}\tfrac{[\left|x - \frac{y}{4}\right|^2 + \left|y - \frac{x}{12}\right|^2]e^{i\theta}}{\left|x - \frac{y}{4}\right| + \left|y - \frac{x}{12}\right|} + \\
&+ \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{3y}{4}\right)e^{i\theta} \\
&+ \tfrac{1}{20}\left[\left|x - \tfrac{y}{4}\right| + \left|y - \tfrac{x}{12}\right|\right]e^{i\theta} + \tfrac{1}{20}(y - x)e^{i\theta} \\
&= \tfrac{a[\{d(x,Ty)\}^2 + \{d(y,Sx)\}^2]}{d(x,Ty) + d(y,Sx)} + \\
&+ b[d(x,Sx) + d(y,Ty)] + \\
&+ c[d(x,Ty) + d(y,Sx)] + ed(x,y).
\end{aligned}$$

(III) When $y \in [0, \frac{1}{2})$ and $x \in [\frac{1}{2}, 1)$, we have $Sx = \dfrac{x}{12}$, $Ty = \dfrac{y}{6}$. If $y \preceq \dfrac{x}{2}$, then

$$\begin{aligned}
&d(Sx, Ty) = \tfrac{1}{6}\left(y - \tfrac{x}{2}\right)e^{i\theta} \leq \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{5y}{6}\right)e^{i\theta} \\
&\leq \tfrac{1}{20}\tfrac{[\left|x - \frac{y}{6}\right|^2 + \left|y - \frac{x}{12}\right|^2]e^{i\theta}}{\left|x - \frac{y}{6}\right| + \left|y - \frac{x}{12}\right|} + \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{5y}{6}\right)e^{i\theta} \\
&+\tfrac{1}{20}\left[\left|x - \tfrac{y}{6}\right| + \left|y - \tfrac{x}{12}\right|\right]e^{i\theta} + \tfrac{1}{20}\left|x - y\right|e^{i\theta} \\
&= \tfrac{a[\{d(x,Ty)\}^2 + \{d(y,Sx)\}^2]}{d(x,Ty) + d(y,Sx)} + \\
&+b[d(x, Sx) + d(y, Ty)] + \\
&+c[d(x, Ty) + d(y, Sx)] + ed(x, y),
\end{aligned}$$

and if $y \succ \dfrac{x}{2}$, then

$$\begin{aligned}
&d(Sx, Ty) = \tfrac{1}{6}\left(\tfrac{x}{2} - y\right)e^{i\theta} \leq \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{5y}{6}\right)e^{i\theta} \\
&\leq \tfrac{1}{20}\tfrac{[\left|x - \frac{y}{6}\right|^2 + \left|y - \frac{x}{12}\right|^2]e^{i\theta}}{\left|x - \frac{y}{6}\right| + \left|y - \frac{x}{12}\right|} + \tfrac{7}{20}e^{i\theta} + \\
&+\tfrac{1}{20}\left[\left|x - \tfrac{y}{6}\right| + \left|y - \tfrac{x}{12}\right|\right]e^{i\theta} + \\
&+\tfrac{1}{20}(y - x)e^{i\theta} \\
&= \tfrac{a[\{d(x,Ty)\}^2 + \{d(y,Sx)\}^2]}{d(x,Ty) + d(y,Sx)} + \\
&+b[d(x, Sx) + d(y, Ty)] + \\
&+c[d(x, Ty) + d(y, Sx)] + ed(x, y).
\end{aligned}$$

(IV) For $x, y \in [\frac{1}{2}, 1)$, we have $Sx = \dfrac{x}{12}$, $Ty = \dfrac{y}{4}$. If $y \preceq \dfrac{x}{3}$, then

$$\begin{aligned}
&d(Sx, Ty) = \tfrac{1}{4}(y - \tfrac{x}{3})e^{i\theta} \leq \tfrac{1}{4}ye^{i\theta} \leq \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{3y}{4}\right)e^{i\theta} \\
&\leq \tfrac{1}{20}\tfrac{[\left|x - \frac{y}{4}\right|^2 + \left|y - \frac{x}{12}\right|^2]e^{i\theta}}{\left|x - \frac{y}{4}\right| + \left|y - \frac{x}{12}\right|} + \\
&+\tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{3y}{4}\right)e^{i\theta} + \\
&+\tfrac{1}{20}\left[\left|x - \tfrac{y}{4}\right| + \left|y - \tfrac{x}{12}\right|\right]e^{i\theta} + \tfrac{1}{20}\left|x - y\right|e^{i\theta} \\
&= \tfrac{a[\{d(x,Ty)\}^2 + \{d(y,Sx)\}^2]}{d(x,Ty) + d(y,Sx)} + \\
&+b[d(x, Sx) + d(y, Ty)] + \\
&+c[d(x, Ty) + d(y, Sx)] + ed(x, y)
\end{aligned}$$

and if $y \succ \frac{x}{3}$, then

$$\begin{aligned}
&d(Sx,Ty) = \tfrac{1}{4}\left(\tfrac{x}{3} - y\right)e^{i\theta} \leq \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{3y}{4}\right)e^{i\theta} \\
&\leq \tfrac{1}{20}\tfrac{[\left|x-\frac{y}{4}\right|^2+\left|y-\frac{x}{12}\right|^2]e^{i\theta}}{\left|x-\frac{y}{4}\right|+\left|y-\frac{x}{12}\right|} + \tfrac{7}{20}\left(\tfrac{11x}{12} + \tfrac{3y}{4}\right)e^{i\theta} + \\
&+\tfrac{1}{20}\left[\left|x - \tfrac{y}{4}\right| + \left|y - \tfrac{x}{12}\right|\right]e^{i\theta} + \tfrac{1}{20}(x-y)e^{i\theta} \\
&= \tfrac{a[\{d(x,Ty)\}^2+\{d(y,Sx)\}^2]}{d(x,Ty)+d(y,Sx)} + \\
&+b[d(x,Sx) + d(y,Ty)] + \\
&+c[d(x,Ty) + d(y,Sx)] + ed(x,y).
\end{aligned}$$

Thus the conditions of Theorem 2.1 are satisfied with $a = c = e = \frac{1}{20}$ and $b = \frac{7}{20}$ where $2a+2b+2c+e = \frac{19}{20} < 1$. Moreover, 0 is the unique common fixed point of S and T.

In Theorem 2.1, take $S = T$, to obtain the following corollary.

Corollary 2.3. Let $(X, \preceq)$ be a partially ordered set such that there exists a complete complex valued metric d on X and let T be a weakly increasing self-map on X. Also, for every comparable $x, y \in X$, suppose

$$d(Tx,Ty) \leq \frac{a[\{d(x,Ty)\}^2 + \{d(y,Tx)\}^2]}{d(x,Ty) + d(y,Tx)} + \\ +b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)] + ed(x,y) \tag{2.3}$$

if $d(x,Ty) + d(y,Tx) \neq 0$, $a, b, c, e \geq 0$ and $2a + 2b + 2c + e < 1$, or

$$d(Tx,Ty) = 0 \text{ if } d(x,Ty) + d(y,Tx) = 0. \tag{2.4}$$

If T is continuous or for a nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \preceq z$ for all $n \in \mathbb{N}$, then T has a fixed point. Moreover, the set of fixed points of T is totally ordered if and only if T has one and only one fixed point.

Theorem 2.4. Let $(X, \preceq)$ be a partially ordered set such that there exists a complete complex valued metric d on X and let S and T be weakly increasing self-maps on X. Also, for every comparable $x, y \in X$, suppose

$$d(Sx,Ty) \leq \frac{a[d(x,Sx)d(x,Ty) + d(y,Ty)d(y,Sx)]}{d(x,Ty) + d(y,Sx)} + \\ +\frac{bd(x,Ty)d(y,Sx)}{d(x,Sx) + d(y,Ty)} \tag{2.5}$$

if $d(x, Ty) + d(y, Sx) \neq 0$ and $d(x, Sx) + d(y, Ty) \neq 0$, where $a, b \geq 0$ with $0 \leq a + b < 1$, or

(2.6)

$$d(Sx, Ty) = 0 \text{ if } d(x, Ty) + d(y, Sx) = 0 \text{ or } d(x, Sx) + d(y, Ty) = 0.$$

If S or T is continuous or for a nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \preceq z$ for all $n \in \mathbb{N}$. Then S and T have a common fixed point. Moreover, the set of common fixed points of S and T is totally ordered if and only if S and T have one and only one common fixed point.

Proof. First we shall show that if S or T has a fixed point, then it is a common fixed point of S and T. Indeed, let u be a fixed point of S. Then from (2.5) with $x = y = u$, we have

$$\begin{aligned}
d(u, Tu) &= d(Su, Tu) \\
&\leq \frac{a[d(u, Su)d(u, Tu) + d(u, Tu)d(u, Su)]}{d(u, Tu) + d(u, Su)} + \\
&\quad + \frac{bd(u, Tu)d(u, Su)}{d(u, Su) + d(u, Tu)} \\
&= \frac{a[d(u, u)d(u, Tu) + d(u, Tu)d(u, u)]}{d(u, Tu) + d(u, u)} + \\
&\quad + \frac{bcd(u, Tu)d(u, u)}{d(u, u) + d(u, Tu)} \\
&= 0.
\end{aligned}$$

Hence $d(u, Tu) = 0$ and so u is a common fixed point of S and T.

Similarly, if u is a fixed point of T, then it is also fixed point of S.

Now let x_0 be an arbitrary point of X. If $Sx_0 = x_0$, then the proof is finished.

Assume that $Sx_0 \neq x_0$. Define a sequence $\{x_n\}$ in X as follows:

$$\begin{aligned}
x_1 &= Sx_0 \preceq TSx_0 = Tx_1 = x_2, \\
x_2 &= Tx_1 \preceq STx_1 = Sx_2 = x_3.
\end{aligned}$$

Continuing this process we have

$$x_1 \preceq x_2 \preceq \ldots \preceq x_n \preceq x_{n+1} \preceq \ldots.$$

We may assume that $d(x_{2n}, x_{2n+1}) > 0$, for every $n \in \mathbb{N}$. If not, then $x_{2n} = x_{2n+1}$ for some n and for all those n, $x_{2n} = x_{2n+1} = Sx_{2n}$ and proof is finished.

Now, since x_{2n} and x_{2n+1} are comparable and taking $d(x_{2n}, x_{2n+1}) > 0$ for $n = 0, 1, 2, 3, \ldots$, we have

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq a[d(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2}) + \\ &+ d(x_{2n+1}, x_{2n+2})d(x_{2n+1}, x_{2n+1})]/ \\ & d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}) + \\ &+ \frac{bd(x_{2n}, x_{2n+2})d(x_{2n+1}, x_{2n+1})}{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})} \\ &= \frac{ad(x_{2n}, x_{2n+1})d(x_{2n}, x_{2n+2})}{d(x_{2n}, x_{2n+2})} \\ &= ad(x_{2n}, x_{2n+1}). \end{aligned}$$

Similarly, $d(x_{2n}, x_{2n+1}) \leq ad(x_{2n-1}, x_{2n})$. It follows that

$$d(x_{n+1}, x_{n+2}) \leq ad(x_n, x_{n+1})$$

and consequently

$$d(x_{n+1}, x_{n+2}) \leq ad(x_n, x_{n+1}) \leq \ldots \leq a^{n+1}d(x_0, x_1)$$

for all $n \geq 0$.

Now for $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m+1}, x_m) \\ &\leq a^n d(x_0, x_1) + a^{n+1}d(x_0, x_1) + \ldots + a^{m-1}d(x_0, x_1) \\ &\leq \frac{a^n}{1-a}d(x_0, x_1). \end{aligned}$$

Therefore,

$$|d(x_n, x_m)| \leq \frac{a^n}{1-a}|d(x_0, x_1)|,$$

and so $|d(x_n, x_m)| \to 0$, as $m, n \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, the sequence $\{x_n\}$ converges to a point u in X.

If S or T is continuous, then it is clear that $Su = u = Tu$.

If neither S nor T is continuous, then $x_n \preceq u$ for all x in $\mathbb{N}$. We claim that u is a fixed point of S. If not, then $d(u, Su) = z > 0$. Now from (2.5), we have

$$\begin{aligned}
z &\le d(u,x_{n+2}) + d(x_{n+2},Su) \\
&\le d(u,x_{n+2}) + d(Su,Tx_{n+1}) \\
&\le d(u,x_{n+2}) + \tfrac{a[d(u,Su)d(u,Tx_{n+1})+d(x_{n+1},Tx_{n+1})d(x_{n+1},Su)]}{d(u,Tx_{n+1})+d(x_{n+1},Su)} \\
&\quad + \tfrac{bd(u,Tx_{n+1})d(x_{n+1},Su)}{d(u,Su)+d(x_{n+1},Tx_{n+1})} \\
&= d(u,x_{n+2}) + \tfrac{a[d(u,Su)d(u,x_{n+2})+d(x_{n+1},x_{n+2})d(x_{n+1},Su)}{d(u,x_{n+2})+d(x_{n+1},Su)} \\
&\quad + \tfrac{bd(u,x_{n+2})d(x_{n+1},Su)}{d(u,Su)+d(x_{n+1},x_{n+2})}.
\end{aligned}$$

Taking the limit as $n \to \infty$ gives $|z| \le 0$, a contradiction and so $u = Su$. Therefore $Su = Tu = u$.

Now suppose that set of common fixed points of S and T is totally ordered. We claim that common fixed point of S and T is unique. Assume on the contrary that u and v are distinct common fixed points of S and T. By supposition, we can replace x by u and y by v in (2.5) to obtain

$$\begin{aligned}
d(u,v) &= d(Su,Tv) \\
&\le \tfrac{a[d(u,Su)d(u,Tv)+d(v,Tv)d(v,Su)]}{d(u,Tv)+d(v,Su)} + \tfrac{bd(u,Tv)d(v,Su)}{d(u,Su)+d(v,Tv)},
\end{aligned}$$

which implies that $d(u,v) = d(Su,Tv) = 0$ and hence $u = v$.

Conversely, if S and T have only one common fixed point then the set of common fixed point of S and T being a singleton is totally ordered. □

In Theorem 2.4, take $S = T$, to obtain the following corollary.

Corollary 2.5. Let $(X,\preceq)$ be a partially ordered set such that there exists a complete complex valued metric d on X and T be weakly increasing self-map on X. Also, for every comparable $x,y \in X$, we have

$$d(Tx,Ty) \le \frac{a[d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)]}{d(x,Ty) + d(y,Tx)} + \\ + \frac{bd(x,Ty)d(y,Tx)}{d(x,Tx) + d(y,Ty)} \tag{2.7}$$

if $d(x,Ty) + d(y,Tx) \ne 0$ and $d(x,Tx) + d(y,Ty) \ne 0$, where $a, b \ge 0$ with $0 \le a + b < 1$, or

$$d(Tx,Ty) = 0 \text{ if } d(x,Ty) + d(y,Tx) = 0 \\ \text{or} \\ d(x,Tx) + d(y,Ty) = 0. \tag{2.8}$$

If T is continuous or for any nondecreasing sequence $\{x_n\}$ with $x_n \to z$ in X we necessarily have $x_n \preceq z$ for all $n \in \mathbb{N}$. Then T has a fixed point. Moreover, the set of fixed points of T is totally ordered if and only if T has one and only one fixed point. □

3. Well-posedness

The notion of well-posedness of a fixed point has evoked much interest to several mathematicians. Recently, Karapinar [13] studied well-posed problem for a cyclic weak $\phi-$contraction mapping on a complete metric space (see also, [15, 19]). We define well-posedness of fixed point and common fixed point problems for order contractive mappings.

Definition 3.1. A fixed point problem of self-map S on X, $FP(S,X)$, is called well-posed if $F(S)$ (a set of fixed point of S) is singleton and for any sequence $\{x_n\}$ in X whose every term is comparable with $x^* \in F(S)$ and $\lim\limits_{n\to\infty} d(Sx_n, x_n) = 0$ implies $x^* = \lim\limits_{n\to\infty} x_n$.

Definition 3.2. A common fixed point problem of selfmap S and T on X, $CFP(S,T,X)$, is called well-posed if $CF(S,T)$ (a set of common fixed points of S and T) is singleton and for any sequence $\{x_n\}$ in X whose every term is comparable with $x^* \in CF(S,T)$ and $\lim\limits_{n\to\infty} d(Sx_n, x_n) = 0$ or $\lim\limits_{n\to\infty} (Tx_n, x_n) = 0$ implies $x^* = \lim\limits_{n\to\infty} x_n$.

Theorem 3.3. Let $(X, \preceq)$ be a partially ordered set such that there exists a complete complex valued metric d on X. Suppose that S and T be self-maps on X as in Theorem 2.1. Then the common fixed point problem of S and T is well-posed.

Proof. From Theorem 2.1, the mappings S and T have a unique common fixed point, say $u \in X$. Let $\{x_n\}$ be a sequence in X whose every term is comparable with u and $\lim\limits_{n\to\infty} d(Sx_n, x_n) = 0$ or $\lim\limits_{n\to\infty} (Tx_n, x_n) = 0$. Without loss of generality, we may suppose that $u \neq x_n$ for every non-negative integer n. Then from the triangle inequality and inequality (2.1) we have

$$\begin{aligned}
&d(u, x_n) \leq d(Sx_n, Tu) + d(Sx_n, x_n)\\
&\leq a[\{d(x_n, Tu)\}^2 + \{d(u, Sx_n)\}^2]/[d(x_n, Tu) + d(u, Sx_n)]+\\
&+b[d(x_n, Sx_n) + d(u, Tu)] + c[d(x_n, Tu) + d(u, Sx_n)]+\\
&+ed(x_n, u) + d(Sx_n, x_n) = \tfrac{a[\{d(x_n,u)\}^2+\{d(u,Sx_n)\}^2]}{d(x_n,u)+d(u,Sx_n)}+\\
&+bd(x_n, Sx_n) + c[d(x_n, u) + d(u, Sx_n)] + ed(x_n, u)+\\
&+d(Sx_n, x_n) \leq \tfrac{a[\{d(x_n,u)\}^2+\{d(u,Sx_n)\}^2]}{d(x_n,u)+d(u,Sx_n)}+\\
&+(1 + b + c)d(x_n, Sx_n) + (2c + e)d(x_n, u),
\end{aligned}$$

since $d(x_n, u) + d(u, Sx_n) \neq 0$ for each n.

If $d(x_n, u) + d(u, Sx_n) = 0$ for some n, then for those n, we have from (2.2), $d(Sx_n, Tu) = 0$ and so

$$d(u, x_n) \leq d(Sx_n, Tu) + d(Sx_n, x_n) = d(Sx_n, x_n).$$

Taking limit as $n \to \infty$ implies $|d(u, x_n)| \to 0$, that is $\lim_{n\to\infty} x_n = u$.

Thus $d(x_n, u) + d(u, Sx_n) \neq 0$ for each n. The following three cases now arise:

(i) If $d(x_n, u) \neq 0$ and $d(u, Sx_n) \neq 0$, then

$$\begin{aligned}
d(u, x_n) &\leq \frac{a\{d(x_n, u)\}^2}{d(x_n, u) + d(u, Sx_n)} + \frac{a\{d(u, Sx_n)\}^2}{d(x_n, u) + d(u, Sx_n)} \\
&\quad + (1 + b + c)d(x_n, Sx_n) + (2c + e)d(x_n, u) \\
&\leq \frac{a\{d(x_n, u)\}^2}{d(x_n, u)} + \frac{a\{d(u, Sx_n)\}^2}{d(u, Sx_n)} \\
&\quad + (1 + b + c)d(x_n, Sx_n) + (2c + e)d(x_n, u) \\
&= (1 + a + b + c)d(x_n, Sx_n) + (a + 2c + e)d(x_n, u),
\end{aligned}$$

which implies that

$$d(u, x_n) \leq \frac{(1 + a + b + c)}{1 - (a + 2c + e)} d(x_n, Sx_n).$$

Taking limit as $n \to \infty$ implies $|d(u, x_n)| \to 0$, that is $\lim_{n\to\infty} x_n = u$.

(ii) If $d(x_n, u) \neq 0$ and $d(u, Sx_n) = 0$, then

$$\begin{aligned}
&d(u, x_n) \leq \tfrac{a\{d(x_n, u)\}^2}{d(x_n, u) + d(u, Sx_n)} + \tfrac{a\{d(u, Sx_n)\}^2}{d(x_n, u) + d(u, Sx_n)} \\
&+ (1 + b + c)d(x_n, Sx_n) + (2c + e)d(x_n, u) \\
&= \tfrac{a\{d(x_n, u)\}^2}{d(x_n, u)} + (1 + b + c)d(x_n, Sx_n) + (2c + e)d(x_n, u) \\
&= (1 + b + c)d(x_n, Sx_n) + (a + 2c + e)d(x_n, u),
\end{aligned}$$

which implies that

$$d(u, x_n) \leq \frac{(1 + b + c)}{1 - (a + 2c + e)} d(x_n, Sx_n).$$

Taking limit as $n \to \infty$ implies $|d(u, x_n)| \to 0$, that is $\lim_{n\to\infty} x_n = u$.

(iii) If $d(x_n, u) = 0$ and $d(u, Sx_n) \neq 0$, then taking the limit as $n \to \infty$ implies $|d(u, x_n)| \to 0$. That is $\lim_{n\to\infty} x_n = u$. This completes the proof. □

4. Periodic point results

Clearly, a fixed point p of T is also a fixed point of T^n for every $n \in \mathbb{N}$. However, the converse is false. For example, consider, $X = [0,1]$, and define T by $Tx = 1 - x$. Then T has a unique fixed point $\frac{1}{2}$ and every even iterate of T is the identity map, which has every point of $[0,1]$ as a fixed point. On the other hand, if $X = [0,\pi]$, $Tx = \cos x$, then every iterate of T has the same fixed point as T (cf. [1, 9, 12]).

If a map T satisfies $F(T) = F(T^n)$ for each $n \in \mathbb{N}$, then it is said to have property P [12]. The set $O(x,\infty) = \{x, Tx, T^2x, ...\}$ is called the *orbit of* x.

Theorem 4.1. Let $(X, \preceq)$ be a partially ordered set such that there exists a complete complex valued metric d on X. Let T be a self-map on X as in Corollary 2.3. If $O(x,\infty)$ is totally ordered, then T has property P.

Proof. From Corollary 2.3, T has a fixed point. Let $u \in F(T^n)$. Now from (2.3), we have

$$\begin{aligned}
d(u,Tu) &= d(T(T^{n-1}u), T(T^n u)) \\
&\le \frac{a[\{d(T^{n-1}u, T^{n+1}u)\}^2 + \{d(T^n u, T^n u)\}^2]}{d(T^{n-1}u, T^{n+1}u) + d(T^n u, T^n u)} + \\
&+ b[d(T^{n-1}u, T^n u) + d(T^n u, T^{n+1}u)] + \\
&+ c[d(T^{n-1}u, T^{n+1}u) + d(T^n u, T^n u)] + ed(T^{n-1}u, T^n u) \\
&= (a+c)d(T^{n-1}u, T^{n+1}u) + (b+e)d(T^{n-1}u, T^n u) + \\
&+ bd(T^n u, T^{n+1}u) \\
&= (a+c)d(T^{n-1}u, Tu) + (b+e)d(T^{n-1}u, u) + bd(u, Tu) \\
&\le (a+b+c+e)d(T^{n-1}u, u) + (a+b+c)d(Tu, u),
\end{aligned}$$

which implies

$$d(u,Tu) \le \frac{(a+b+c+e)}{1-(a+b+c)} d(T^{n-1}u, u).$$

Put $\lambda = \frac{(a+b+c+e)}{1-(a+b+c)}$. Obviously $0 \le \lambda < 1$ and we have

$$\begin{aligned}
d(u,Tu) &= d(Tu, T^n u) \le \lambda d(T^{n-1}u, T^n u) \le \\
&\le \lambda^2 d(T^{n-2}u, T^{n-1}u) \le ... \le \lambda^n d(u, Tu).
\end{aligned}$$

Since $0 \le \lambda < 1$, this implies that $d(u,Tu) = 0$ and so $u = Tu$. $\square$

Theorem 4.2. Let $(X, \preceq)$ be a partially ordered set such that there exists a complete complex valued metric d on X. Let T be a self-map on X as in Corollary 2.5. If $O(x, \infty)$ is totally ordered, then T has property P.

Proof. From Corollary 2.5, T has a fixed point. Let $u \in F(T^n)$. Now from (2.7), we have

$$\begin{aligned}
&d(u, Tu) = d(T(T^{n-1}u), T(T^n u)) \\
&\leq \tfrac{a[d(T^{n-1}u,T^n u)d(T^{n-1}u,T^{n+1}u)+d(T^n u,T^{n+1}u)d(T^n u,T^n u)]}{[d(T^{n-1}u,T^{n+1}u)+d(T^n u,T^n u)]} + \\
&+ \tfrac{bd(T^{n-1}u,T^{n+1}u)d(T^n u,T^n u)}{d(T^{n-1}u,T^n u)+d(T^n u,T^{n+1}u)} \\
&= \tfrac{ad(T^{n-1}u,u)d(T^{n-1}u,Tu)}{d(T^{n-1}u,Tu)} \\
&= ad(T^{n-1}u, u).
\end{aligned}$$

Thus we have

$$\begin{aligned}
d(u, Tu) &= d(Tu, T^n u) \leq ad(T^{n-1}u, T^n u) \\
&\leq a^2 d(T^{n-2}u, T^{n-1}u) \leq \ldots \leq a^n d(u, Tu).
\end{aligned}$$

Since $0 \leq a < 1$, this implies that $d(u, Tu) = 0$ and so $u = Tu$. $\square$

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