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GENERAL RANDERS MECHANICAL SYSTEMS

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Abstract. The general Randers spaces were introduced by R. Miron [2]. These are some generalizations of Randers spaces denoted $GR^n = (M, F + \beta)$, equipped with the Lorentz nonlinear connection. In the present paper we define the General Randers Mechanical System as a triple (M, T, F_e), where T is the energy of the GR^n space. We obtain the expressions for the curvature and the torsion of GR^n and we give the formula for the local coefficients of the canonical connection.

1. Preliminaries on finsler spaces

Let M be a n-dimensional C^∞ manifold. Denote by (TM, τ, M) the tangent bundle of M. Let $\widetilde{F}^n = \left(M, \widetilde{F}(x,y)\right)$ be a Finsler space where $F : TM \to \mathbb{R}$ is its fundamental function and the Hessian given by $\widetilde{g_{ij}}(x,y) = \frac{1}{2}\frac{\partial^2 \widetilde{F}}{\partial y^i \partial y^j}$ called the fundamental tensor field of $\widetilde{F}^n$ is positive defined. The Cartan nonlinear connection $\widetilde{N}$ of the space $\widetilde{F}^n$ has the coefficients $\widetilde{N}^i_j = \frac{1}{2}\frac{\partial}{\partial y^j}\left(\widetilde{\gamma}^i_{kh}(x,y)\, y^k y^h\right)$, where we denoted by $\widetilde{\gamma}^i_{kh}$ the Christoffel symbols of the metric tensor field $\widetilde{g_{ij}}$. The Cartan nonlinear connection determines the horizontal distribution which is supplementary to the vertical distribution. The adapted basis to this distribution is $\left(\frac{\widetilde{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ with $\frac{\widetilde{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \widetilde{N}^i_j \frac{\partial}{\partial y^j}$ and the dual adapted basis is given by $\left(dx^i, \widetilde{\delta}\, y^i\right)$, with $\widetilde{\delta}\, y^i = dy^i + \widetilde{N}^i_j\, dx^j$.

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The Cartan connection $C\Gamma(N) = \left(\widetilde{F}^i_{jk}, \widetilde{C}^i_{jk}\right)$ is given by

$$\begin{cases} \widetilde{F}^i_{jk} = \frac{1}{2}\,\widetilde{g}^{is}\left(\frac{\widetilde{\delta}\,\widetilde{g}_{sj}}{\delta x^k} + \frac{\widetilde{\delta}\,\widetilde{g}_{sk}}{\delta x^j} - \frac{\widetilde{\delta}\,\widetilde{g}_{jk}}{\delta x^s}\right) \\ \widetilde{C}^i_{jk} = \frac{1}{2}\,\widetilde{g}^{is}\left(\frac{\partial\,\widetilde{g}_{sj}}{\partial y^k} + \frac{\partial\,\widetilde{g}_{sk}}{\partial y^j} - \frac{\partial\,\widetilde{g}_{jk}}{\partial y^s}\right). \end{cases}$$

2. General randers spaces

Let $\widetilde{F}^n = \left(M, \widetilde{F}(x,y)\right)$ be a Finsler space and $\beta(x,y) = b_i(x)\,y^i$ an 1-forms field on TM, where $b_i(x)$ is a covector field on M or on an open set of M. We shall consider the real function $L : TM \to \mathbb{R}$, $L(x,y) = \widetilde{F}(x,y) + \beta(x,y)$. The pair $GR^n = (M, L(x,y))$ is called the General Randers space. The tensor field $g_{ij}$of GR^n is $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$. Let us consider the nonlinear connection N whose local coefficients are $N^i_j = \widetilde{N}^i_j - F^i_j$, where $\widetilde{N}^i_j$ is the Cartan nonlinear connection of the Finsler space $\widetilde{F}^n$and $F^i_j = g^{ik}(x,y)\,F_{kj}(x)$, with $F_{kj}(x) = \frac{\partial b_j}{\partial x^k} - \frac{\partial b_k}{\partial x^j}$, the electromagnetic tensor field of the electromagnetic potentials $b_i(x)$. N is called the Lorentz nonlinear connection of the space GR^n. The local basis adapted to the Lorentz connection is $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, $\frac{\delta}{\delta x^i} = \frac{\widetilde{\delta}}{\delta x^i} + F^j_i \frac{\partial}{\partial y^j}$.

It is well known that using the Lorentz nonlinear connection it can be constructed an unique d-connection $D\Gamma(N) = \left(L^i_{jk}, C^i_{jk}\right)$, called the canonical metrical d-connection of the General Randers space, with the properties:

$$1. \nabla^H_k g_{ij} = 0;$$

$$2. \nabla^V_k g_{ij} = 0;$$

$$3. T^i_{jk} = 0;$$

$$4. S^i_{jk} = 0.$$

Its coefficients are given by the generalized Christoffel symbols:

$$\begin{cases} L^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\delta g_{sk}}{\delta x^j} + \frac{\delta g_{js}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^s}\right) \\ C^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s}\right) \end{cases}$$

3. General randers mechanical systems

Definition 3.1. A General Randers Mechanical System (GRMS) is a triple $\sum_{GR} = (M, T, F_e)$, where:
i) M is an n-dimensional, real, differentiable manifold, called configuration space;
ii) $T = y^i \frac{\partial L^2}{\partial y^i} - L^2 = L^2$ is the energy of the General Randers space GR^n;
iii) $F_e(x, y) = L^i(x, y) \frac{\partial}{\partial y^i}$ are the external forces given as a vertical vector field on TM and $L_i(x, y) = g_{ij} L^i(x, y)$, $i = \overline{1, n}$ are the covariant components of the F_e.

The nonlinear connection $\overset{M}{N}$ of the GRMS has the coefficients

$$\overset{M}{N}{}^i_j = N^i_j - \frac{1}{4} \frac{\partial F^i}{\partial y^j},$$

where N^i_j are the local coefficients of the Lorentz nonlinear connection. So,

$$\overset{M}{N}{}^i_j = \tilde{N}{}^i_j - \left(F^i_j + \frac{1}{4} \frac{\partial F^i}{\partial y^j} \right).$$

$\overset{M}{N}$ is called the Lorentz nonlinear connection of *the* GRMS and determines the horizontal distribution which is supplementary to the natural vertical distribution on TM.

A local adapted basis to these distribution is $\left(\frac{\overset{M}{\delta}}{\delta x^i}, \frac{\partial}{\partial y^i} \right)$, $i = \overline{1, n}$, where

$$\frac{\overset{M}{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \overset{M}{N}{}^i_j \frac{\partial}{\partial y^j} = \frac{\overset{\sim}{\delta}}{\delta x^i} + \left(F^i_j + \frac{1}{4} \frac{\partial F^j}{\partial y^i} \right) \frac{\partial}{\partial y^j}.$$

The dual adapted basis is $\left(dx^i, \overset{M}{\delta} y^i \right)$ with

$$\overset{M}{\delta} y^i = dy^i + \overset{M}{N}{}^i_j dx^j = \overset{\sim}{\delta} y^i - \left(F^i_j + \frac{1}{4} \frac{\partial F^i}{\partial y^j} \right) dx^j.$$

We calculate the curvature $\overset{M}{R}{}^{i}_{jk}$ and the torsion $\overset{M}{T}{}^{i}_{jk}$ of the Lorentz nonlinear connection $\overset{M}{N}$ of the GRMS:

$$\begin{aligned}\overset{M}{T}{}^{i}_{jk} &= \frac{\partial \overset{M}{N}{}^{i}_{j}}{\partial y^k} - \frac{\partial \overset{M}{N}{}^{i}_{k}}{\partial y^j} = \left[\frac{\partial \tilde{N}{}^{i}_{j}}{\partial y^k} - \frac{\partial}{\partial y^k}\left(F^i_j + \frac{1}{4}\frac{\partial F^i}{\partial y^j}\right)\right] - \\ &- \left[\frac{\partial \tilde{N}{}^{i}_{k}}{\partial y^j} - \frac{\partial}{\partial y^j}\left(F^i_k + \frac{1}{4}\frac{\partial F^i}{\partial y^k}\right)\right] = 0.\end{aligned}$$

$$\begin{aligned}&\overset{M}{R}{}^{i}_{jk} = \frac{\overset{M}{\delta}\overset{M}{N}{}^{i}_{j}}{\delta x^k} - \frac{\overset{M}{\delta}\overset{M}{N}{}^{i}_{k}}{\delta x^j} = \frac{\overset{M}{\delta}}{\delta x^k}\left[\tilde{N}{}^{i}_{j} - F^i_j - \tfrac{1}{4}\tfrac{\partial F^i}{\partial y^j}\right] - \frac{\overset{M}{\delta}}{\delta x^j}\left[\tilde{N}{}^{i}_{k} - F^i_k - \tfrac{1}{4}\tfrac{\partial F^i}{\partial y^k}\right] = \\ &= \frac{\overset{\sim}{\delta}}{\delta x^i}\left(\tilde{N}{}^{i}_{j} - F^i_j - \tfrac{1}{4}\tfrac{\partial F^i}{\partial y^j}\right) + \left(F^i_j + \tfrac{1}{4}\tfrac{\partial F^j}{\partial y^i}\right)\frac{\partial}{\partial y^j}\left(\tilde{N}{}^{i}_{j} - F^i_j - \tfrac{1}{4}\tfrac{\partial F^i}{\partial y^j}\right) - \\ &- \frac{\overset{\sim}{\delta}}{\delta x^j}\left(\tilde{N}{}^{i}_{k} - F^i_k - \tfrac{1}{4}\tfrac{\partial F^i}{\partial y^k}\right) + \left(F^i_k + \tfrac{1}{4}\tfrac{\partial F^k}{\partial y^i}\right)\frac{\partial}{\partial y^k}\left(\tilde{N}{}^{i}_{k} - F^i_k - \tfrac{1}{4}\tfrac{\partial F^i}{\partial y^k}\right) = \\ &= \tilde{R}{}^{i}_{jk} - \left(\frac{\overset{\sim}{\delta} F^i_j}{\delta x^i} - \frac{\overset{\sim}{\delta} F^i_k}{\delta x^j}\right) - \frac{1}{4}\left(\frac{\overset{\sim}{\delta}}{\delta x^i}\frac{\partial F^i}{\partial y^j} - \frac{\overset{\sim}{\delta}}{\delta x^j}\frac{\partial F^i}{\partial y^k}\right) + \left(F^i_j\frac{\partial \tilde{N}{}^{i}_{j}}{\partial y^j} - F^i_k\frac{\partial \tilde{N}{}^{i}_{k}}{\partial y^k}\right) - \\ &- \left(F^i_j\frac{\partial F^i_j}{\partial y^j} - F^i_k\frac{\partial F^i_k}{\partial y^k}\right) + \frac{1}{4}\left(\frac{\partial F^j}{\partial y^i}\frac{\partial \tilde{N}{}^{i}_{j}}{\partial y^j} - \frac{\partial F^k}{\partial y^i}\frac{\partial \tilde{N}{}^{i}_{k}}{\partial y^k}\right) - \frac{1}{4}\left(\frac{\partial F^j}{\partial y^i}\frac{\partial \tilde{F}{}^{i}_{j}}{\partial y^j} - \frac{\partial F^k}{\partial y^i}\frac{\partial \tilde{F}{}^{i}_{k}}{\partial y^k}\right).\end{aligned}$$

So, we can state the following theorem:

Theorem 3.1. *The torsion* $\overset{M}{T}{}^{i}_{jk}$ *of the Lorentz nonlinear connection* $\overset{M}{N}$ *of the GRMS vanishes and the curvature tensor is given by*

$$\begin{aligned}R^{i}_{jk}{}^{M} = R^{i}_{jk}{}^{\sim} &- \left(\frac{\overset{\sim}{\delta} F^i_j}{\delta x^i} - \frac{\overset{\sim}{\delta} F^i_k}{\delta x^j}\right) - \frac{1}{4}\left(\frac{\overset{\sim}{\delta}}{\delta x^i}\frac{\partial F^i}{\partial y^j} - \frac{\overset{\sim}{\delta}}{\delta x^j}\frac{\partial F^i}{\partial y^k}\right) + \\ &+ \left(F^i_j\frac{\partial \tilde{N}{}^{i}_{j}}{\partial y^j} - F^i_k\frac{\partial \tilde{N}{}^{i}_{k}}{\partial y^k}\right) - \left(F^i_j\frac{\partial F^i_j}{\partial y^j} - F^i_k\frac{\partial F^i_k}{\partial y^k}\right) + \\ &+ \frac{1}{4}\left(\frac{\partial F^j}{\partial y^i}\frac{\partial \tilde{N}{}^{i}_{j}}{\partial y^j} - \frac{\partial F^k}{\partial y^i}\frac{\partial \tilde{N}{}^{i}_{k}}{\partial y^k}\right) - \frac{1}{4}\left(\frac{\partial F^j}{\partial y^i}\frac{\partial \tilde{F}{}^{i}_{j}}{\partial y^j} - \frac{\partial F^k}{\partial y^i}\frac{\partial \tilde{F}{}^{i}_{k}}{\partial y^k}\right)\end{aligned}$$

We fix the Lorentz nonlinear connection of GRMS and we consider a d-connection $D\overset{M}{\Gamma}\left(\overset{M}{N}\right) = \left(\overset{M}{L}{}^{i}_{jk}, \overset{M}{C}{}^{i}_{jk}\right)$ which is uniquely determined by the following axioms:

1. $\overset{M}{\nabla}{}^H_k g_{ij} = 0$ (D is h-metrical);
2. $\overset{M}{\nabla}{}^V_k g_{ij} = 0$(D is v-metrical);
3. $\overset{M}{T}{}^i_{jk} = 0$ (D is h-torsion free);
4. $\overset{M}{S}{}^i_{jk} = 0$ (D is v-torsion free), where

$$\overset{M}{\nabla}{}^H_k g_{ij} = \frac{\overset{M}{\delta} g_{ij}}{\delta x^k} - \overset{M}{L}{}^s_{ik} g_{sj} - \overset{M}{L}{}^s_{jk} g_{is}, \quad \overset{M}{\nabla}{}^V_k g_{ij} = \frac{\partial g_{ij}}{\partial y^k} - \overset{M}{C}{}^s_{ik} g_{sj} - \overset{M}{C}{}^s_{jk} g_{is}.$$

The local coefficients of $D\overset{M}{\Gamma}\left(\overset{M}{N}\right) = \left(\overset{M}{L}{}^i_{jk}, \overset{M}{C}{}^i_{jk}\right)$ are expressed by the generalized Christoffel symbols:

$$\begin{cases} \overset{M}{L}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\overset{M}{\delta} g_{sj}}{\delta x^k} + \frac{\overset{M}{\delta} g_{sk}}{\delta x^j} - \frac{\overset{M}{\delta} g_{jk}}{\delta x^s}\right) \\ \overset{M}{C}{}^i_{jk} = \frac{1}{2} g^{is}\left(\frac{\partial g_{sj}}{\partial y^k} + \frac{\partial g_{sk}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^s}\right). \end{cases}$$

Through a direct calculation and using the results from [2] we get the explicit form of these coefficients in the following theorem:

Theorem 3.2. *The coefficients* $\overset{M}{L}{}^i_{jk}, \overset{M}{C}{}^i_{jk}$ *of* $D\overset{M}{\Gamma}\left(\overset{M}{N}\right) = \left(\overset{M}{L}{}^i_{jk}, \overset{M}{C}{}^i_{jk}\right)$ *are given by*

$$\overset{M}{L}{}^i_{jk} = L^i_{jk} + A^i_{jk}, \quad \overset{M}{C}{}^i_{jk} = C^i_{jk},$$

where $C\Gamma(N) = \left(L^i_{jk}, C^i_{jk}\right)$ *is the canonical metrical d-connection of* GR^n, *with*

$$L^i_{jk} = \tilde{F}{}^i_{jk} + \tilde{C}{}^i_{js} F^s_k + B^i_{jk}, C^i_{jk} = \tilde{C}{}^i_{jk} + E^i_{jk}$$

and

$$A^i_{jk} = \frac{1}{4} g^{is}\left(C_{skh}\frac{\partial F^h}{\partial y^j} + C_{jsh}\frac{\partial F^h}{\partial y^k} - C_{jkh}\frac{\partial F^h}{\partial y^s}\right)$$
$$B^i_{jk} = \frac{1}{2} g^{ir}\left(\overset{\sim}{\nabla}{}^H_k g_{jr} + \overset{\sim}{\nabla}{}^H_j g_{rk} - \overset{\sim}{\nabla}{}^H_r g_{jk}\right) + F^s_k E^i_{sj} + F^s_j C^i_{sk} - g^{ir} F^s_r C_{sjk}$$
$$E^i_{jk} = -\frac{F}{L}\frac{y^i}{F} b^s \tilde{C}_{sjk} + \frac{1}{2F^2} g^{is}\left[\left(Fb_k - \beta\frac{\partial F}{\partial y^i}\right) F\frac{\partial^2 F}{\partial y^s \partial y^j} + \right.$$
$$\left. + \left(Fb_s - \beta\frac{\partial F}{\partial y^s}\right) F\frac{\partial^2 F}{\partial y^j \partial y^k} + \left(Fb_j - \beta\frac{\partial F}{\partial y^j}\right) F\frac{\partial^2 F}{\partial y^s \partial y^k}\right]$$

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