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FIXED POINT THEOREMS IN FUZZY METRIC SPACES THROUGH WEAK COMPATIBILITY

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Abstract. We prove a common fixed point theorem for six self-maps on a complete fuzzy metric space that generate some compatible and weakly compatible pairs of maps. Our result extends and unifies corresponding fixed point theorems of Sessa [9], Jungck [5], [6], Singh and Chauhan [10], that were proved for commuting and weakly commuting self maps of metric spaces or of probabilistic metric spaces.

1. Introduction and preliminaries

Zadeh's [11] introduction of the notion of fuzzy set laid the foundation of fuzzy mathematics. Erceg [2], Kramosil and Michalek [7] have introduced the concept of fuzzy metric spaces in different ways. George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [7] and defined a Hausdorff topology on this fuzzy metric spaces and gave a relation $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ in which every metric induces a fuzzy metric. Sessa [9] defined a generalization of commutativity, which is called weak commutativity.

Jungck [5] gave the concept of compatibility, that is more general than commutativity and weak commutativity in metric space and proved common fixed point theorems. Singh and Chauhan [10] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric spaces in the sense of George and Veeramani [3].

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Jungck and Rhoades [6] introduced the notion of coincidentally commuting (or weakly compatible) mappings and obtained fixed point theorems for set-valued mappings. The purpose of this paper is to prove a common fixed point theorem for compatible, weakly compatible maps in a fuzzy metric space which extends, generalizes, improves and unifies corresponding results of Sessa [9], Jungck [5], [6], Singh and Chauhan [10] for commuting and weakly commuting mappings on metric spaces and probabilistic metric spaces.

Now, we give some definitions and lemmas.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(1) [(a)]
(2) $*$ is associative and commutative,
(3) $*$ is continuous,
(4) $a * 1 = a$ for all $a \in [0,1]$,
(5) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $(a, b, c, d \in [0,1])$.

Examples:

(1) [(i)]
(2) $a * b = ab$
(3) $a * b = \min\{a, b\}$.

Definition 1.2. [8] The triplet $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$.

(1) [(1)]
(2) $M(x, y, t) = 0$,
(3) $M(x, y, t) = 1$ if and only if $x = y$,
(4) $M(x, y, t) = M(y, x, t)$,
(5) $M(x, y, t) * M(y, z, s) = M(x, z, t + s)$,
(6) $M(x, y, \cdot) : [0, \infty) \rightarrow [0,1]$ is left continuous,
(7) $\lim_{t \to \infty} M(x, y, t) = 1$.

Remark 1.2. For all $x, y \in X$ the function $M(x, y, \cdot)$ is non-decreasing and $M(x, y, t) > 0$ for all $x, y \in X$ and $t > 0$.

Example 1.1. Let (X, d) be a metric space, and let $a * b = ab$ or $a * b = \min\{a, b\}$.
Let $M(x, y, t) = \dfrac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space, and this fuzzy metric M induced by d is called the standard fuzzy metric [6].

Definition 1.3. Let $(X, M, *)$ be fuzzy metric space:
A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ (denoted by $\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} M(x_n, x, t) = 1$, for all $t > 0$.
Definition 1.4. Two self mappings A and B of a fuzzy metric space $(X, M, *)$ are said to be compatible if $\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.
Definition 1.5. Two self maps of a fuzzy metric space $(X, M, *)$ are said to be weakly compatible if $ABx = BAx$ when $Ax = Bx$ for some $x \in X$.
It is easy to see that if self mappings A and B of a fuzzy metric space $(X, M, *)$ are compatible then these are weakly compatible.

The following examples shows that the converse of above statement does not hold.
Example 1.2. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$ with the usual distance $d(x, y) = |x - y|$, and let the t-norm defined by $a * b = \min\{a, b\}$. Consider the fuzzy metric M induced on X by d, namely $M(x, y, t) = \dfrac{t}{t + d(x, y)}$.

Define self maps $A, B : X \to X$ as follows:

$$Ax = \begin{cases} 2 & \text{if } \ x \in [0, 1] \\ \frac{x}{2} & \text{if } \ x \in (1, 2] \end{cases}$$

respectively

$$Bx = \begin{cases} 2 & \text{if } \ x = 1 \\ \frac{x+3}{5} & \text{if } \ x \in [0, 2] - \{1\} \end{cases}$$

Clearly, $Ax = Bx$ iff $x \in \{1, 2\}$, that implies $ABx = BAx$. Hence A and B are weak compatible.

On the other hand, A and B are not compatible since for the sequence defined by $x_n = 2 - \dfrac{1}{(2n)}, n \geq 1$, we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = 1,$$

but

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = \lim_{n\to\infty} M(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{5t}{5t+6} \neq 1,$$

Remark 1.3. If $\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1$, then

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$$

Lemma 1.1. [11] *Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, *)$ with continuous t-norm and $t * t \geq t$. If there exists a constant $k \in (0, 1)$ such that*

$$M(x_n, x_{n+1}, kt) \geq M(x_{n-1}, x_n, t)$$

for all $t > 0$ and $n = 1, 2, \ldots,$ then $\{x_n\}$ is a Cauchy sequence in X.
Lemma 1.2. [9] *Let $(X, M, *)$ be a fuzzy metric space. If there exists a constant $k \in (0, 1)$ such that*

$$M(x, y, kt) \geq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$, then $x = y$.
Lemma 1.3. *Let U, V be compatible self-maps of a fuzzy metric space $(X, M, *)$. Assume that $\lim_{n\to\infty} Ux_n = \lim_{n\to\infty} Vx_n = y$ and $\lim_{n\to\infty} UVx_n = Uy$ for some sequence $\{x_n\}$ in X and some $y \in X$. Then $\lim_{n\to\infty} VUx_n = Uy$*
Proof. Let $t > 0$. Since U, V are compatible self-maps,

$$\lim_{n\to\infty} M(UVx_n, VUx_{n,}, t) = 1.$$

On the other hand

$$\lim_{n\to\infty} M(UVx_n, Uy, t) = 1.$$

By (3) and (4) of Definition 1.2,
$1 \geq M(VUx_n, Uy, t) \geq M(UVx_n, VUx_n, t) * M(UVx_n, Uy, t)$ for all $n \geq 1$,
hence $\lim_{n\to\infty} M(VUx_n, Uy, t) = 1$.

2. Main results

Theorem 2.1. *Let A, B, S, T, P and Q be self maps on a complete fuzzy metric space $(X, M, *)$ with $t * t \geq t$ for all $t > 0$, satisfying:*
a)

$$P(X) \subset ST(X), Q(X) \subset AB(X); \tag{2.1}$$

b) There exists a constant $k \in (0, 1)$ such that

$$\begin{aligned} M^2(Px, Qy, kt) &: [M(ABx, Px, kt) * M(STy, Qy, kt)] \\ &\geq [pM(ABx, Px, t) + qM(ABx, STy, t)] \cdot M(ABx, Qy, 2kt) \end{aligned} \tag{2.2}$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$;

c)

$$AB = BA, ST = TS, PB = BP, QT = TQ; \tag{2.3}$$

d) Either AB or P is continuous;

e) The pair (P, AB) is compatible and the pair (Q, ST) is weakly compatible. Then A, B, S, T, P and Q have a unique common fixed point.

Proof. Let $t > 0$ and x_0 be an arbitrary point of X. By (2.1.1), there exist $x_1, x_2 \in X$ such that $Px_0 = STx_1 = y_0$ and $Qx_1 = ABx_1 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Px_{2n} = STx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = ABx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2, \ldots$

Step 1. By taking $x = x_{2n}$ and $y = x_{2n+1}$ in (2.1.2), we have

$$\begin{aligned} &M^2(Px_{2n}, Qx_{2n+1}, kt).[M(ABx_{2n}, Px_{2n}, kt) * M(STx_{2n+1}, Qx_{2n+1}, kt)] \\ &\quad \geq [pM(ABx_{2n}, Px_{2n}, t) + qM(ABx_{2n,}STx_{2n+1}, t)] \\ &\qquad M(ABx_{2n}, Qx_{2n+1}, 2kt), \end{aligned}$$

$$\begin{aligned} &M^2(y_{2n}, y_{2n+1}, kt).[M(y_{2n-1}, y_{2n}, kt) * M(y_{2n}, y_{2n+1}, kt)] \\ &\quad \geq [pM(y_{2n}, y_{2n-1}, t) + qM(y_{2n-1}, y_{2n}, t)]M(y_{2n-1,}y_{2n+1}, 2kt), \\ &M(y_{2n}, y_{2n+1}, kt).[M(y_{2n-1}, y_{2n}, kt) * M(y_{2n}, y_{2n+1}, kt)] \\ &\quad \geq [(p+q)M(y_{2n}, y_{2n-1}, t)]M(y_{2n-1,}y_{2n+1}, 2kt), \\ &M(y_{2n}, y_{2n+1}, kt)[M(y_{2n-1}, y_{2n+1}, 2kt)] \\ &\quad \geq [M(y_{2n-1}, y_{2n}, t)]M(y_{2n-1,}y_{2n+1}, 2kt), \end{aligned}$$

Hence, we have

$$M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$$

Similarly, by taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (2.1.2), we also have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n+1}, y_{2n}, t)$$

In general, for all n even or odd, we have

$$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$$

for $k \in (0, 1)$ and all $t > 0$. Thus, by Lemma 1.1, $\{y_n\}$ is a Cauchy sequence in X. Since $(X, M, *)$ is complete, $\{y_n\}$ converges to a point z in X.

Also its subsequences converge as follows: $\{Px_{2n}\} \to z$, $\{ABx_{2n}\} \to z$, $\{Qx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$.

Case I. Assume that AB is continuous. Since AB is continuous, $AB(AB)x_{2n} \to ABz$ and $(AB)Px_{2n} \to ABz$. Since (P, AB) is compatible, $P(AB)x_{2n} \to ABz$ (using Lemma 1.3).

Step 2. By taking $x = ABx_{2n}$ and $y = x_{2n+1}$ in (2.1.2), we have

$$\begin{aligned} &M^2(P(AB)x_{2n}, Qx_{2n+1}, kt) \cdot [M(AB(AB)x_{2n}, P(AB)x_{2n}, kt) \\ &* M(STx_{2n+1}, Qx_{2n+1}, kt)] \\ &\qquad \geq [pM(AB(AB)x_{2n}, P(AB)x_{2n}, t) \\ &\qquad\qquad + qM(AB(AB)x_{2n}, STx_{2n+1}, t)]M(AB(AB)x_{2n}, Qx_{2n+1}, 2kt) \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} M^2(z, ABz, kt) &\cdot [M(ABz, ABz, kt) * M(z, z, kt)] \\ &\geq [pM(ABz, ABz, t) + qM(z, ABz, t)]M(z, ABz, 2kt) \\ &\geq [p + qM(z, ABz, t)]M(z, ABz, kt), \end{aligned}$$

$$\begin{aligned} M(z, ABz, kt) &\geq p + qM(z, ABz, kt) \\ &\geq p + qM(z, ABz, kt), \\ M(z, ABz, kt) &\geq \frac{p}{1-q} = 1 \end{aligned}$$

for $k \in (0, 1)$ and all $t > 0$. Thus, we have $z = ABz$.

Step 3. By taking $x = z$ and $y = x_{2n+1}$ in (2.1.2) and letting n tend to infinity, we obtain

$$\begin{aligned} M(z, Pz, kt) &\geq pM(z, Pz, t) + q \\ &\geq pM(z, Pz, kt) + q, \\ M(z, Pz, kt) &\geq \frac{q}{1-p} = 1 \end{aligned}$$

for $k \in (0, 1)$ and all $t > 0$.

Thus, we have $z = Pz = ABz$.

Step 4. By taking $x = Bz, y = x_{2n+1}$ in (2.1.2) and letting n tend to infinity, we have

$$\begin{aligned} M(z, Bz, kt) &\geq p + qM(z, Bz, t), \\ &\geq p + qM(z, Bz, kt), \\ M(z, Bz, kt) &\geq \frac{p}{1-q} = 1 \end{aligned}$$

for $k \in (0, 1)$ and all $t > 0$. Thus, we have $z = Bz$. Since $z = ABz$, we also have $z = Az$. Therefore, $z = Az = Bz = Pz$.

Step 5. Since $P(X) \subseteq ST(X)$, there exists $v \in X$ such that $z = Pz = STv$.
By taking $x = x_{2n}, y = v$ in (2.1.2) and letting n tend to infinity, we have

$$M^3(z, Qv, kt) \geq M(z, Qv, 2kt),$$

But $M(z, Qv, 2kt) \geq M(z, Qv, kt)$,
Hence

$$M^3(z, Qv, kt) \geq M(z, Qv, t), \quad \text{therefore} \quad M(z, Qv, t) \geq 1.$$

Thus, by lemma 1.2, we have $z = Qv$ and so $z = Qv = STv$. Since (Q, ST) is weakly compatible, we have $STQv = QSTv$. Thus, $STz = Qz$.
Step 6. By taking $x = x_{2n}, y = z$ in (2.1.2) and using Step 5 and letting n tend to infinity, we have

$$\begin{aligned} M(z, Qz, kt) &\geq p + qM(z, Qz, t) \\ &\geq p + qM(z, Qz, kt), \\ M(z, Qz, kt) &= \frac{p}{1-q} = 1. \end{aligned}$$

Thus, we have $z = Qz$ and therefore $z = Az = Bz = Pz = Qz = STz$.
Step 7. By taking $x = x_{2n}, y = Tz$ in (2.1.2) and letting n tend to infinity, we have

$$\begin{aligned} M(z, Tz, kt) &\geq p + qM(z, Tz, t) \\ &\geq p + qM(z, Tz, kt), \\ M(z, Tz, kt) &= \frac{p}{1-q} = 1. \end{aligned}$$

Thus, we have $z = Tz$. Since $Tz = STz$, we also have $z = Sz$. Therefore, $z = Az = Bz = Pz = Qz = Sz = Tz$, that is, z is the common fixed point of the six maps.
Case II. Assume that P is continuous. Since P is continuous, $PPx_{2n} \to Pz$ and $P(AB)x_{2n} \to Pz$. Since (P, AB) is compatible, $(AB)Px_{2n} \to Pz$.
Step 8. By taking $x = Px_{2n}, y = x_{2n+1}$ in (2.1.2) and letting n tend to infinity, we get

$$\begin{aligned} M(z, Pz, kt) &\geq p + qM(z, Pz, t) \\ &\geq p + qM(z, Pz, kt), \\ M(z, Pz, kt) &\geq \frac{p}{1-q} = 1. \end{aligned}$$

Thus, we have $z = Pz$ and using Steps 5-7, it follows that $z = Pz = Qz = Sz = Tz$.

Step 9. Since $Q(X) \subseteq AB(X)$, there exists $v \in X$ such that $z = Qz = ABv$.

By taking $x = u, y = x_{2n+1}$ in (2.1.2) and letting n tend to infinity, we get

$$M(z, Qu, kt) \geq pM(z, Pu, kt) + q,$$
$$M(z, Qu, kt) \geq \frac{q}{1-p} = 1.$$

Thus, we have $z = Pu = ABu$. Since (P, AB) is weakly compatible, we have $Pz = ABz$ and using Step 4, we also have $z = Bz$. Therefore $z = Az = Bz = Sz = Tz = Pz = Qz$, that is, z is the common fixed point of the six maps in this case also.

Step 10. For uniqueness, let w be a common fixed point of A, B, S, T, P and Q. Taking $x = z, y = w$ in (2.1.2), we obtain

$$\begin{aligned} M(z, w, kt) &\geq p + qM(z, w, t) \\ &\geq p + qM(z, w, kt), \\ M(z, w, kt) &\geq \frac{p}{1-q} = 1. \end{aligned}$$

Thus, we have $z = w$. This completes the proof of the theorem.

If we take $B = T = I_X$ (the identity map on X) in the main Theorem, we have the following:

Corollary 2.2 *Let A, S, P and Q be self maps on a complete fuzzy metric space $(X, M, *)$ with $t * t \geq t$ for all $t \in [0, 1]$, satisfying:*

(1) *[(a)]*
(2) $P(X) \subseteq S(X)$, $Q(X) \subseteq A(X)$;
(3) *there exists a constant $k \in (0, 1)$ such that*

$$\begin{aligned} &M^2(Px, Qy, kt) \cdot [M(Ax, Px, kt) * M(Sy, Qy, kt)] \\ &\quad \geq [pM(Ax, Px, t) + qM(Ax, Sy, t)] \cdot M(Ax, Qy, 2kt) \end{aligned}$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$;
(4) *either A or P is continuous;*
(5) *the pair (P, A) is compatible and (Q, S) is weakly compatible.*

Then A, S, P and Q have a unique common fixed point.

If we take $A = S, P = Q$ and $B = T = I_X$ in the main theorem, we have the following:

Corollary 2.3 *Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$ for all $t \in [0,1]$ and let A and P be compatible maps on X such that $P(X) \subseteq A(X)$.*

If A is continuous and there exists a constant $k \in (0,1)$ such that

$$M^2(Px, Py, kt) \cdot [M(Ax, Px, kt) * M(Ay, Py, kt)]$$
$$\geq [pM(Ax, Px, t) + qM(Ax, Ay, t)] \cdot M(Ax, Py, 2kt)$$

for all $x, y \in X$ and $t > 0$ where $0 < p, q < 1$ such that $p + q = 1$, then A and P have a unique fixed point.

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