"Vasile Alecsandri" University of Bacău
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CALCULUS WITH WEAK UPPER GRADIENTS BASED ON BANACH FUNCTION SPACES

MARCELINA MOCANU

Abstract. In this paper we extend some results regarding the properties of weak upper gradients, from the cases when $\mathbf{B}$ is an Orlicz space or a Lorentz space to the general case of a Banach function space. We provide methods to cut and paste $\mathbf{B}-$weak upper gradients and give extensions to the case of $\mathbf{B}-$weak upper gradients for the product rule and the chain rule. These results require no additional assumptions on the Banach function space $\mathbf{B}$. We also prove the existence of a norm minimizing $\mathbf{B}-$weak upper gradient for a function possessing at least one $\mathbf{B}-$weak upper gradient that belongs to $\mathbf{B}$, under the assumption that $\mathbf{B}$ is reflexive or $\mathbf{B}$ has an absolutely continuous norm. If in addition the norm of $\mathbf{B}$ is strictly monotone, it turns out that a norm minimizing $\mathbf{B}-$weak upper gradient of a function is also minimal pointwise $\mu-$almost everywhere among all the $\mathbf{B}-$weak upper gradients of that function.

1. Introduction

In this paper (X, d, μ) is a metric measure space, where d is a metric and μ is outer regular Borel measure, that is assumed to be positive and finite on balls. $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is a Banach function space corresponding to the measure space (X, μ).

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The extension of the first-order calculus to the setting of metric measure spaces with no a priori smooth structure is an active area of research since 1996. An important step in the development of this research area, motivated by the study on quasiconformal mappings defined on metric spaces, was the introduction in 1999 of the Newtonian spaces $N^{1,p}(X)$, $1 \leq p < \infty$ by Shanmugalingam [20], [21]. Together with Hajłasz-Sobolev spaces [7] and Cheeger type Sobolev spaces [4], the Newtonian spaces extend Sobolev spaces of order one to metric measure spaces. Roughly speaking, $N^{1,p}(X)$ is the space of functions $u \in L^p(X)$ admitting at least one upper gradient $g_u \in L^p(X)$.

Definition 1. *Let $u : X \to \mathbb{R}$. A Borel function $g : X \to [0, +\infty]$ is said to be an upper gradient of u in X if*

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds \tag{1.1}$$

whenever $\gamma : [a, b] \to X$ is a rectifiable curve.

An upper gradient of a function $u : X \to \mathbb{R}$ is a substitute for the length of the gradient of a real-valued function of class C^1 defined on a Riemannian manifold. Since upper gradients are nor stable under changes $\mu-$a.e. and under limits, the notion of upper gradient was generalized to the notion of $p-$weak upper gradient, which is more flexible. The notion of upper gradient was introduced by Heinonen and Koskela in [8, 2.9] and the notion of $p-$weak upper gradient was first defined by Koskela and MacManus [9].

In the definition of $p-$weak upper gradients the notion of $p-$modulus of a curve family plays an essential role. Note that a function $u : X \to \mathbb{R}$ has a $p-$integrable upper gradient if and only if it has a $p-$integrable $p-$weak upper gradient. If $\Omega \subset \mathbb{R}^n$ is a domain and $1 \leq p < \infty$, then $N^{1,p}(\Omega)$ agrees with the Sobolev space $W^{1,p}(\Omega)$, in the following sense: $N^{1,p}(\Omega)$ is continuously embedded in $W^{1,p}(\Omega)$ and every function $u \in W^{1,p}(\Omega)$ has a representative belonging to $N^{1,p}(\Omega)$ [6, Theorem 7.13]. The length of the distributional gradient of a function $u \in W^{1,p}(\Omega)$ is the least $p-$weak upper gradient of u [6, Corollary 7.15].

Two independent generalizations of Newtonian spaces are the Orlicz-Sobolev spaces introduced in 2004 by Aïssaoui [1] and Tuominen [23], on one hand, and the Sobolev-Lorentz spaces introduced in 2011 by Costea and Miranda [5], on the other hand. Orlicz-Sobolev spaces and Sobolev-Lorentz spaces are special cases of Newtonian spaces based on Banach function spaces introduced in [18]. Given a Banach function space $\mathbf{B}$ on X, the Newtonian space $N^{1,\mathbf{B}}(X)$ based on $\mathbf{B}$ contains the functions $u \in \mathbf{B}$ that possess at least one ($\mathbf{B}-$weak) upper gradient belonging to $\mathbf{B}$. Very recently, L. Malý began the study of a more general case, that of Newtonian spaces based on quasi-Banach function lattices [13].

In the study of Newtonian functions, the basic properties of their (weak) upper gradients play a fundamental role. Therefore, we need to extend some results regarding the properties of weak upper gradients, from the cases when $\mathbf{B}$ is an Orlicz space or a Lorentz space to the general case of Banach function spaces. We provide methods to cut and paste $\mathbf{B}-$weak upper gradients and give extensions to the case of $\mathbf{B}-$weak upper gradients for the product rule and the chain rule. These results require no additional assumptions on the Banach function space $\mathbf{B}$. We also prove the existence of a norm minimizing $\mathbf{B}-$weak upper gradient for a function possessing at least one $\mathbf{B}-$weak upper gradient that belongs to $\mathbf{B}$, under the assumption that $\mathbf{B}$ is reflexive or $\mathbf{B}$ has an absolutely continuous norm. If in addition the norm of $\mathbf{B}$ is strictly monotone, it turns out that a norm minimizing $\mathbf{B}-$weak upper gradient of a function is also minimal pointwise $\mu-$almost everywhere among all the $\mathbf{B}-$weak upper gradients of that function.

2. Preliminaries

The notion of Banach function space represents an axiomatic framework for the study of several spaces of measurable functions, such as Lebesgue spaces $L^p(X)$, $1 \leq p \leq \infty$, Orlicz spaces, Lorentz spaces, Marcinkiewicz spaces [2]. Some Lebesgue spaces with variable exponent $L^{p(\cdot)}$ are Banach function spaces [12].

Let (X, Σ, μ) be a complete $\sigma-$finite measure space and let $\boldsymbol{M}^+(X)$ be the collections of all measurable functions $f : X \to [0, +\infty]$.

Definition 2. [2] *A function $N : \boldsymbol{M}^+(X) \to [0, \infty]$ is called a Banach function norm if, for all f, g, f_n ($n \geq 1$) in $\boldsymbol{M}^+(X)$, for all constants $a \geq 0$ and for all measurable sets $E \subset X$, the following properties hold:*

(P1) $N(f) = 0 \Leftrightarrow f = 0$ $\mu-$a.e.; $N(af) = aN(f)$; $N(f+g) \leq N(f) + N(g)$.

(P2) If $0 \leq g \leq f$ $\mu-$a.e., then $N(g) \leq N(f)$.

(P3) If $0 \leq f_n \uparrow f$ $\mu-$a.e., then $N(f_n) \uparrow N(f)$.

(P4) If $\mu(E) < \infty$, then $N(\chi_E) < \infty$.

(P5) If $\mu(E) < \infty$, *then* $\int_E f \; d\mu \leq C_E N(f)$, *for some constant* $C_E \in (0, +\infty)$ *depending only on* E *and* ρ.

The collection $\mathbf{B}$ of the $\mu-$measurable functions $f : X \to [-\infty, +\infty]$ for which $N(|f|) < \infty$ is called a *Banach function space* on X. For $f \in \mathbf{B}$ define

$$\|f\|_{\mathbf{B}} = N(|f|).$$

Remark 1. *By (P5), every function in* $\mathbf{B}$ *is locally integrable, hence finite* $\mu-$*a.e. in* X, *since* μ *is* $\sigma-$*finite.*

Remark 2. *The Fatou property (P3) implies the lower semicontinuity of the norm* $\|\cdot\|_{\mathbf{B}}$ [2, Theorem 1.4, Theorem 1.7]*: if the sequence* $f_n \in \mathbf{B}$ $(n \geq 1)$ *satisfies the conditions* $f_n \to f$ $\mu-$*a.e. and* $\liminf\limits_{n\to\infty} \|f_n\|_{\mathbf{B}} < \infty$, *then* $f \in \mathbf{B}$ *and* $\|f\|_{\mathbf{B}} \leq \liminf\limits_{n\to\infty} \|f_n\|_{\mathbf{B}}$.

A fundamental fact is that $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is a complete normed space, see [2, Theorem 1.6].

The following special cases of Banach function spaces will be repeatedly invoked in this paper. $\mathbf{B} = L^{\Psi}(X)$ represents an Orlicz space endowed with the Luxemburg norm (or with the equivalent Orlicz norm), see [19], [23]. $\mathbf{B} = L^{p,q}(X)$, where $1 < p < \infty$ and $1 \leq q \leq \infty$, represents a Lorentz space endowed with the norm $\|\cdot\|_{(p,q)}$ (alternatively, with the equivalent norm $\|\cdot\|_{p,q}$ if $1 \leq q \leq p$), see [5]. Note that $\|\cdot\|_{p,q}$ is a quasinorm in the general case.

We mean by a curve in the metric space (X, d) any continuous mapping $\gamma : I \to X$, where $I \subset \mathbb{R}$ is an interval. The image of γ will be denoted as $|\gamma| := \gamma(I)$. A curve is said to be compact if it is defined on a compact interval $I = [a, b]$. Any restriction of a curve $\gamma : I \to X$ to a subinterval J of I is called a subcurve of γ.

The length of a compact curve $\gamma : [a, b] \to X$ is defined as $l(\gamma) = \sup \left\{ \sum\limits_{k=1}^{n} d(\gamma(t_k), \gamma(t_{k-1})) \right\}$, where the supremum is taken over all partitions $a = t_0 < t_1 < \ldots < t_n = b$. If the interval I is not compact, then the length of a curve $\gamma : I \to X$ is defined as the supremum of lengths of all compact subcurves of γ.

We say that a curve γ is rectifiable if $l(\gamma) < \infty$. A curve is said to be locally rectifiable if each of its compact subcurves is rectifiable.

We will denote by Γ_{rec} the family of all rectifiable compact curves in X.

For any compact rectifiable curve $\gamma : [a, b] \to X$ there is an associated *length function* $s_\gamma : [\, a, b] \to [0, l(\gamma)]$ defined by $s_\gamma(t) = l\left(\gamma|_{[a,t]}\right)$. We may extend this definition to the case when $\gamma : I \to X$ is rectifiable and I is not necessarily compact, so that $s_\gamma(t) = \sup\limits_{[a,t]\subset I} l\left(\gamma|_{[a,t]}\right)$. For each compact rectifiable curve $\gamma : [a, b] \to X$ there exists a unique curve $\widetilde{\gamma} : [0, l(\gamma)] \to X$ such that $\gamma = \widetilde{\gamma} \circ s_\gamma$, called the *arc length parameterization* of γ, which is also rectifiable and is $1-$Lipschitz. [6, Theorem 3.2]. This results holds also for rectifiable noncompact curves [8, p. 8]. A compact rectifiable curve $\gamma : [a, b] \to X$ will be said to be parameterized by arc length if $\widetilde{\gamma} = \gamma$, i.e. $a = 0$, $b = l(\gamma)$ and $s_\gamma(t) = t$ for all $t \in [a, b]$.

Let $\gamma : I \to X$ be a rectifiable curve and let $\rho : |\gamma| \to [0, +\infty]$ be a Borel measurable function. Then the line integral of ρ along γ is defined by $\int\limits_\gamma \rho ds = \int\limits_0^{l(\gamma)} \rho(\widetilde{\gamma}(t))\, dt$, where $\widetilde{\gamma}$ is the arc length parameterization of γ.

If $\gamma : I \to X$ is locally rectifiable, we set $\int\limits_\gamma \rho ds = \sup\limits_{\gamma'} \int\limits_{\gamma'} \rho ds$, where the supremum is taken over all the rectifiable subcurves γ' of γ. If γ is not locally rectifiable, no line integrals are defined.

Let Γ be a family of curves in X. A nonnegative Borel function $\rho : X \to [0, +\infty]$ is called an *admissible function* for Γ if

$$\int\limits_\gamma \rho ds \geq 1 \tag{2.1}$$

for all locally rectifiable curves $\gamma \in \Gamma$. Denote by $F(\Gamma)$ the family of all functions that are admissible for Γ.

Definition 3. [18, p. 255] *The* $\mathbf{B}-$*modulus of a family* Γ *of curves in* X *is*

$$M_{\mathbf{B}}(\Gamma) = \inf_{\rho \in F(\Gamma)} \|\rho\|_{\mathbf{B}}.$$

It is said that a property holds for $\mathbf{B}-$*almost every curve* if it holds for every curve except a family of curves of zero $\mathbf{B}-$modulus.

Note that the $\mathbf{B}-$modulus of the family of curves that are not rectifiable is zero.

Remark 3. *If* $\mathbf{B} = L^p(X)$, $1 \leq p < \infty$, *then the* $\mathbf{B}-$*modulus* $M_{\mathbf{B}}$ *and the* $p-$*modulus* Mod_p [21] *are related by* $M_{\mathbf{B}}(\Gamma) = (Mod_p(\Gamma))^{1/p}$.

If $\mathbf{B} = L^{\Psi}(X)$ *is an Orlicz space, then the* $\mathbf{B}-$*modulus* $M_{\mathbf{B}}$ *coincides with the* $\Psi-$*modulus* Mod_{Ψ} [23], [1].

Remark 4. *In the case* $\mathbf{B} = L^{p,q}(X)$ *the* $p,q-$*modulus of a curve family was introduced in* [5] *by* $Mod_{p,q}(\Gamma) = \inf_{\rho \in F(\Gamma)} \|\rho\|^{p}_{\mathbf{L}^{p,q}(X)}$, *where* $1 < p < \infty$ *and* $1 \leq q \leq \infty$. *If* $1 \leq q \leq p$ *we have two choices:* $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{L}^{p,q}(X)}$, *in which case* $M_{\mathbf{B}}(\Gamma) = (Mod_{p,q}(\Gamma))^{1/p}$ *or* $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{L}^{(p,q)}(X)}$. *If* $1 < p < q \leq \infty$ *we have to take* $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{L}^{(p,q)}(X)}$. *In the cases when* $\|\cdot\|_{\mathbf{B}} = \|\cdot\|_{\mathbf{L}^{(p,q)}(X)}$ *we have* $(Mod_{p,q}(\Gamma))^{1/p} \leq M_{\mathbf{B}}(\Gamma) \leq \frac{p}{p-1}(Mod_{p,q}(\Gamma))^{1/p}$ [5, p. 3]. *Note that in all cases* $M_{\mathbf{B}}(\Gamma)$ *and* $Mod_{p,q}(\Gamma)$ *are simultaneously equal to zero.*

The following basic properties of the $\mathbf{B}-$modulus have been proved in [18] as natural extensions of the corresponding properties known in the case when $\mathbf{B}$ is an Orlicz space.

Lemma 1. [18, Proposition 1]

(a) The $\mathbf{B}-$*modulus is an outer measure on the collection of all curves in* X.

(b) $M_{\mathbf{B}}(\Gamma) = 0$ *if and only if there is a non-negative Borel function* $\rho \in \mathbf{B}$ *so that* $\int_{\gamma} \rho \, ds = \infty$ *for all* $\gamma \in \Gamma$. *In particular, if* $\rho \in \mathbf{B}$ *is a non-negative Borel function,* $\int_{\gamma} \rho \, ds < \infty$ *for* $\mathbf{B}-$*almost every* $\gamma \in \Gamma_{rec}$.

(c) If $\mu(E) = 0$, *then* $M_{\mathbf{B}}(\Gamma_i) = 0$ *for* $i = 1, 2$, *where* Γ_1 *(*Γ_2*) is the family of all rectifiable curves* γ *with* $L^1(\gamma^{-1}(E)) > 0$ *(respectively,* $H^1(|\gamma| \cap E) > 0$*).*

(d) (Fuglede's Lemma) If g_i $(i \in \mathbb{N})$ *and* g *are Borel functions in* $\mathbf{B}$ *such that* $\lim_{i\to\infty} \|g_i - g\|_{\mathbf{B}} = 0$, *then there is a subsequence* $(g_{i_k})_k$ *such that* $\lim_{k\to\infty} \int_{\gamma} |g_{i_k} - g| \, ds = 0$ *for all* $\mathbf{B}-$*almost every* $\gamma \in \Gamma_{rec}$.

Remark 5. *If* Γ_1 *and* Γ_2 *are curve families such that every curve* $\gamma_1 \in \Gamma_1$ *has a subcurve* $\gamma_2 \in \Gamma_2$, *then* $M_{\mathbf{B}}(\Gamma_1) \leq M_{\mathbf{B}}(\Gamma_2)$, *since every function admissible for* Γ_2 *is also admissible for* Γ_2.

The definition of a $\mathbf{B}-$weak upper gradient is obtained from that of an upper gradient, by admitting some exceptions for the condition (1.1), namely this condition holds for $\mathbf{B}-$almost every compact rectifiable curve.

Definition 4. *Let $u : X \to \mathbb{R}$. A Borel function $g : X \to [0, +\infty]$ is said to be a $\mathbf{B}-$weak upper gradient of u in X if the inequality*

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\, ds$$

holds for every $\mathbf{B}-$almost every rectifiable curve $\gamma : [a, b] \to X$.

Taking account of Remark 3 it follows that the notion of $\mathbf{B}-$weak upper gradient is a generalization of the notions of $\Psi-$weak upper gradient [23, Definition 4.1] and $p-$weak upper gradient [21, Definition 2.3].

The concepts of upper gradient and $\mathbf{B}-$weak upper gradient can be extended to functions $u : X \to [-\infty, +\infty]$. In this case, it is said that (1.1) holds if its right-hand side is infinite whenever its left-hand side is infinite or is not well-defined. This extended concept of upper gradient was used in [4], while the extended concept of $\mathbf{B}-$weak upper gradient for $\mathbf{B} = L^{p,q}(X)$ a Lorentz space was introduced and studied in [5], under the name of p, q-upper gradient. Note that for a function $u : X \to \mathbb{R}$ a Borel function $g : X \to [0, +\infty]$ is a $\mathbf{B}-$weak upper gradient with $\mathbf{B} = L^{p,q}(X)$ if and only if g is a $p, q-$weak upper gradient of u, by Remark 4.

In the following we will deal only with $\mathbf{B}-$weak upper gradients of real-valued functions.

Let us mention some elementary properties of $\mathbf{B}-$weak upper gradients, that follow by the triangle inequality.

Remark 6. *If g_k is a $\mathbf{B}-$weak upper gradient of $u_k : X \to \mathbb{R}$, $k = 1, .., n$ and λ_k,$k = 1, .., n$ are real constants, then $\sum\limits_{k=1}^{n} |\lambda_k| g_k$ is a $\mathbf{B}-$weak upper gradient of $\sum\limits_{k=1}^{n} \lambda_k u_k$.*

Remark 7. *The set of the functions $u : X \to \mathbb{R}$ having at least one $\mathbf{B}-$weak upper gradient is a lattice. If g_k is a $\mathbf{B}-$weak upper gradient of $u_k : X \to \mathbb{R}$, $k = 1, 2$, then $g := \max(g_1, g_2)$ is a $\mathbf{B}-$weak upper gradient of $\max(u_1, u_2)$ and $\min(u_1, u_2)$. See [18, p.257].*

We recall some basic properties of $\mathbf{B}-$weak upper gradients.

Lemma 2. [18, Proposition 2] *(a) For every $\mathbf{B}-$weak upper gradient $g \in \mathbf{B}$ of a function $u : X \to \mathbb{R}$ there is a decreasing sequence $(g_i)_{i\geq 1}$ of upper gradients of u such that $\lim\limits_{i\to\infty} \|g_i - g\|_{\mathbf{B}} = 0$.*

(b) If $(g_i)_{i\geq 1}$ is a decreasing sequence of $\mathbf{B}-$*weak upper gradients of* $u : X \to \mathbb{R}$*, then* $g := \lim\limits_{i\to\infty} g_i$ *is a* $\mathbf{B}-$*weak upper gradient. of* u

(c) If $(u_i)_{i\geq 1}$ is a sequence of measurable functions with corresponding $\mathbf{B}-$*weak upper gradients* $(g_i)_{i\geq 1}$ *and if* $u := \sup\limits_{i\geq 1} u_i$ *is finite* $\mu-$*a.e., then* $g := \sup\limits_{i\geq 1} g_i$ *is a* $\mathbf{B}-$*weak upper gradient of* u.

(d) Each function that is $\mu-$*a.e. equal with a* $\mathbf{B}-$*weak upper gradient of a function* $u : X \to \mathbb{R}$ *is also a* $\mathbf{B}-$*weak upper gradient of* u.

(e) Assume that for each $i \geq 1$, $u_i : X \to \mathbb{R}$ *is a measurable function with a* $\mathbf{B}-$*weak upper gradient* $g_i \in \mathbf{B}$. *Let* $E \subset X$ *be a set such that* $M_{\mathbf{B}}(\Gamma_E) = 0$. *If there exist a measurable function* $u : X \to \mathbb{R}$ *and a Borel function* $g : X \to [0, +\infty]$ *such that* $\lim\limits_{i\to\infty} u_i(x) = u(x)$ *for all* $x \in X \setminus E$ *and* $\lim\limits_{i\to\infty} g_i = g$ *in* $\mathbf{B}$, *then* g *is a* $\mathbf{B}-$*weak upper gradient of* u.

We denoted by Γ_E the family of curves $\gamma \in \Gamma_{rec}$ for which $|\gamma| \cap E \neq \varnothing$.

Definition 5. *Let* $u : X \to \mathbb{R}$. *The function* u *is said to be absolutely continuous on a compact rectifiable curve* γ *if* $u \circ \widetilde{\gamma} : [0, l(\gamma)] \to \mathbb{R}$ *is absolutely continuous. The function* u *is said to be absolutely continuous on* $\mathbf{B}-$*almost every curve if there exists a family* $\Gamma_0 \subset \Gamma_{rec}$ *with* $M_{\mathbf{B}}(\Gamma_0) = 0$, *such that* u *is absolutely continuous on each curve* $\gamma \in \Gamma_{rec} \setminus \Gamma_0$.

We will denote by $ACC_{\mathbf{B}}(X)$ the family of all functions $u : X \to \mathbb{R}$ that are absolutely continuous on $\mathbf{B}-$almost every curve. If $\mathbf{B} = L^p(X)$ with $1 \leq p < \infty$ ($\mathbf{B} = L^\Psi(X)$ is an Orlicz space) we have $u \in ACC_{\mathbf{B}}(X)$ if and only if u has the ACC_p property [21, Definition 2.2] (respectively, $u \in ACC_\Psi(X)$ [23, 4.1]).

Assume that $u : X \to \mathbb{R}$ has a $\mathbf{B}-$weak upper gradient $g \in \mathbf{B}$ in X. The family $\Gamma_0 \subset \Gamma_{rec}$ of curves γ for which $\int\limits_\gamma g\, ds = \infty$ has zero $\mathbf{B}-$modulus, by Lemma 1 (b). Let $\Gamma_1 \subset \Gamma_{rec}$ be the family of curves having at least one subcurve γ for which the inequality (1.1) is not satisfied. By the definition of a $\mathbf{B}-$weak upper gradient and by Remark 5, Γ_1 has zero $\mathbf{B}-$modulus. Then $M_{\mathbf{B}}(\Gamma_0 \cup \Gamma_1) \leq M_{\mathbf{B}}(\Gamma_0) + M_{\mathbf{B}}(\Gamma_1) = 0$. Using the absolute continuity of the integral $\int\limits_\gamma g\, ds$, one proves that u is absolutely continuous on each $\gamma \in \Gamma_{rec} \setminus (\Gamma_0 \cup \Gamma_1)$. We obtain the following

Lemma 3. *If $u : X \to \mathbb{R}$ has a $\mathbf{B}-$weak upper gradient $g \in \mathbf{B}$ in X, then $u \in ACC_{\mathbf{B}}(X)$.*

The following characterization of $\mathbf{B}-$weak upper gradients extends from the case $\mathbf{B} = L^p(X)$, $1 \le p < \infty$ to the general case (see Lemmas 3.1 and 3.3 from [14, Lemmas 3.1 and 3.3]).

Lemma 4. [17, Lemma 2] *Let $u : X \to \mathbb{R}$ and let $g \in \mathbf{B}$ be a Borel measurable nonnegative function. For each compact rectifiable curve γ parameterized by arc length define $h(s) = u(\gamma(s))$, $s \in [0, l(\gamma)]$.*

a) Assume that for $\mathbf{B}-$almost every curve $\gamma \in \Gamma_{rec}$ the function h is absolutely continuous on $[0, l(\gamma)]$ and

$$|h'(s)| \le g(\gamma(s)) \text{ for almost every } s \in [0, l(\gamma)]. \tag{2.2}$$

Then g is a $\mathbf{B}-$weak upper gradient of u.

b) Conversely, if g is a $\mathbf{B}-$weak upper gradient of u, then (2.2) holds for $\mathbf{B}-$almost every curve $\gamma \in \Gamma_{rec}$.

3. Cutting and pasting weak upper gradients

The "cut and paste" results on weak upper gradients allow us to build new weak upper gradients from old ones.

Lemma 5. *Assume that $u_k : X \to \mathbb{R}$, $k \in \{1, 2, 3\}$, where $u_1 \in ACC_{\mathbf{B}}(X)$ and u_k has a $\mathbf{B}-$weak upper gradient $g_k \in \mathbf{B}$ in X for $k \in \{2, 3\}$. If $F \subset X$ is a Borel set such that $u_1|_F = u_2|_F$ and $u_1|_{X\setminus F} = u_3|_{X\setminus F}$, then the function $g_1 := g_2\chi_F + g_3\chi_{X\setminus F}$ is a $\mathbf{B}-$weak upper gradient of u_1 in X.*

Proof. We have $u_k \in ACC_{\mathbf{B}}(X)$ for $k \in \{2, 3\}$, since u_k has a $\mathbf{B}-$weak upper gradient that belongs to $\mathbf{B}$. Denote by Γ_k the family of curves on which u_k is not absolutely continuous, $k \in \{1, 2, 3\}$. Note that $M_{\mathbf{B}}(\Gamma_k) = 0$ for $k \in \{1, 2, 3\}$.

Let $\gamma \in \Gamma_{rec}\setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ parameterized by arc length. For each $k \in \{1, 2, 3\}$ there is a set $E_k \subset [0, l(\gamma)]$ with $\mathcal{L}^1(E_k) = 0$ such that the derivative $\frac{d}{ds}(u_k \circ \gamma)(s)$ is defined whenever $s \in [0, l(\gamma)] \setminus E_k$. Moreover, for $k \in \{2, 3\}$ we may assume by Lemma 4 (b) that $\left|\frac{d}{ds}(u_k \circ \gamma)(s)\right| \le g_k(\gamma(s))$ for all $s \in [0, l(\gamma)] \setminus E_k$.

Let $s_0 \in [0, l(\gamma)] \setminus (E_1 \cup E_2 \cup E_3)$. If s_0 is an interior point of $\gamma^{-1}(F) \cap [0, l(\gamma)]$ relative to $[0, l(\gamma)]$, then $\frac{d}{ds}(u_1 \circ \gamma)(s_0) =$

$\frac{d}{ds}(u_2 \circ \gamma)(s_0)$, hence $\left|\frac{d}{ds}(u_1 \circ \gamma)(s_0)\right| \leq g_2(\gamma(s_0)) = g_1(\gamma(s_0))$. Similarly, if s_0 is an interior point of $\gamma^{-1}(X \setminus F) \cap [0, l(\gamma)]$ relative to $[0, l(\gamma)]$, then $\frac{d}{ds}(u_1 \circ \gamma)(s_0) = \frac{d}{ds}(u_3 \circ \gamma)(s_0)$, hence $\left|\frac{d}{ds}(u_1 \circ \gamma)(s_0)\right| \leq g_3(\gamma(s_0)) = g_1(\gamma(s_0))$. Now consider the remaining case, when s_0 is a boundary point of both $\gamma^{-1}(F) \cap [0, l(\gamma)]$ and $\gamma^{-1}(X \setminus F) \cap [0, l(\gamma)]$ relative to $[0, l(\gamma)]$. The existence of the derivatives $\frac{d}{ds}(u_k \circ \gamma)(s_0)$, $k \in \{1, 2, 3\}$ implies $\frac{d}{ds}(u_1 \circ \gamma)(s_0) = \frac{d}{ds}(u_k \circ \gamma)(s_0)$ for $k \in \{2, 3\}$, hence $\left|\frac{d}{ds}(u_1 \circ \gamma)(s_0)\right| \leq \min\{g_2(\gamma(s_0)), g_3(\gamma(s_0))\} \leq g_1(\gamma(s_0))$.

We proved that $\left|\frac{d}{ds}(u_1 \circ \gamma)(s)\right| \leq g_1(\gamma(s))$ for all $s \in [0, l(\gamma)] \setminus (E_1 \cup E_2 \cup E_3)$. Note that $\mathcal{L}^1(E_1 \cup E_2 \cup E_3) = 0$ and $M_{\mathbf{B}}(\Gamma_1 \cup \Gamma_2 \cup \Gamma_3) = 0$. By Lemma 4 (a) it follows that g_1 is a $\mathbf{B}-$weak upper gradient of u_1 in X. $\square$

Corollary 1. *Assume that $g, h \in \mathbf{B}$ are two $\mathbf{B}-$weak upper gradients of a function $u : X \to \mathbb{R}$ in X and that $F \subset X$ is a Borel set. Then $\rho := g\chi_F + h\chi_{X \setminus F}$ is a $\mathbf{B}-$weak upper gradient of u in X.*

Proof. Since u has a $\mathbf{B}-$weak upper gradient that belongs to $\mathbf{B}$, we have $u \in ACC_{\mathbf{B}}(X)$, by Lemma 3. We apply Lemma 5 with $u_1 = u_2 = u_3 = u$, $g_2 = g$ and $g_3 = h$. $\square$

Corollary 2. *Assume that $u : X \to \mathbb{R}$, $c_0 \in \mathbb{R}$ and $F \subset \{x \in X : u(x) = c_0\}$ is a Borel set. If u has a $\mathbf{B}-$weak upper gradient $g \in \mathbf{B}$ in X, then $g\chi_{X \setminus F}$ is also a $\mathbf{B}-$weak upper gradient of u in X.*

Proof. As in the above corollary, $u \in ACC_{\mathbf{B}}(X)$. We apply Lemma 5 with $u_1 = u_2 = u$, $u_3 \equiv c_0$, $g_2 = g$ and $g_3 \equiv 0$. It follows that $g_1 = g\chi_{X \setminus F}$ is a $\mathbf{B}-$weak upper gradient of $u_1 = u$ in X. $\square$

We also give a slight generalization of Corollary 2, as follows.

Corollary 3. *Assume that $F \subset X$ is a Borel set and $u : X \to \mathbb{R}$ is constant $\mu-$a.e. on F. If u has a $\mathbf{B}-$weak upper gradient $g \in \mathbf{B}$ in X, then $g\chi_{X \setminus F}$ is also a $\mathbf{B}-$weak upper gradient of u in X.*

Proof. Denote by c_0 the constant value of u on F, except a set of zero measure, namely $E = \{x \in F : u(x) \neq c_0\}$. By the Borel regularity of the measure μ, there exists a Borel set $E' \subset X$ such that $E \subset E'$ and $\mu(E') = \mu(E) = 0$. Then $F \setminus E'$ is a Borel set and $F \setminus E' \subset \{x \in X : u(x) = c_0\}$. By Corollary 2, a $\mathbf{B}-$weak upper gradient of u in X is $g_1' = g\chi_{(X \setminus F) \cup E'}$. Let $g_1 = g\chi_{X \setminus F}$. Since $\mu(E') = 0$, we have

$g_1 = g_1'$ $\mu-$a.e. in X, therefore g_1 is also a $\mathbf{B}-$weak upper gradient of u in X, by Lemma 2 (d). □

Remark 8. *Lemma 5 generalizes Lemma 4.11 from* [23]*. Corollary 1 was also proved as an independent result in* [17] *and generalizes Lemma 4.10 from* [23] *and Lemma 4.7 from* [5]*. Corollary 2 generalizes Corollary 4.12 from* [23] *and Corollary 3 generalizes Lemma 4.6 from* [5]*.*

Comparing Corollary 3 in the case $\mathbf{B} = L^p(X)$ with Lemma 4.3 from [21], it is natural to ask if we can remove the assumption $g \in \mathbf{B}$ from the statement of this corollary. The answer is positive and we arrive at the following stronger form of Corollary 3.

Lemma 6. *Assume that $F \subset X$ is a Borel set and that the function $u \in ACC_{\mathbf{B}}(X)$ is constant $\mu-$a.e. on F. If u has a $\mathbf{B}-$weak upper gradient g in X, then $g\chi_{X\setminus F}$ is also a $\mathbf{B}-$weak upper gradient of u in X.*

Proof. We use some techniques from the proof of Lemma 4.3 [21] and the proof of Lemma 4.6 [5]. Denote by c_0 the constant value of u on F, except a set of zero measure. As in the proof of Corollary 3, denote $E = \{x \in F : u(x) \neq c_0\}$ and let By the Borel regularity of the measure μ, there exists a Borel set $E' \subset X$ be a Borel set such that $E \subset E'$ and $\mu(E') = \mu(E) = 0$.

Denote by $\Gamma^+_{E'}$ the family of all rectifiable compact curves that intersect E' on a set of positive linear measure. By Lemma 1 (c), $M_{\mathbf{B}}\left(\Gamma^+_{E'}\right) = 0$.

Let Γ_0 be the family of all rectifiable curves $\gamma : [a,b] \to X$ on which u is not absolutely continuous or on which the inequality

$$|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\, ds \tag{3.1}$$

is not satisfied. By our assumptions $u \in ACC_{\mathbf{B}}(X)$ and g is a $\mathbf{B}-$weak upper gradient of u in X, $M_{\mathbf{B}}(\Gamma_0) = 0$. Denote by Γ_1 the family of all curves having a subcurve in Γ_0. Then every nonnegative Borel function which is admissible for Γ_0 is also admissible for Γ_1, therefore $M_{\mathbf{B}}(\Gamma_1) \leq M_{\mathbf{B}}(\Gamma_0) = 0$, i.e. $M_{\mathbf{B}}(\Gamma_1) = 0$.

Let $\gamma : [a,b] \to X$, $\gamma \in \Gamma_{rec} \setminus \left(\Gamma^+_{E'} \cup \Gamma_1\right)$. We shall analyze several cases.

Case I. If $\gamma(a), \gamma(b) \in F \setminus E'$, then $|u(\gamma(a)) - u(\gamma(b))| = |c_0 - c_0| \leq \int_\gamma g\chi_{X\setminus F}\, ds$.

Case II. If $\gamma([a,b]) \subset (X \setminus F) \cup E'$, then $|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\, ds$, since $\gamma \notin \Gamma_0$ and $\int_\gamma g\, ds = \int_\gamma g\chi_{X\setminus F}\, ds$, since $\gamma \notin \Gamma^+_{E'}$. We obtain $|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\chi_{X\setminus F}\, ds$.

Case III. Assume that $\gamma([a,b])$ is not included in $(X \setminus F) \cup E'$ and that $\{\gamma(a), \gamma(b)\}$ is not included in $F \setminus E'$. It suffices to assume that $\gamma(a) \in (X \setminus F) \cup E'$.

Consider the set $K := (u \circ \gamma)^{-1}(\{c_0\})$, that is nonempty by the assumption that $\gamma([a,b])$ has at least one point in $F \setminus E'$. Since $u \circ \gamma$ is continuous on $[a,b]$ [6, Lemma 7.3], K is relatively closed in $[a,b]$, therefore K is compact in $\mathbb{R}$. Then $a_0 := \inf K$ and $b_0 := \sup K$ belong to K.

Since $u(\gamma(a_0)) = u(\gamma(b_0)) = c_0$, we have by the triangle inequality $|u(\gamma(a)) - u(\gamma(b))| \leq |u(\gamma(a)) - u(\gamma(a_0))| + |u(\gamma(b_0)) - u(\gamma(b))|$.

Since $\gamma \notin \Gamma_1$, $|u(\gamma(a)) - u(\gamma(a_0))| \leq \int_{\gamma|_{[a,a_0]}} g\, ds$ and $|u(\gamma(b_0)) - u(\gamma(b))| \leq \int_{\gamma|_{[b_0,b]}} g\, ds$. Since $\gamma(t) \in (X \setminus F) \cup E'$ for $t \in [a, a_0) \cup (b_0, b]$, we have $\int_{\gamma|_{[a,a_0]}} g\, ds = \int_{\gamma|_{[a,a_0]}} g\chi_{(X\setminus F)\cup E'}\, ds$ and $\int_{\gamma|_{[b_0,b]}} g\, ds = \int_{\gamma|_{[b_0,b]}} g\chi_{(X\setminus F)\cup E'} ds$. Moreover, since $\gamma \notin \Gamma^+_{E'}$, each of the images of the restrictions $\gamma|_{[a,a_0]}$ and $\gamma|_{[b_0,b]}$ intersects E' only on a set of zero linear measure, therefore $\int_{\gamma|_{[a,a_0]}} g\chi_{(X\setminus F)\cup E'}\, ds = \int_{\gamma|_{[a,a_0]}} g\chi_{X\setminus F}\, ds$ and $\int_{\gamma|_{[b_0,b]}} g\chi_{(X\setminus F)\cup E'} ds = \int_{\gamma|_{[b_0,b]}} g\chi_{(X\setminus F)} ds$.

We conclude that

$$\begin{aligned} |u(\gamma(a)) - u(\gamma(b))| &\leq \int_{\gamma|_{[a,a_0]}} g\chi_{X\setminus F}\, ds + \int_{\gamma|_{[b_0,b]}} g\chi_{(X\setminus F)} ds \\ &\leq \int_\gamma g\chi_{X\setminus F}\, ds, \end{aligned}$$

hence $|u(\gamma(a)) - u(\gamma(b))| \leq \int_\gamma g\chi_{X\setminus F}\, ds$. □

Remark 9. *Lemma 4.3 from* [21] *follows by the above lemma, when* $\mathbf{B} = L^p(X)$, *F is a closed set and g is an upper gradient of u.*

4. Minimal weak upper gradient

For $u : X \to \mathbb{R}$ denote by G_u the family of all $\mathbf{B}-$weak upper gradients $g \in \mathbf{B}$ of u in X.

Lemma 7. *For every $u : X \to \mathbb{R}$ the set G_u is closed and convex in $\mathbf{B}$.*

Proof. If G_u is empty, there is nothing to prove. Assume that G_u is nonempty. By Remark 6, G_u is convex.

In order to prove that G_u is closed, we will show that for each sequence $(g_j)_{j\geq 1}$ in G_u, that is convergent in $\mathbf{B}$ to some function g, the limit g is a $\mathbf{B}-$weak upper gradient of u in X. By Fuglede's lemma (Lemma 1 (d)), there exists a subsequence $(g_{j_k})_{k\geq 1}$ such that $\lim\limits_{k\to\infty} \int\limits_\gamma |g_{j_k} - g|\, ds = 0$ for $\mathbf{B}-$a.e. rectifiable curve γ. Using a renumbering, we may assume that $\lim\limits_{j\to\infty} \int\limits_\gamma |g_j - g|\, ds = 0$ for all $\gamma \in \Gamma_{rec} \setminus \Gamma_0$, where $M_{\mathbf{B}}(\Gamma_0) = 0$.

For each $j \geq 1$ we find $\Gamma_j \subset \Gamma_{rec}$ a family of compact curves, with $M_{\mathbf{B}}(\Gamma_j) = 0$, such that

$$|u(x) - u(y)| \leq \int\limits_\gamma g_j\, ds$$

for all compact curves $\gamma \in \Gamma_{rec} \setminus \Gamma_j$ with endpoints x and y.

Let $\gamma \in \Gamma_{rec} \setminus \bigcup\limits_{j=0}^{\infty} \Gamma_j$ be a compact curve with endpoints x and y. Taking $j \to \infty$ in the above inequality, we get $|u(x) - u(y)| \leq \int\limits_\gamma g ds$. Since $M_{\mathbf{B}}\left(\bigcup\limits_{j=0}^{\infty} \Gamma_j\right) = 0$, this proves that g is a $\mathbf{B}-$weak upper gradient of u in X. $\square$

Note that we proved in [17, Lemma 4] that G_u is closed and convex in $\mathbf{B}$, but under the assumption $u \in \mathbf{B}$, using a version of Mazur's lemma [18, Theorem 1] that is not applicable here.

Proposition 1. *Assume that $\mathbf{B}$ is reflexive. Then for every $u : X \to \mathbb{R}$ for which G_u is nonempty, there exists $g_u \in G_u$ such that $\|g_u\|_{\mathbf{B}} = \inf\limits_{g\in G_u} \|g\|_{\mathbf{B}}$.*

Proof. Let $u : X \to \mathbb{R}$ be a function such that G_u is nonempty. By Lemma 7, G_u is closed and convex in $\mathbf{B}$.

As a consequence of James' characterization of reflexivity of Banach spaces, it follows that each nonempty closed convex subset of a reflexive Banach space has at least one element of smallest norm (see [3, Proposition 4]). Since $\mathbf{B}$ is reflexive, the nonempty closed convex subset G_u of $\mathbf{B}$ has an element g_u of smallest norm. □

Remark 10. *If a Banach space is not reflexive, then there exists at least one nonempty closed convex subset of the space that has no element of smallest norm. As an Erratum, let us note that we have to replace the assumption "$\mathbf{B}$ is strictly convex" by "$\mathbf{B}$ is reflexive" in the statements of Lemma 5 and Theorem 1 from* [17], *as it was hinted in* [13].

Remark 11. *An Orlicz space $L^{\Psi}(X)$ is reflexive if and only if Ψ and its complementary function are doubling (satisfy the Δ_2-condition). Note that the complementary function of a Young function Ψ is doubling if and only if Ψ satisfies a ∇_2-condition.*

The Lorentz space $L^{p,q}(X)$ is reflexive when $1 < q < \infty$, provided that the measure μ is nonatomic, see Costea and Miranda [5, p. 3], *Bennet and Sharpley* [2, Theorem IV.4.7 and Corollaries I.4.3 and IV.4.8].

The reflexivity of the Banach space $\mathbf{B}$ is a strong condition. Fortunately, we may replace the condition of reflexivity by the condition that $\mathbf{B}$ has absolutely continuous norm.

Definition 6. [2, Definition I.3.1] *A function f in a Banach function space $\mathbf{B}$ is said to have absolutely continuous norm if $\lim_{n\to\infty} \|f\chi_{E_n}\|_{\mathbf{B}} = 0$ for every sequence $(E_n)_{n\geq 1}$ of measurable functions satisfying the condition $\mu\left(\limsup_{n\to\infty} E_n\right) = 0$. If each function in $\mathbf{B}$ has absolutely continuous norm, then the space $\mathbf{B}$ itself is said to have absolutely continuous norm.*

Example 1. *Every Lebesgue space $L^p(X)$, $1 \leq p < \infty$ has absolutely continuous (AC) norm.*

If the Young function Ψ is doubling, then the Orlicz space $L^{\Psi}(X)$ has AC norm. Moreover, if Ψ is doubling, then for each $f \in L^{\Psi}(X)$ and every $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $\mu(A) < \delta(\varepsilon)$ implies $\|f\chi_A\|_B < \varepsilon$. See [**?**, Lemma 3] *for a proof.*

If $1 < p < \infty$ and $1 \leq q < \infty$, then the Lorentz space $L^{p,q}(X)$ has absolutely continuous norm, provided that the measure μ is $\sigma-$finite (see [5, p.16], [2, Proposition I.3.6], [2, Corollary IV.4.8]*).*

Proposition 2. *Assume that* $\mathbf{B}$ *has absolutely continuous norm. For every* $u : X \to \mathbb{R}$ *for which* G_u *is nonempty, there exists* $g_u \in G_u$ *such that* $\|g_u\|_{\mathbf{B}} = \inf_{g \in G_u} \|g\|_{\mathbf{B}}$.

Proof. Let $u : X \to \mathbb{R}$ be a function such that G_u is nonempty. By Lemma 7, G_u is closed and convex in $\mathbf{B}$.

We will use a technique of Hajłasz [6, Theorem 7.16]. Let $m := \inf_{g \in G_u} \|g\|_{\mathbf{B}}$. Consider a sequence of positive numbers $(\varepsilon_i)_{i \geq 1}$ converging to zero. Let $(g_i)_{i \geq 1}$ be a sequence in G_u such that $\|g_i\|_{\mathbf{B}} < m + \varepsilon_i$ for every $i \geq 1$. The sequence $(g_i)_{i \geq 1}$ will be modified in order to obtain a decreasing sequence $(\rho_i)_{i \geq 1}$ with $\rho_i \in \rho_i$ and $\|\rho_i\|_{\mathbf{B}} < m + 2\varepsilon_i$ for every $i \geq 1$. The sequence $(\rho_i)_{i \geq 1}$ is defined inductively, as follows. Set $\rho_1 = g_1$. Suppose that $\rho_1, ..., \rho_k$ have already been chosen such that $\rho_1(x) \geq ... \geq \rho_k(x)$ for $x \in X$ and $\|\rho_i\|_{\mathbf{B}} < m + 2\varepsilon_i$ for $i = 1, ..., k$. We will define ρ_k such that $\rho_k \in G_u$, $\rho_k(x) \geq \rho_{k+1}(x)$ for every $x \in X$ and $\|\rho_{k+1}\|_{\mathbf{B}} < m + 2\varepsilon_{k+1}$. Consider $E_k = \{x \in X : g_{k+1}(x) < \rho_k(x)\}$. Note that E_k is a Borel set, since g_{k+1} and ρ_k are Borel functions. By the Borel regularity of μ, for each $\delta_k > 0$ there exists a closed set $F_k = F_k(\delta_k) \subset E_k$ such that $\mu(E_k \setminus F_k) < \delta_k$. We will choose a convenient $\delta_k > 0$ later. Define

$$\rho_{k+1} = g_{k+1}\chi_{F_k} + \rho_k \chi_{X \setminus F_k}.$$

By Corollary 1, ρ_{k+1} is a $\mathbf{B}-$weak upper gradient of u in X. Obviously, $\rho_{k+1} \in \mathbf{B}$, hence $\rho_{k+1} \in G_u$. For $x \in F_k$ we have $\rho_{k+1}(x) = g_{k+1}(x) < \rho_k(x)$, while $x \in X \setminus F_k$ implies $\rho_{k+1}(x) = \rho_k(x)$. Then $\rho_{k+1}(x) \leq \rho_k(x)$ for every $x \in X$.

Let $h_k := g_{k+1}\chi_{(X \setminus E_k) \cup F_k} + \rho_k \chi_{E_k \setminus F_k}$. We see that $x \in X \setminus E_k \subset X \setminus F_k$ implies $\rho_{k+1}(x) = \rho_k(x) \leq g_{k+1}(x) = h_k(x)$, and $x \in F_k$ implies $\rho_{k+1}(x) = g_{k+1}(x) < h_k(x)$, while $x \in E_k \setminus F_k$ implies $\rho_{k+1}(x) = g_{k+1}(x) < \rho_k(x) = h_k(x)$. Then $\rho_{k+1} \leq h_k$ on X, hence $\|\rho_{k+1}\|_{\mathbf{B}} \leq \|h_k\|_{\mathbf{B}}$ by the monotonicity of the norm of $\mathbf{B}$.

By the triangle inequality and the monotonicity of the norm of $\mathbf{B}$,

$$\begin{aligned} \|h_k\|_{\mathbf{B}} &\leq \left\|g_{k+1}\chi_{(X \setminus E_k) \cup F_k}\right\|_{\mathbf{B}} + \left\|\rho_k \chi_{E_k \setminus F_k}\right\|_{\mathbf{B}} \leq \\ &\leq \|g_{k+1}\|_{\mathbf{B}} + \left\|\rho_k \chi_{E_k \setminus F_k}\right\|_{\mathbf{B}} \leq m + \varepsilon_{k+1} + \left\|\rho_k \chi_{E_k \setminus F_k}\right\|_{\mathbf{B}}. \end{aligned} \tag{4.1}$$

Since $\mathbf{B}$ has absolutely continuous, for each $f \in \mathbf{B}$ and every $\varepsilon > 0$ there exists $\delta = \delta(f, \varepsilon) > 0$ such that $\mu(E) < \delta$ implies $\|f\chi_E\|_{\mathbf{B}} < \varepsilon$ [2, Lemma I. 3.4.]. We choose $\delta_k = \delta(\rho_k, \varepsilon_{k+1})$. Then $\left\|\rho_k \chi_{E_k \setminus F_k}\right\|_{\mathbf{B}} < \varepsilon_{k+1}$. Using this inequality and (4.1), we get $\|h_k\|_{\mathbf{B}} < m + 2\varepsilon_{k+1}$, hence $\|\rho_{k+1}\|_{\mathbf{B}} \leq m + 2\varepsilon_{k+1}$, q.e.d.

Since $(\rho_i)_{i\geq 1}$ is decreasing on X, there exists the pointwise limit $\rho := \lim_{i\to\infty} \rho_i$. Since $\liminf_{i\to\infty} \|\rho_i\|_{\mathbf{B}} < \infty$, it follows by the lower semicontinuity of the norm $\|\cdot\|_{\mathbf{B}}$ (see Remark 2) that $\rho \in \mathbf{B}$ and $\|\rho\|_{\mathbf{B}} \leq \liminf_{i\to\infty} \|\rho_i\|_{\mathbf{B}}$. But $m \leq \|\rho_i\|_{\mathbf{B}} < m + 2\varepsilon_i$ for every $i \geq 1$ and $\lim_{i\to\infty} \varepsilon_i = 0$ implies $\lim_{i\to\infty} \|\rho_i\|_{\mathbf{B}} = m$. Then $\|\rho\|_{\mathbf{B}} \leq m$.

On the other hand, by Lemma 2 (b), ρ is a $\mathbf{B}-$weak upper gradient of u in X, as a pointwise limit of a decreasing sequence of $\mathbf{B}-$weak upper gradients of u. Then $\rho \in G_u$ and consequently $\|\rho\|_{\mathbf{B}} \geq m$.

Now the claim follows for $g_u := \rho$. □

Definition 7. *Let $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ be a Banach function space. The norm $\|\cdot\|_{\mathbf{B}}$ is said to be strictly monotone if for every $f, g \in \mathbf{B}$ with $0 \leq g \leq f$ $\mu-$almost everywhere. in X, $\|g\|_{\mathbf{B}} = \|f\|_{\mathbf{B}}$ implies $g = f$ $\mu-$almost everywhere.*

Example 2. *If the Young function $\Psi : [0, \infty) \to [0, \infty)$ is strictly increasing and doubling, then the Luxemburg norm of the Orlicz space $L^{\Psi}(X)$ is strictly monotone (see* [17, p. 127]*)*

If $1 < p < \infty$ and $1 \leq q < \infty$, then the Lorentz space $L^{p,q}(X)$ has strictly monotone norm (see [5, p.16], [11, Proposition 2.1]*)*

Lemma 8. *Assume that $\mathbf{B}$ has a strictly monotone norm. Let $u : X \to \mathbb{R}$. If there exists $g_u \in G_u$ such that $\|g_u\|_{\mathbf{B}} = \inf_{g\in G_u} \|g\|_{\mathbf{B}}$, then $g_u(x) \leq g(x)$ for $\mu-$ a.e. $x \in X$, whenever $g \in G_u$.*

Note that the existence of $g_u \in G_u$ such that $g_u(x) \leq g(x)$ for $\mu-$a.e. $x \in X$ implies $\|g_u\|_{\mathbf{B}} = \inf_{g\in G_u} \|g\|_{\mathbf{B}}$.

Proof. Assume that there exists $g_u \in G_u$ such that $\|g_u\|_{\mathbf{B}} = \inf_{g\in G_u} \|g\|_{\mathbf{B}}$. Let $g \in G_u$. We have to prove that $F := \{x \in X : g_u(x) > g(x)\}$ is of measure zero. F is a Borel set, since g_u and g are Borel measurable functions. By Corollary 1, the function $\rho = g_u \chi_{X\setminus F} + g\chi_F$ belongs to G_u.

By the definition of g_u, we have $\|g_u\|_B \leq \|\rho\|_B$. By the definition of ρ, we have $\rho(x) \leq g_u(x)$ for every $x \in X$, hence $\|\rho\|_B \leq \|g_u\|_B$, by the monotonicity of the norm in a Banach function space. It follows that $\|\rho\|_B = \|g_u\|_B$.

Since the norm $\|\cdot\|_B$ is strictly monotone, from $\rho \leq g_u$ $\mu-$a.e. in X and $\|\rho\|_B = \|g_u\|_B$ we get $\rho = g_u$ $\mu-$a.e. in X. Then $(g - g_u)\chi_F = 0$ $\mu-$a.e. in X, but $g - g_u < 0$ on F, hence $\mu(F) = 0$, q.e.d. □

Remark 12. *Assume that* $\mathbf{B}$ *is reflexive or* $\mathbf{B}$ *has absolutely continuous norm. Assume also that* $\mathbf{B}$ *is strictly convex or* $\mathbf{B}$ *has strictly monotone norm. Then there exists a unique (up to a set of zero measure)* $g_u \in G_u$ *such that* $\|g_u\|_{\mathbf{B}} = \inf\limits_{g \in G_u} \|g\|_{\mathbf{B}}$.

The existence of g_u follows from Proposition 1 and Proposition 2. If $\mathbf{B}$ has a strictly monotone norm, the uniqueness (up to a set of zero measure) of g_u is a straightforward consequence of Lemma 8. Assume now that $\mathbf{B}$ is strictly convex, i.e. $\|f+g\| < 2$ whenever $f, g \in \mathbf{B}$ are distinct norm one elements. Denote $m := \inf\limits_{g \in G_u} \|g\|_{\mathbf{B}}$. Let $g_1, g_2 \in G_u$ such that $\|g_1\|_{\mathbf{B}} = \|g_2\|_{\mathbf{B}} = m$. If $m = 0$, then g_1 and g_2 are null $\mu-$a.e., hence $g_1 = g_2$ $\mu-$a.e in X. Suppose that $m > 0$. For every $\lambda \in [0,1]$, $\|(1-\lambda)\, g_1 + \lambda g_2\|_{\mathbf{B}} \leq m$ and $(1-\lambda)\, g_1 + \lambda g_2 \in G_u$, hence $\|(1-\lambda)\, g_1 + \lambda g_2\|_{\mathbf{B}} = m$. In particular, for $\lambda = \frac{1}{2}$ we get $\left\|\frac{1}{m} g_1 + \frac{1}{m} g_2\right\|_{\mathbf{B}} = 2$, Since $\left\|\frac{1}{m} g_1\right\|_{\mathbf{B}} = \left\|\frac{1}{m} g_2\right\|_{\mathbf{B}} = 1$, if $g_1 \neq g_2$ in $\mathbf{B}$, then we deduce by the strict convexity of $\mathbf{B}$ that $\left\|\frac{1}{m} g_1 + \frac{1}{m} g_2\right\|_{\mathbf{B}} < 2$, a contradiction. Then $g_1 = g_2$ in $\mathbf{B}$, hence $g_1 = g_2$ $\mu-$a.e in X.

By Proposition 1, Proposition 2 and Lemma 8 we obtain

Theorem 1. *Assume that* $\mathbf{B}$ *is reflexive or* $\mathbf{B}$ *has an absolutely continuous norm and that* $\mathbf{B}$ *has a strictly monotone norm. Then for every* $u : X \to \mathbb{R}$ *for which* G_u *is nonempty, there exists* $g_u \in G_u$ *such that* $g_u(x) \leq g(x)$ *for* $\mu-$ *a.e.* $x \in X$*, whenever* $g \in G_u$.

Remark 13. *In the case when* $\mathbf{B} = L^{\Psi}(X)$ *is an Orlicz space, with* Ψ *a doubling and strictly increasing Young function, Theorem 1 shows that every function in* $\mathbf{B}$ *admitting at least one* $\mathbf{B}-$*weak upper gradient* $g \in \mathbf{B}$ *has a norm minimizing* $\mathbf{B}-$*weak upper gradient* g_u*, which is also minimal pointwise* $\mu-$*almost everywhere. This was proved in* [23]*, see Theorem 6.6, Theorem 6.7, Lemma 6.8, Corollary 6.9 and Theorem 6.11.*

In the case when $\mathbf{B} = L^{p,q}(X)$ *is a Lorentz space, with* $1 < p < \infty$ *and* $1 \leq q < \infty$*, Theorem 1 implies Theorem 4.8 from* [5]*, in the case of real-valued functions.*

5. Product rule and chain rule

The following theorems provide counterparts of the product rule and of the chain rule for derivatives, extending some corresponding results of Costea and Miranda [5] from the case when $\mathbf{B}$ is a Lorentz space to the general case. We will show that the proofs of Lemma 6.7 and

Proposition 3.11 from [5] still work in the setting of general Banach function spaces.

Theorem 2. *Assume that $u_k : X \to \mathbb{R}$ is a Borel function which has a $\mathbf{B}-$weak upper gradient $g_k \in \mathbf{B}$ in X, for $k \in \{1,2\}$. Then the function $g := |u_1|\, g_2 + |u_2|\, g_1$ is a $\mathbf{B}-$weak upper gradient of $u := u_1 u_2$ in X and $g \in \mathbf{B}$.*

Proof. Since u_k and g_k are Borel functions for $k = 1, 2$, we see that g is a Borel nonnegative function. By the properties (P1) and (P2) of a Banach function norm, $g \in \mathbf{B}$.

Let $\varepsilon > 0$. Denote $h_\varepsilon := (|u_1| + \varepsilon)\, g_2 + (|u_2| + \varepsilon)\, g_1$. Using a technique of Cheeger [4, Lemma 1.7] we will prove that

$$|u(\gamma(0)) - u(\gamma(l(\gamma)))| \leq \int_\gamma h_\varepsilon\, ds < \infty, \tag{5.1}$$

for $\mathbf{B}-$almost every $\gamma \in \Gamma_{rec}$ parameterized by arc length.

Let $\Gamma_0 \subset \Gamma_{rec}$ be the family of curves γ for which $\int_\gamma (g_1 + g_2)\, ds = \infty$. Since $g_k \in \mathbf{B}$ for $k = 1, 2$ we have $M_{\mathbf{B}}(\Gamma_0) = 0$ by Proposition 1 (b). For $k = 1, 2$ let $\Gamma_k \subset \Gamma_{rec}$ be the family of curves γ parameterized by arc length for which the inequality $|u_k(\gamma(0)) - u_k(\gamma(l(\gamma)))| \leq \int_\gamma g_k\, ds$ is not satisfied. Since g_k is a $\mathbf{B}-$weak upper gradient of u_k, $M_{\mathbf{B}}(\Gamma_k) = 0$ for $k = 1, 2$. Let $\Gamma_3 \subset \Gamma_{rec}$ be the family of curves that have a subcurve in $\Gamma_1 \cup \Gamma_2$. Then $M_{\mathbf{B}}(\Gamma_3) \leq M_{\mathbf{B}}(\Gamma_1 \cup \Gamma_2) \leq M_{\mathbf{B}}(\Gamma_1) + M_{\mathbf{B}}(\Gamma_2) = 0$, hence $M_{\mathbf{B}}(\Gamma_3) = 0$. Note that $M_{\mathbf{B}}(\Gamma_0 \cup \Gamma_3) = 0$.

Fix $\gamma \in \Gamma_{rec} \setminus (\Gamma_0 \cup \Gamma_3)$ parameterized by arc length and denote $l = l(\gamma)$. Fix a positive integer n and let $t_i = i\frac{l}{n}$, $i = 0, 1, ..., n$. We have $|u(\gamma(t_i)) - u(\gamma(t_{i-1}))| \leq |u_1(\gamma(t_i))| \cdot |u_2(\gamma(t_i)) - u_2(\gamma(t_{i-1}))| + |u_2(\gamma(t_{i-1}))| \cdot |u_1(\gamma(t_i)) - u_1(\gamma(t_{i-1}))|$ for $i = 1, ..., n$. But $|u_k(\gamma(t_i)) - u_k(\gamma(t_{i-1}))| \leq \int_{t_{i-1}}^{t_i} g_k(\gamma(t))\, dt$ for $k = 1, 2$ and $i = 1, ..., n$. Then

$$|u(\gamma(t_i)) - u(\gamma(t_{i-1}))| \leq \int_{t_{i-1}}^{t_i} [|u_1(\gamma(t_i))|\, g_2(t) + |u_2(\gamma(t_{i-1}))|\, g_1(t)]\, dt$$

for $i = 1, ..., n$.

Summing the above inequalities over i we obtain
(5.2)

$$|u(\gamma(0)) - u(\gamma(l))| \le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left[|u_1(\gamma(t_i))| \, g_2(t) + |u_2(\gamma(t_{i-1}))| \, g_1(t) \right] dt.$$

Let $k \in \{1, 2\}$. Since $u_k \circ \gamma$ is absolutely continuous on $I := [0, l]$, it is uniformly continuous on I. There exists $\delta_k > 0$ such that $|u_k(\gamma(v)) - u(\gamma(w))| < \varepsilon$ whenever $|v - w| < \delta_k$, $v, w \in I$.

Choose n such that $\frac{l}{n} < \min\{\delta_1, \delta_2\}$. Then $|u_1(\gamma(t_i))| < |u_1(\gamma(t))| + \varepsilon$ and $|u_2(\gamma(t_{i-1}))| < |u_2(\gamma(t))| + \varepsilon$ for all $t \in [t_{i-1}, t_i]$, for $i = 1, ..., n$. Using these inequalities in (5.2) we obtain inequality (5.1).

Taking advantage of the integrability of $h_\varepsilon \circ \gamma$ and letting $\varepsilon \to 0$ we obtain, by Lebesgue's dominated convergence theorem, $|u(\gamma(0)) - u(\gamma(l))| \le \int_\gamma [|u_1| \, g_2 + |u_2| \, g_1] \, ds$ and the claim follows. □

We give another variant to the product rule, with the assumptions u_1 Borel, u_2 bounded and $g_k \in \mathbf{B}$, $k = 1, 2$ removed and a slightly weaker claim.

Proposition 3. *Assume that $u_k : X \to \mathbb{R}$ is has a $\mathbf{B}-$weak upper gradient g_k in X, for $k \in \{1, 2\}$, u_1 is bounded and u_2 is Borel. Then the function $g := \|u\|_\infty g_2 + |u_2| \, g_1$ is a $\mathbf{B}-$weak upper gradient of $u := u_1 u_2$ in X.*

Proof. We will use a technique of Shanmugalingam [22, Lemma 4.10], where a similar claim was proved for upper gradients. As in the proof of Theorem 2 let $\Gamma_k \subset \Gamma_{rec}$ be the family of curves γ parameterized by arc length for which the inequality $|u_k(\gamma(0)) - u_k(\gamma(l(\gamma)))| \le \int_\gamma g_k \, ds$ is not satisfied, for $k = 1, 2$. Let $\Gamma_3 \subset \Gamma_{rec}$ be the family of curves that have a subcurve in $\Gamma_1 \cup \Gamma_2$. Then $M_{\mathbf{B}}(\Gamma_3) = 0$.

Fix $\gamma \in \Gamma_{rec} \setminus \Gamma_3$ parameterized by arc length. Denote $x = \gamma(0)$ and $y = \gamma(l(\gamma))$. By the triangle inequality

$$|u_1(x)u_2(x) - u_1(y)u_2(y)| \le \|u_1\|_\infty \int_\gamma g_2 ds + |u_2(y)| \int_\gamma g_1 ds.$$

For each $z = \gamma(c)$, $c \in I := [0, l(\gamma)]$, denote by γ_{xz} and γ_{zy} the restrictions of γ to $[0, c]$ and $[c, l(\gamma)]$, respectively. Applying the above inequality for γ_{xz} and γ_{zy} we get

$$\begin{aligned}|u_1(x)u_2(x)-u_1(y)u_2(y)| &\le |u_1(x)u_2(x)-u_1(z)u_2(z)|+\\ &+ |u_1(z)u_2(z)-u_1(y)u_2(y)|\\ &\le \|u_1\|_\infty \int\limits_{\gamma_{xz}} g_2 ds+|u_2(z)|\int\limits_{\gamma_{xz}} g_1 ds+\\ &+ \|u_1\|_\infty \int\limits_{\gamma_{zy}} g_2 ds+|u_2(z)|\int\limits_{\gamma_{zy}} g_1 ds.\end{aligned}$$

Then $|u_1(x)u_2(x)-u_1(y)u_2(y)| \le \|u_1\|_\infty \int\limits_\gamma g_2 ds+|u_2(z)| \int\limits_\gamma g_1 ds$ for each $z \in \gamma(I) =: |\gamma|$, hence

$$|u_1(x)u_2(x)-u_1(y)u_2(y)| \le \int\limits_\gamma (\|u_1\|_\infty g_2+\inf_{z\in|\gamma|}|u_2(z)|\; g_1)ds.$$

This implies

$$|u_1(x)u_2(x)-u_1(y)u_2(y)| \le \int\limits_\gamma (\|u_1\|_\infty g_2+|u_2|\; g_1)ds,$$

which completes the proof. □

Theorem 3. *Let $F:\mathbb{R}\to\mathbb{R}$ be a C^1 function. If $u:X\to\mathbb{R}$ has a $\mathbf{B}-$weak upper gradient $g\in\mathbf{B}$ in X, then $|F'(u)|\,g$ is a $\mathbf{B}-$weak upper gradient of the function $F\circ u:X\to\mathbb{R}$.*

Proof. Let Γ_0 be the family of all rectifiable curves $\gamma:[0,l(\gamma)]\to X$ for which $\int\limits_\gamma g\,ds=\infty$. Then $M_{\mathbf{B}}(\Gamma_0)=0$, by Proposition 1. Let Γ_1 be the family of all rectifiable curves $\gamma:[0,l(\gamma)]\to X$ on which the inequality

$$|u(\gamma(a))-u(\gamma(b))|\le\int\limits_\gamma g\,ds$$

is not satisfied. Then $M_{\mathbf{B}}(\Gamma_1)=0$. Denoting by Γ_2 the family of all curves having a subcurve in Γ_1, we have $M_{\mathbf{B}}(\Gamma_2)=0$. Then $M_{\mathbf{B}}(\Gamma_0\cup\Gamma_2)=0$ and u is absolutely continuous on γ for each $\gamma\in\Gamma_{rec}\setminus(\Gamma_0\cup\Gamma_2)$.

Let $\gamma\in\Gamma_{rec}\setminus(\Gamma_0\cup\Gamma_2)$ a curve parameterized by arc length, $\gamma:[0,l]\to X$, where $l:=l(\gamma)$. For every $0\le t_1<t_2\le l$ we have $|u(\gamma(t_1))-u(\gamma(t_2))|\le\int\limits_{\gamma|_{[t_1,t_2]}} g\,ds=\int\limits_{t_1}^{t_2} g(\gamma(t))\,dt$. The function

$u\circ\gamma$ is absolutely continuous on $[a,b]$, in particular it is uniformly continuous. Note that F' is uniformly continuous on the compact interval $I := (u\circ\gamma)([0,l])$.

Fix a positive integer n. We consider the points $t_k := k\frac{l}{n}$, $k = 0,1,..,n$. We will estimate $|(F\circ u)(\gamma(0)) - (F\circ u)(\gamma(l))| \leq \sum\limits_{k=0}^{n-1} |(F\circ u)(\gamma(t_{k+1})) - (F\circ u)(\gamma(t_k))|$ in terms of a line integral involving $|F'(u)|\, g$.

By Lagrange's mean value theorem for F and the intermediate value theorem for $u\circ\gamma$, for each $k \in \{1,...,n\}$ there exists a point $\tau_k \in (t_{k-1},t_k)$ such that $(F\circ u)(\gamma(t_k)) - (F\circ u)(\gamma(t_{k-1})) = F'((u\circ\gamma)(\tau_k))\,[(u\circ\gamma)(t_k) - (u\circ\gamma)(t_{k-1})]$. Then

$$\begin{aligned}
&|(F\circ u)(\gamma(t_k)) - (F\circ u)(\gamma(t_{k-1}))| \leq \\
&\leq |F'((u\circ\gamma)(\tau_k))| \cdot |(u\circ\gamma)(t_k) - (u\circ\gamma)(t_{k-1})| \leq \\
&\leq |F'((u\circ\gamma)(\tau_k))| \int\limits_{t_{k-1}}^{t_k} g(\gamma(t))\,dt.
\end{aligned}$$

Since F' is uniformly continuous on I, for each $\varepsilon > 0$ there exists $\delta > 0$ such that $|F'(v) - F'(w)| < \varepsilon$ whenever $|v-w| < \delta$, $v,w \in I$. Since $u\circ\gamma$ is uniformly continuous on $[0,l]$, for each $\delta > 0$ there exists $\eta > 0$ such that $|(u\circ\gamma)(t) - (u\circ\gamma)(\tau)| < \delta$ whenever $|t-\tau| < \eta$, $t,\tau \in [0,l]$.

Fix $\varepsilon > 0$. Choose a positive integer n such that $\frac{l}{n} < \eta$. Then $|F'((u\circ\gamma)(\tau_k)) - F'((u\circ\gamma)(t))| < \varepsilon$ for all $t \in [t_{k-1},t_k]$, $k = 1,...,n$, hence $|F'((u\circ\gamma)(\tau_k))| \int\limits_{t_{k-1}}^{t_k} g(\gamma(t))\,dt \leq \int\limits_{t_{k-1}}^{t_k} (|F'((u\circ\gamma)(t))| + \varepsilon)\, g(\gamma(t))\,dt$, $k = 1,...,n$. Then

$$\begin{aligned}
&|(F\circ u)(\gamma(t_k)) - (F\circ u)(\gamma(t_{k-1}))| \leq \\
&\leq \int\limits_{t_{k-1}}^{t_k} (|F'((u\circ\gamma)(t))| + \varepsilon)\, g(\gamma(t))\,dt, k = 1,...,n.
\end{aligned}$$

Summing over k it follows that

$$\begin{aligned}
&|(F\circ u)(\gamma(0)) - (F\circ u)(\gamma(l))| \leq \\
&\leq \int\limits_0^l (|F'((u\circ\gamma)(t))| + \varepsilon)\, g(\gamma(t))\,dt,
\end{aligned} \tag{5.3}$$

for every $\varepsilon > 0$.

Since $\int_0^l g(\gamma(t))\,dt < \infty$ and $F' \circ u \circ \gamma$ is bounded on $[0, l]$, we have $\int_0^l (|F'((u \circ \gamma)(t))| + \varepsilon)\, g(\gamma(t))\,dt < \infty$ for every $\varepsilon > 0$. Letting $\varepsilon \to 0$ in (5.3) we get by Lebesgue's dominated convergence theorem the inequality $|(F \circ u)(\gamma(0)) - (F \circ u)(\gamma(l))| \leq \int_0^l |F'((u \circ \gamma)(t))|\, g(\gamma(t))\,dt$, which completes the proof. $\square$

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Marcelina Mocanu
Department of Mathematics and Informatics, Faculty of Sciences, “Vasile Alecsandri” University of Bacău, Calea Mărăşeşti 157, Bacău 600115, ROMANIA, e-mail: mmocanu@ub.ro