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A NOTE ON α-COMPACT FUZZY TOPOLOGICAL SPACES

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Abstract. It is widely accepted that one of the most satisfactory generalizations of the concept of compactness to fuzzy topological spaces is α-compactness, first initiated by Gantner et al.[3], followed by further investigations by many others. In this article, we propose to characterize the said notion of α-compactness in terms of ordinary nets and power-set filters, and this seems to be quite a new approach to the study of α-compactness.

1. Introduction

The concept of compactness for a fuzzy topological space (henceforth abbreviated as fts) has been introduced and studied by many mathematicians in different ways. The first among them was Chang [2] who proved only two results regarding his compact fts. This attempt has been followed by Goguen [4] who after pointing out a drawback in the definition of Chang, proved Alexander's Subbase Theorem, but could establish Tychonoff theorem only for finite products. Thereafter Wong [8], Weiss [7] and Lowen [5] treated compactness for fts's in different ways. But it is seen that all those approaches had many limitations. In 1978, Gantner et al. [3] initiated a new definition of fuzzy compactness, termed α-compactness, by which they could assign degrees to compactness and with this definition they finally generalized Tychonoff theorem and even 1-point compactification. It seems from the subsequent investigations that the theory of α-compactness by Gantner et al. is the most satisfactory one.

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This view has also been endorsed by Malghan and Benchalli [6] who also treated α-compactness vis-a-vis α-perfect maps in fts's and studied a version of local α-compactness weaker than that of Gantner et al. Again in [1], Benchalli and Siddapur studied and characterized α-compactness by means of the α-level topology.

In this note, we propose to characterize α-compactness in terms of ordinary nets and filters. This seems to be quite a new approach in as much as to our knowledge, in almost none of the theories concerning the investigations of numerous fuzzy topological concepts, ordinary nets and filters have been involved so far.

In what follows, by X we mean an fts [2], and to denote A to be a fuzzy set in X, we shall sometimes write $A \in I^X$, where $I = [0,1]$. For A, $B \in I^X$, we write $A \leq B$ if $A(x) \leq B(x)$, for each $x \in X$. For a family $\mathcal{U} = \{A_\alpha : \alpha \in \Lambda\}$ (here and henceforth also, Λ denotes an indexing set) of fuzzy sets in X, the union $\bigcup\limits_{\alpha \in \Lambda} A_\alpha$ and the intersection $\bigcap\limits_{\alpha \in \Lambda} A_\alpha$ are the fuzzy sets defined respectively by $(\bigcup\limits_{\alpha \in \Lambda} A_\alpha)(x) = \sup\limits_{\alpha \in \Lambda} A_\alpha(x)$ and $(\bigcap\limits_{\alpha \in \Lambda} A_\alpha)(x) = \inf\limits_{\alpha \in \Lambda} A_\alpha(x)$, for each $x \in X$ [9].

2. Main results

Let us start recalling from [3] the definitions of α-shading and α-compactness for an fts.

Definition 2.1. Let X be an fts. A collection $\mathcal{U} \subset I^X$ is called an α-shading of X, where $0 < \alpha < 1$, if for each $x \in X$ there exists some $U_x \in \mathcal{U}$ such that $U_x(x) > \alpha$. A subcollection $\mathcal{U}_0$ of an α-shading $\mathcal{U}$ of X, that is also an α-shading of X, is called an α-subshading of $\mathcal{U}$.

Remark 2.2. It is clear from the above definition that a collection $\mathcal{U}$ of fuzzy sets in an fts X is an α-shading iff $\sup\{U(x) : U \in \mathcal{U}\} > \alpha$, for each $x \in X$.

Definition 2.3. [3] An fts X is said to be α-compact if each α-shading of X by fuzzy open sets of X has a finite α-subshading.

The following definition and the next theorem are also given in [3] for an L-fuzzy space, in slightly different forms. We prefer to incorporate here, a complete proof of the said theorem in our setting.

Definition 2.4. A family $\{F_i : i \in \Lambda\}$ of fuzzy sets in an fts X is said to have α-finite intersection property (to be abbreviated as α-FIP) if for each finite subset Λ_0 of Λ, there is some $x \in X$ such that $\inf\limits_{i\in\Lambda_0} F_i(x) \geq 1-\alpha$.

Theorem 2.5. An fts X is α-compact iff for every family $\mathcal{F} = \{F_i : i \in \Lambda\}$ of fuzzy closed sets in X with α-FIP, there is some $x \in X$ such that $\inf\limits_{i\in\Lambda} F_i(x) \geq 1-\alpha$.

Proof. Let X be α-compact and let $\mathcal{F} = \{F_i : i \in \Lambda\}$ be a family of fuzzy closed sets in X with α-FIP. If possible, let for each $x \in X$, $(\bigcap\limits_{i\in\Lambda} F_i)(x) < 1-\alpha$, i.e., $\bigcup\limits_{i\in\Lambda}(1-F_i)(x) > \alpha$, for each $x \in X$. Hence $\mathcal{U} = \{1-F_i : i \in \Lambda\}$ is an α-shading of X by fuzzy open sets. By α-compactness of X, there exists a finite subset Λ_0 of Λ such that $\bigcup\limits_{i\in\Lambda_0}(1-F_i)(x) > \alpha$, for each $x \in X$, i.e., $(\bigcap\limits_{i\in\Lambda_0} F_i)(x) < 1-\alpha$, for each $x \in X$, so that $\mathcal{F}$ does not have α-FIP, a contradiction.

Conversely, let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be an α-shading of X by fuzzy open sets. Then $\mathcal{F} = \{1-U_i : i \in \Lambda\}$ is a family of fuzzy closed sets with $\bigcap\limits_{i\in\Lambda}(1-U_i) < 1-\alpha$. Then in view of the hypothesis, $\mathcal{F}$ cannot have α-FIP. Thus for a finite subset Λ_0 of Λ and for each $x \in X$, we have $1-(\bigcup\limits_{i\in\Lambda_0} U_i)(x) = \bigcap\limits_{i\in\Lambda_0}(1-U_i)(x) < 1-\alpha$, for each $x \in X$, i.e., $(\bigcup\limits_{i\in\Lambda_0} U_i)(x) > \alpha$. Hence X is α-compact.

We now define the concept of a sort of cluster points of ordinary nets and filters in an fts and ultimately use them to characterize α-compactness of an fts.

Definition 2.6. A point $x \in X$ is said to be an α-cluster point of an ordinary net $\{S_n : n \in (D, \leq)\}$ ($(D, \leq)$ being any directed set) in an fts X if for each fuzzy open set U with $U(x) > \alpha$ and for each $m \in D$, there exists a $k \in D$ such that $k \geq m$ in D and $U(S_k) > \alpha$.

Theorem 2.7. An fts X is α-compact iff every ordinary net in X has an α-cluster point in X.

Proof. Let X be α-compact. If possible, let there be a net $\{y_n : n \in (D, \leq)\}$ in X having no cluster point in X. Then for each $x \in X$, there exist a fuzzy open set U_x with $U_x(x) > \alpha$ and an $m_x \in D$ such that $U_x(y_n) \leq \alpha$, for all $n \geq m_x$ $(n \in D)$. Then $\mathcal{U} = \{U_x : x \in X\}$ is a collection of fuzzy open sets such that for any finite subcollection

$\{U_{x_1},...,U_{x_k}\}$ of $\mathcal{U}$, there exists $m \in D$ such that $m \geq m_{x_1},...,m_{x_k}$ and $(\bigcup_{i=1}^{k} U_{x_i})(y_n) \leq \alpha$ for all $n \geq m$ $(n \in D)$. This implies that $\inf_{1\leq i\leq k}(1-U_{x_i})(y_m) \geq 1-\alpha$. Then the collection $\mathcal{U}' = \{1-U_x : x \in X\}$ of fuzzy closed sets has α-FIP. By Theorem 2.5, there exists $y \in X$ such that $[\bigcap_{x\in X}(1-U_x)](y) \geq 1-\alpha$, i.e., $[\bigcup_{x\in X} U_x](y) \leq \alpha$, i.e., $U_x(y) \leq \alpha$, for all $U_x \in \mathcal{U}$. In particular, $U_y(y) \leq \alpha$, contradicting the definition of U_y. Hence the given set in X has an α-cluster point in X.

Conversely, let every net in X have an α-cluster point in X. Let $\mathcal{U} = \{U_i : i \in \Lambda\}$ be an arbitrary collection of fuzzy closed sets in X with α-FIP. Let Λ^* denote the collection of all finite subsets of Λ. Then Λ^* is a directed set, directed with the ordinary inclusion $\subseteq$. Put $F_\mu = \cap\{U_i : i \in \mu\}$ for each $\mu \in \Lambda^*$. As $\mathcal{U}$ has α-FIP, for each $\mu \in \Lambda^*$, there is a point x_μ (say) in X such that $\inf_{i\in\mu} U_i(x_\mu) \geq 1-\alpha$ (1). By virtue of Theorem 2.5 it is now only to be shown that $\inf_{i\in\Lambda} U_i(z) \geq 1-\alpha$ for some $z \in X$. If not, then $\inf_{i\in\Lambda} U_i(z) < 1-\alpha$, for each $z \in X$ (2).

Now, $S = \{x_\mu : \mu \in (\Lambda^*, \leq)\}$ is clearly a net in X, and hence by hypothesis, has an α-cluster point $y(\in X)$. Then by (2), $\inf_{i\in\Lambda} U_i(y) < 1-\alpha$, and hence there exists $i_0 \in \Lambda$ such that $U_{i_0}(y) < 1-\alpha$, i.e., $(1-U_{i_0})(y) > \alpha$. Since $\{i_0\} \in \Lambda^*$, there exists $\mu_0 \in \Lambda^*$ with $\mu_0 \geq \{i_0\}$ (i.e., $i_0 \in \mu_0$) such that $(1-U_{i_0})(x_{\mu_0}) > \alpha$. Then $U_{i_0}(x_{\mu_0}) < 1-\alpha$. As $i_0 \in \mu_0$, $\inf_{i\in\mu_0} U_i(x_{\mu_0}) \leq U_{i_0}(x_{\mu_0}) < 1-\alpha$, contradicting (1).

Definition 2.8. A point $x \in X$, where X is an fts, is said to be an α-cluster point of a filterbase $\mathcal{F}$ on X if for each fuzzy open set U with $U(x) > \alpha$ and for each $F \in \mathcal{F}$, there exists $x_F \in F$ such that $U(x_F) > \alpha$.

Theorem 2.9. An fts X is α-compact iff every filterbase $\mathcal{F}$ on X has an α-cluster point in X.

Proof. Let X be α-compact and let there exist, if possible, a filterbase $\mathcal{F}$ on X having no α-cluster point in X. Then for each $x \in X$, there exists a fuzzy open set U_x with $U_x(x) > \alpha$, and there exists an $F_x \in \mathcal{F}$ such that $U_x(y) \leq \alpha$ for each $y \in F_x$. Thus $\mathcal{U} = \{U_x : x \in X\}$ is an α-shading of X by fuzzy open sets. By α-compactness of X, there exist finitely many points $x_1, x_2, ..., x_n \in X$ such that $\mathcal{U}_0 = \{U_{x_1},...,U_{x_n}\}$ is again an α-shading of X. Now, let $F \in \mathcal{F}$ such that $F \subseteq F_{x_1} \bigcap F_{x_2} \bigcap ... \bigcap F_{x_n}$. Then $U_{x_i}(y) \leq \alpha$, for each $y \in F$ and for

$i = 1, 2, ..., n$. Thus $\mathcal{U}_0$ fails to be an α-shading of X, a contradiction. Conversely, let the condition hold and suppose, if possible, $\{y_n : n \in (D, \leq)\}$ be a net in X having no α-cluster point in X. Then for each $x \in X$, there is a fuzzy open set U_x with $U_x(x) > \alpha$ and there exists some $m_x \in D$ such that $U_x(y_n) \leq \alpha$, for all $n \geq m_x$ $(n \in D)$. Thus $\mathcal{B} = \{F_x : x \in X\}$, where $F_x = \{y_n : n \geq m_x\}$, is a subbase for a filterbase $\mathcal{F}$ on X, where $\mathcal{F}$ consists of all finite intersections of members of $\mathcal{B}$. By hypothesis, $\mathcal{F}$ has an α-cluster point $z \in X$. But there is a fuzzy open set U_z with $U_z(z) > \alpha$ and there exists some $m_z \in D$ such that $U_z(y_n) \leq \alpha$, for all $n \geq m_z$, i.e., for all $p \in F_z$ $(\in \mathcal{B} \subset \mathcal{F})$, $U_z(p) \leq \alpha$. Hence z cannot be an α-cluster point of the filterbase $\mathcal{F}$, a contradiction.

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References

[1] Benchalli, S. S. and Siddapur, G. P.; **On level spaces and fuzzy topological spaces**, Bull. Math. Anal. Appl. 1(No. 2) (2009), 57-65.

[2] Chang, C.L.; **Fuzzy topological spaces**, J. Math. Anal. Appl. 24 (1968), 182-190.

[3] Gantner, T.E., Steinlage, R.C. and Warren, R.H.; **Compactness in fuzzy topological spaces**, J. Math. Anal. Appl. 62(1978), 547-562.

[4] Goguen, J.A.; **The fuzzy Tychonoff theorem**, J. Math. Anal. Appl. 43 (1973), 734-741.

[5] Lowen, R.; **Topologies floues**, C.R. Acad. Sc. Paris (sèrie A) 278 (1974), 925-928.

[6] Malghan, S.R. and Benchalli, S.S.; **Open maps, closed maps and local compactness in fuzzy topological spaces**, J. Math. Anal. Appl. 99 (1984), 338-349.

[7] Weiss, M.D.; **Fixed points, separation and induced topologies for fuzzy sets**, J. Math. Anal. Appl. 50 (1975), 142-150.

[8] Wong, C.K.; **Product and quotient theorems**, J. Math. Anal. Appl. 45 (1974), 512-521.

[9] Zadeh, L.A.; **Fuzzy Sets**, Inform. Control 8 (1965), 338-353.

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