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ON THE LOCALLY MINKOWSKI GL^n SPACE

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Abstract. We continue the investigations of generalized Lagrangian mechanical system [5] with the study of the GL-metric when the Finsler space F^n is a locally Minkowski space. Moreover we replace the nonlinear connection with a new one.

1. Generalized Lagrangian mechanical systems

Definition 1.1. A GL^n-mechanical system is a triple $\sum = (M, g_{ij}, F_i)$ where g_{ij} is the fundamental tensor of the Finsler space F^n from [5] and $F_i(x, y)$ are the external forces.
We assume that the evolution equation of the GL^n-mechanical system $\sum$ are:

$$\frac{d^2x^i}{dt^2} + \gamma^i_{jk}\left(x, \frac{dx}{dt}\right)\frac{dx^j}{dt}\frac{dx^k}{dt} = \lambda^{ij}F_j\left(x, \frac{dx}{dt}\right)\left(1 - \frac{1}{n^2\left(x, \frac{dx}{dt}\right)}\right) \tag{1.1}$$

The reason for considering this system of second order differential equation is that it is determined only by the mechanical system $\sum$. But the evolution equation (1.1) give us a local dynamical system.

In the case when the Finsler fundamental function $F(x, y)$, the refractive index $n(x, y)$ and the external forces $F_i(x, y)$ are globally defined, then the dynamical system (1.1) can be characterized by a vector field S on the phase space TM.

We have

Theorem 1.1. *The following properties hold:*

1°. *There exists a semispray S on the manifold TM, which depends only on the mechanical system $\sum$.*

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2°. *The integral curves of S the solution curves of the evolution equations (1.1).*

Proof. 1°. A semispray S on TM given in a local coordinates (x^i, y^i) has the form: $S = y^i \frac{\partial}{\partial x^i} + S^i(x,y) \frac{\partial}{\partial y^i}$.

A change of local coordinates $(x^i, y^i) \to (\tilde{x}^i, \tilde{y}^i)$ on TM, changes (S^i) to $\left(\tilde{S}^i\right)$ such that $\tilde{S}^i = \frac{\partial \tilde{x}^i}{\partial x^j} S^j + \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} y^j y^k$.

Conversely, if sets of locally given functions $(S^i(x,y))$ verify this rule under a change of coordinates, then they define a semispray.

Consider the system of functions:

$$2G^i(x,y) = \gamma^i_{jk}(x,y)\, y^j y^k - \left(1 - n^2(x,y)\right) F^i(x,y) \tag{1.2}$$

where

$$F^i(x,y) = \gamma^{ij}(x,y)\, F_j(x,y) \tag{1.3}$$

Taking into account that $\left(\gamma^i_{jk}(x,y)\right)$ transform like the coefficients of a linear connection on M and $(F^i(x,y))$ transform like the components of a vector field on M, by a direct computation one finds that $-2G^i$ are the coefficients of a semispray S:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i} \tag{1.4}$$

Evidently, the coefficients G^i depend only on the mechanical system $\sum$.

2°. The integral curves of S are expressed by

$$\frac{dx^i}{dt} = y^i,\ \frac{dy^i}{dt} + 2G^i = 0 \tag{1.5}$$

By virtue of (1.2), this system of differential equations is equivalent to system (1.1).

This is an important result, since the geometrical theory of the mechanical system $\sum$ is the geometry of the triple (T, M, S). So, we have

Theorem 1.2. *The nonlinear connection N of the mechanical system $\sum$ has the coefficients*

$$N^i_j = \frac{\partial G^i}{\partial y^j} = \overset{o}{N}{}^i_j + 2u \frac{\partial u}{\partial y^j} F^i - \left(1 - u^2\right) \frac{\partial F^i}{\partial y^j} \tag{1.6}$$

Theorem 1.3. *The canonical metrical connection $CT(N)$ of the mechanical system $\sum$ has the coefficients*

$$L^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\delta g_{sk}}{\partial x^j} + \frac{\delta g_{js}}{\partial x^k} - \frac{\delta g_{jk}}{\delta x^s} \right) \tag{1.7}$$

$$C^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sk}}{\partial y^j} + \frac{\partial g_{js}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^s} \right)$$

where the operators $\frac{\delta}{\delta x^i}$ are

$$\frac{\delta}{\delta x_i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} = \frac{\overset{0}{\delta}}{\delta x^i} - \left[2u \frac{\partial u}{\partial y^i} F^j - \left(1 - u^2\right) \frac{\partial F^j}{\partial y^i} \right] \frac{\partial}{\partial y^j} \tag{1.8}$$

All important geometrical properties of system $\sum$ can be derived from the connections N and $CT(N)$.

2. Main result

We consider the case when F^n is a locally Minkowski space around every point $u = (x, y) \in T_0 M := TM/\{0\}$ there exists a system of local coordinates in which the function F does not depend on (x^i). In such a system of coordinates, the metric (γ_{ij}) depends only on (y^i), the Christoffel symbols vanish and so the coefficients of the nonlinear connection vanish, too.

If, moreover, one assumes that the refractive index does not depend on (x^i), that is $\frac{\partial n}{\partial x^i} = 0 \ \forall i = \overline{1, n}$ it follows that the Finsler-Synge metric [5] has the property $\frac{\partial g_{ij}}{\partial x^k} = 0$.

We give some properties of the space $GL^n = (M, g_{ij}(y))$ with

$$g_{ij}(y) = \gamma_{ij}(y) + \left(1 - \frac{1}{n^2(y)}\right) y_i y_j, \ y_i = \gamma_{ij}(y) y^j \tag{2.1}$$

Properties

1° The nonlinear connection N is integrable.

2° The autoparallel curves of N are solutions of the differential system $\frac{d^2 x^i}{dt^2} = 0$,

3° The refractive index is h-covariant constant, that is $\frac{\delta n}{\delta x^k} = 0$.

4° The canonical metrical connection $CT(N) = \left(L^i_{jk}, C^i_{jk}\right)$ has the coefficients

$$L^i_{jk} = 0, \ C^i_{jk} = \overset{o}{C}{}^i_{jk} + \overset{1}{C}{}^i_{jk} \tag{2.2}$$

with $\overset{o}{C}{}^i_{jk} = \frac{1}{4} \gamma^{ih} \frac{\partial^3 F^2}{\partial y^h \partial y^j \partial y^k}$.

5° The h-paths of the space GL^n are solutions of the differential system $\frac{d^2 x^i}{dt} = 0$, $\frac{dy^i}{dt} = 0$.

6° The v-paths of the space GL^n in a point $x_0 \in M$ are characterized by $x^i = x_0^i$, $\frac{d^2y^i}{dt^2} + \left[\overset{o}{C}{}^i_{jk}(y) + \overset{1}{C}{}^i_{jk}(y) \right] \frac{dy^i}{dt}\frac{dy^k}{dt} = 0$.

7° The Einstein equations of the space GL^n reduce to $S_{ij} - \frac{1}{2} S g_{ij} = x\breve{T}_{ij}$and $\underset{ij}{\overset{\vee}{T}} |_i = 0$.

Recall that we postulated that the GL-space with the Finsler-Synge metric is equipped with the nonlinear connection N with the coefficients N^i_j. This nonlinear connection does not depend on the refractive index $n(x, y)$.

In order to involve also the function $n(x, y)$ at the nonlinear level we replace N with a nonlinear connection N^* of local coefficients

$$N^{*i}_j = N^i_j - a(x, y)\, \delta^i_j \tag{2.3}$$

where $a(x, y) = 1 + \left(1 - \frac{1}{n^2(x,y)}\right) F^2$

Then we have

Theorem 2.1. *The autoparallel curves of N^* are solutions of the diferential system*

$$\frac{dy^i}{dt} + \gamma^i_{jk}(x, y)\, y^j y^k = a(x, y)\, y^i,\ y^i = \frac{dx^i}{dt} \tag{2.4}$$

Proof. The autoparallel curves of N^* are solutions of the differential system $\frac{\delta^* y^i}{dt} := \frac{dy^i}{dt} + N^{*i}_j \frac{dx^j}{dt} = 0,\ y^i = \frac{dx^i}{dt}$.

By (2.3) this reduces to (2.4).

Theorem 2.2. *The metrical N^*-linear connection $CT(N^*) = \left(L^{*i}_{jk}, C^{*i}_{jk}\right)$ has the coefficients*

$$L^{*i}_{jk} = L^i_{jk} + aC^i_{jk},\ C^{*i}_{jk} = C^i_{jk} \tag{2.5}$$

Proof. The said coefficients have the same form as in (1.7) with $\frac{\delta}{\delta x^i}$ replaced with $\frac{\delta^*}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{*i}_j \frac{\partial}{\partial y^k}$. The second formula in (2.5) is clear. The first follows by a direct computation using (2.3).

Now is again of interest the case $1 - \frac{1}{n^2} = \frac{1}{c^2}$.

Also, it is of interest to review the case when F^n is a locally Minkowski space when N was replaced by N^*. We have (assuming that n depends only on y, hence $a(x, y)$ becomes $a(y)$) the following

Properties

1° g_{ij} depends only on (y^i),

2° the nonlinear connection has the coefficients $N^{ki}_j = -a(y)\, \delta^i_j$, 3° N^* has the weak torsion $t^{*i}_{jk} : \frac{\partial N^{ki}_j}{\partial y^k} - \frac{\partial N^{ki}_k}{\partial y^j}$ of the form $t^{*i}_{jk} = \delta^i_k \frac{\partial a}{\partial y^j} - \delta^i_j \frac{\partial a}{\partial y^k}$,

4° The curvature takes the form $R^{*i}_{jk} = -a t^{*i}_{jk}$,

5° N^* is integrable if and only if $t^{*i}_{jk} = 0$,

6° $t^{*i}_{jk} = 0$ if and only if $\frac{\partial a}{\partial y^i} = 0 \Leftrightarrow y^i \frac{\partial u^2}{\partial y^i} = 2\left(1 - u^2\right)$.

7° The autoparallel curves of N^* are solutions of the differential system

$$\frac{dy^i}{dt} - a\left(y\right) y^i = 0,\ y^i = \frac{dx^i}{dt}$$

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