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A GENERAL FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS AND APPLICATIONS

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Abstract. In this paper a general fixed point theorem for two pairs of owc mappings satisfying an implicit relation using a generalization of the notion of distance introduced in [4], without triangle inequality and symmetry is proved. As application some results in quasi - metric and G - metric spaces are obtained. For two mappings we obtain some similar results to Theorems 2.1, 2.2, 2.3 [4].

1. Introduction

Let A and S be self mappings of a metric space (X, d). Jungck [21] defined A and S to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

A point $x \in X$ is called a coincidence point of A and S if $Sx = Ax$. We denote by $C(A, S)$ the set of all coincidence points of A and S. In [35], Pant defined A and S to be pointwise R - weakly commuting if for each $x \in X$, there exists $R > 0$ such that $d(SAx, ASx) \le Rd(Ax, Sx)$. It is proved in [36] that pointwise R - weakly commuting is equivalent with the commuting at coincidence points.

Definition 1.1. *A and S is said to be weakly compatible* [22] *if $ASu = SAu$ for every $u \in C(A, S)$.*

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Definition 1.2. *A and S are said to be occasionally weakly compatible (briefly owc)* [3] *if $ASu = SAu$ for some $u \in C(A, S)$.*

Remark 1.1. *If $C(A, S) \neq \emptyset$ and A and S are weakly compatible, then A and S are owc, but the converse is not true (Example* [3]*).*

Some fixed point theorems for owc mappings are proved in [2], [23], [39], [42] and in other papers.

In [23] and [39] some fixed point theorems for owc mappings on symmetric spaces are proved without using triangle inequality and symmetry.

Quite recently, Bhatt, Chandra and Sahu [4] obtained some common fixed point theorems for a pair of owc mappings on a set X together with the function $d : X \times X \to [0, \infty)$, without using the triangle inequality and symmetry.

The study of fixed points for mappings satisfying an implicit relation was initiated in [37], [38].

In this paper a general fixed point theorem for two pairs of owc mappings satisfying an implicit relation using the distance introduced in [4], without triangle inequality and symmetry, is proved. As application, some results in quasi - metric spaces and G - metric spaces are obtained. For two mappings we obtain results similar to Theorems 2.1, 2.2, 2.3 [4].

2. Preliminaries

First we recall the definitions of some generalizations of the notions of metric and symmetric spaces.

Definition 2.1. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is said to be a metric on X if for each $x, y, z \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$,

3) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called metric space.

Definition 2.2. *Let X be a nonempty set. A function $d : X \times X \to \mathbb{R}_+$ is said to be a symmetric on X if for each $x, y \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$.

The pair (X, d) is called symmetric space.

There exists a vast literature concerning fixed points in symmetric spaces.

Definition 2.3. *Let X be a nonempty set. A function $d : X\times X \to \mathbb{R}_+$ is said to be a quasi - metric on X [48] if for each $x, y, z \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) \leq d(x, z) + d(z, y)$.

The pair (X, d) is called quasi - metric space.

Some fixed point theorems in quasi - metric spaces are obtained in [18], [19], [20], [44], [45], [7], [26] and in other papers.

Definition 2.4. *Let X be a nonempty set and $s \geq 1$ be a real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be a b - metric on X [27] if for each $x, y, z \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$,

3) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

The pair (X, d) is called b - metric space.

Some fixed point theorems in b - metric spaces are proved in [8], [9], [10], [11], [34], [46] and in other papers.

Definition 2.5. *Let X be a nonempty set. A function $d : X\times X \to \mathbb{R}_+$ is said to be a generalized metric on X [5] if for each $x, y, z \in X$:*

1) $d(x, y) = 0$ *if and only if* $x = y$,

2) $d(x, y) = d(y, x)$,

3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, y)$.

The pair (X, d) is called generalized metric space.

Some fixed point theorems in generalized metric spaces are proved in [5], [12], [13], [14], [25], [17] and in other papers.

Remark 2.1. *In Definitions 2.1 - 2.5 the condition 1) is single common condition. In [4] the authors proved some fixed point theorems for two owc mappings on a nonempty set where $d : X \times X \to \mathbb{R}_+$ satisfies only condition 1).*

In the following we will call the class of functions which satisfy only condition 1) as a class of minimal condition (m.c) metric.

Definition 2.6. *Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}_+$ is said to be a minimal condition metric $m(x, y) = 0$ if and only if $x = y$,*

The pair (X, m) is said to be a minimal condition metric space (briefly mc metric space).

Remark 2.2. *1) By Definitions 2.1 - 2.5 it follows that metrics, symmetrics, quasi - metrics, b - metrics, generalized metrics are mc - metrics.*

2) The metric spaces, symmetric spaces, quasi - metric spaces, b - metric spaces, generalized metric spaces are all mc - metric spaces.

The following theorem is proved in [4].

Theorem 2.1. *Let* X *be a nonempty set and* $d : X \times X \to [0, \infty)$ *be a function satisfying the condition* $d(x, y) = 0$ *if and only if* $x = y$*. If* f *and* g *are owc self mappings on* X *and satisfy the following condition:*

$$d(fx, fy) \leq \phi(\max\{d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)\}) \tag{2.1}$$

for all $x, y \in X$*, where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies* $\phi(t) < t$*,* $\forall t > 0$*, then* f *and* g *have a unique common fixed point.*

Theorem 2.2. *Let* X *be a nonempty set and* $d : X \times X \to \mathbb{R}_+$ *be a function satisfying the condition* $d(x, y) = 0$ *if and only if* $x = y$*. If* f *and* g *are owc self mappings on* X *and satisfy the following condition:*

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Theorem 2.3. *Let* X *be a nonempty set and* $d : X \times X \to \mathbb{R}_+$ *be a function satisfying the condition* $d(x, y) = 0$ *if and only if* $x = y$*. If* f *and* g *are owc self mappings on* X *and satisfy the following condition:*

$$\begin{aligned} d(fx, fy) \leq\ & ad(gx, gy) + b\max\{d(fx, gx), d(fy, gy)\}+ \\ & +c\max\{d(gx, gy), d(gx, fx), d(gy, fy)\} \end{aligned} \tag{2.3}$$

for all $x, y \in X$*, where* $a, b, c \geq 0$ *and* $a + c < 1$*, then* f *and* g *have a unique common fixed point.*

3. Implicit relations

Definition 3.1. *Let* $\mathcal{F}_m$ *be the set of all real functions* $F(t_1, ..., t_6) : \mathbb{R}_+^6 \to \mathbb{R}$ *satisfying the condition:*

(F_m): $F(t, t, 0, 0, t, t) > 0$, $\forall t > 0$.

Example 3.1. $F(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_2, t_5, t_6\}$*, where* $a, b, c \geq 0$ *and* $a + c < 1$*.*

Example 3.2. $F(t_1, ..., t_6) = t_1 - k\max\{t_2, t_3, ..., t_6\}$*, where* $k \in (0, 1)$*.*

Example 3.3. $F(t_1, ..., t_6) = t_1 - \phi(\max\{t_2, t_3, ..., t_6\})$*, where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$*.*

Example 3.4. *$F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \frac{t_5+t_6}{2}\right\}$, where $k \in (0, 1)$.*

Example 3.5. *$F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\right\}$, where $k \in (0, 1)$.*

Example 3.6. *$F(t_1, ..., t_6) = t_1^2 - at_1(t_2+t_3+t_4) - bt_5t_6$, where $a, b \geq 0$ and $a + b < 1$.*

Example 3.7. *$F(t_1, ..., t_6) = t_1^2 - k \max\{t_2t_3, t_2t_4, t_3t_4, t_5t_6\}$, where $k \in (0, 1)$.*

Example 3.8. *$F(t_1, ..., t_6) = t_1 - k \max\{t_2, \sqrt{t_3t_4}, \sqrt{t_5t_6}$, where $k \in (0, 1)$.*

Example 3.9. *$F(t_1, ..., t_6) = t_1^2 - at_2^2 - bt_3t_4 - ct_5t_6$, where $a, b, c \geq 0$ and $a + c < 1$.*

Example 3.10. *$F(t_1, ..., t_6) = t_1^2 - at_2^2 - b\frac{t_3t_4}{1+t_5+t_6}$, where $a, b \geq 0$ and $a < 1$.*

Example 3.11. *$F(t_1, ..., t_6) = t_1 - k \max\left\{\frac{2t_3+t_4}{2}, \frac{2t_3+t_5}{2}, \frac{2t_3+t_6}{2}\right\}$, where $k \in (0, 1)$.*

Example 3.12. *$F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1-\alpha)(at_5+bt_6)$, where $0 < \alpha < 1$, $a, b \geq 0$ and $a + b < 1$.*

4. General fixed point theorem

Theorem 4.1. *Let (X, m) be a mc - metric space and f, g, S and T be self mappings of X such that*

$$F(m(fx, gy), m(Sx, Ty), m(fx, Sx), \\ m(gy; Ty), m(fx; Ty), m(Sx; gy)) \leq 0 \tag{4.1}$$

for all $x, y \in X$ with $fx \neq gy$, where $F \in \mathcal{F}_m$. If (f, S) and (g, T) are owc, then f, g, S and T have a unique common fixed point.

Proof. Since (f, S) and (g, T) are owc, there exists $x, y \in X$ such that $fx = Sx$, $gy = Ty$ and $fSx = Sfx$ and $gTy = Tgy$. First we prove that $fx = gy$. Suppose that $fx \neq gy$. Then by (4.1) we obtain

$$F(m(fx, gy), m(fx, gy), 0, 0, m(fx, Ty), m(Sx, gy)) \leq 0,$$

a contradiction of (F_m). Hence $fx = gy$ which implies $fx = Sx = gy = Ty$ and $f^2x = fSx = Sfx$. Next, we prove that $fx = f^2x$. Suppose that $f^2x \neq fx = gy$. Then by (4.1) we have successively

$$F(m(f^2x, gy), m(Sfx, gy), 0, 0, m(f^2x, Ty), m(Sfx, gy)) \leq 0,$$

$$F(m(f^2x, fx), m(Sfx, fx), 0, 0, m(f^2x, fx), m(f^2x, fx)) \leq 0,$$

a contradiction of (F_m). Hence $fx = f^2x$ and fx is a fixed point of f. Similarly, $g^2y = gy$. Therefore, $fx = f^2x = gy = g^2y = gfx$. Hence, fx is a fixed point of g.

On the other hand, $fx = f^2x = gy = g^2y = gTy = Tgy = Tfx$. Hence, fx is a fixed point of T.

Therefore, $w = fx$ is a common fixed point of f, g, S and T.

Suppose that $w' \neq w$ is another common fixed point of f, g, S and T. Then, by (4.1) we have successively

$$F(m(fw, gw'), m(Sw, Tw'), 0, 0, m(fw, Tw'), m(Sw, gw')) \leq 0,$$

$$F(m(w, w'), m(w, w'), 0, 0, m(w, w'), m(w, w')) \leq 0,$$

a contradiction of (F_m). Hence $w' = w$ and $w = fx$ is the unique common fixed point of f, g, S and T. □

Corollary 4.1. *Let (X, d) be a metric space (resp. symmetric space, quasi - metric space, b - metric space, generalized metric space) and f, g, S and T be self mappings of X such that the inequality (4.1) holds for all $x, y \in X$ with $fx \neq gy$, $m = d$ and $F \in \mathcal{F}_m$. Then f, g, S and T have a unique common fixed point.*

Remark 4.1. *If (X, d) is a symmetric space, by Corollary 4.1 we obtain the result of Theorem 4.1* [39].

By Examples 3.1 - 3.12 we obtain

Corollary 4.2. *Let (X, m) be a mc - metric space and f, g, S and T be self mappings satisfying one of the following inequalities for all $x, y \in X$ and $fx \neq gy$:*

1)

$$\begin{aligned} m(fx, gy) \leq\ & am(Sx, Ty) + b\max\{m(fx, Sx), m(gy, Ty)\} + \\ & + c\max\{m(Sx, Ty), m(fx, Ty), m(Sx, gy)\}, \end{aligned}$$

where $a, b, c \geq 0$ and $a + c < 1$,

2)

$$\begin{aligned} m(fx, gy) \leq\ & k\max\{m(Sx, Ty), m(fx, Sx), m(gy, Ty), \\ & m(fx, Ty), m(Sx, gy)\}, \end{aligned}$$

where $k \in (0, 1)$,

3)

$$\begin{aligned} m(fx, gy) \leq\ & \phi(\max\{m(Sx, Ty), m(fx, Sx), \\ & m(gy, Ty), m(fx, Ty), m(Sx, gy)\}), \end{aligned}$$

where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\phi(t) < t$, $\forall t > 0$,

4)

$$\begin{aligned} m(fx, gy) \leq\ & k \max\{m(Sx, Ty), m(fx, Sx), m(gy, Ty), \\ & \frac{1}{2}[m(fx, Ty) + m(Sx, gy)]\}, \end{aligned}$$

where $k \in (0, 1)$,

5)

$$\begin{aligned} m(fx, gy) \leq\ & k \max\{m(Sx, Ty), \frac{1}{2}[m(fx, Sx) + m(gy, Ty)], \\ & \frac{1}{2}[m(Sx, Ty) + m(Sx, gy)]\}, \end{aligned}$$

where $k \in (0, 1)$,

6)

$$\begin{aligned} m^2(fx, gy) \leq\ & am(fx, gy)[m(Sx, Ty) + m(fx, Sx) + m(gy, Ty)] + \\ & + bm(fx, Ty) \cdot m(Sx, gy), \end{aligned}$$

where $a, b \geq 0$ *and* $a + b < 1$,

7)

$$\begin{aligned} m^2(fx, gy) \leq\ & k \max\{m(Sx, Ty) \cdot m(fx, Sx), m(Sx, Ty) \cdot m(gy, Ty), \\ & m(fx, Sx) \cdot m(gy, Ty), m(fx, Ty) \cdot m(Sx, gy)\}, \end{aligned}$$

where $k \in (0, 1)$,

8)

$$\begin{aligned} m(fx, gy) \leq\ & k \max\{m(Sx, Ty), \sqrt{m(fx, Sx) \cdot m(gy, Ty)}, \\ & \sqrt{m(fx, Ty) \cdot m(Sx, gy)}\}, \end{aligned}$$

where $k \in (0, 1)$,

9)

$$\begin{aligned} m^2(fx, gy) \leq\ & am^2(Sx, Ty) + bm(fx, Sx) \cdot m(gy, Ty) + \\ & + cm(fx, Ty) \cdot m(Sx, gy), \end{aligned}$$

where $a, b, c \geq 0$ *and* $a + c < 1$,

10)

$$m^2(fx, gy) \leq am^2(Sx, Ty) + b\frac{m(fx, Sx) \cdot m(gy, Ty)}{1 + m(fx, Ty) + m(fx, gy)},$$

where $a, b \geq 0$ *and* $a < 1$,

11)

$$m(fx,gy) \leq k\max\{m(Sx,Ty), \frac{2m(fx,Sx)+m(gy,Ty)}{2}, \frac{2m(fx,Sx)+m(fx,Ty)}{2}, \frac{2m(gy,Ty)+m(Sx,gy)}{2}\},$$

where $k \in (0,1)$,

12)

$$m(fx,gy) \leq \alpha\max\{m(Sx,Ty), m(fx,Sx), m(gy,Ty)\} + +(1-\alpha)(am(fx,Ty)+bm(fx,gy)),$$

where $0 < \alpha < 1$, $a,b \geq 0$ and $a+b < 1$.

If (f,S) and (g,T) are owc, then f,g,S and T have a unique common fixed point.

If $f = g$ and $S = T$ by Theorem 4.1 we obtain

Theorem 4.2. *Let (X,m) be a mc - metric space and f,S be self mappings of X such that*

$$F(m(fx,fy), m(Sx,Sy), m(fx,Sx), m(fy,Sy), m(fx,Sy), m(Sx,fy)) \leq 0 \tag{4.2}$$

for all $x,y \in X$ with $fx \neq gy$ and $F \in \mathcal{F}_m$. If (f,S) are owc, then f and S have a unique common fixed point.

Remark 4.2. *By Corollary 4.2, 1), 2), 3) we obtain results similar to Theorems 2.1 - 2.3.*

5. Applications

a) Fixed point results for mappings satisfying a contractive condition of integral type

In [5], Branciari established the following point fix theorem, which opened the way to the study of mappings satisfying a contractive condition of integral type.

Theorem 5.1 (Branciari [5]). *Let (X,d) be a complete metric space, $c \in (0,1)$ and $f : (X,d) \to (X,d)$ be a mapping such that for each $x,y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \leq c\int_0^{d(x,y)} h(t)dt$$

where $h : [0,\infty) \to [0,\infty)$ is a Lebesgue measurable mapping which is summable (i.e. with finite integral) on each compact subset of $[0,\infty)$, *such that, for each* $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. *Then* f *has a unique fixed point* $z \in X$ *such that, for each* $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Some fixed point theorems for compatible weakly compatible and occasionally weakly compatible mappings satisfying contractive conditions of integral type are proved in [1], [24], [28], [41], [39], [42], [43], [47] and in other papers.

Let (X, d) be a metric space and $D(x, y) = \int_0^{d(x,y)} h(t)dt$, where $h(t)$ is as in Theorem 5.1. In [28] and [41] is proved that $D(x, y)$ is a symmetric on X and the study of some fixed point problems for mappings satisfying contractive conditions of integral type is reduced to the study of fixed point problems in symmetric spaces.

Let (X, d) be a metric space and (X, D) be the symmetric space defined by $D(x, y)$. Then

$$\begin{aligned} &D(Ax, By) = \int_0^{d(Ax,By)} h(t)dt, D(Sx, Ty) = \int_0^{d(Sx,Ty)} h(t)dt, \\ &D(Sx, Ax) = \int_0^{d(Sx,Ax)} h(t)dt, D(Ty, By) = \int_0^{d(Ty,By)} h(t)dt, \\ &D(Sx, By) = \int_0^{d(Sx,By)} h(t)dt, D(Ax, Ty) = \int_0^{d(Ax,Ty)} h(t)dt, \end{aligned} \tag{5.1}$$

where $h(t)$ is as in Theorem 5.1.

Theorem 5.2. *Let* (X, d) *be a metric space and* A, B, S *and* T *be self mappings of* X, *where* (A, S) *and* (B, T) *are owc such that*

$$\begin{aligned} &F(\int_0^{d(Ax,By)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sx,Ax)} h(t)dt, \\ &\int_0^{d(Ty,By)} h(t)dt, \int_0^{d(Sx,By)} h(t)dt, \int_0^{d(Ax,Ty)} h(t)dt) \leq 0, \end{aligned} \tag{5.2}$$

for all $x, y \in X$, *with* $Ax \neq By$, *where* $h(t)$ *is as in Theorem 5.1 and* $F \in \mathcal{F}_m$. *Then* A, B, S *and* T *have a unique common fixed point.*

Proof. By (5.1) and (5.2) we obtain

$$\begin{gathered} F(D(Ax, By), D(Sx, Ty), D(Sx, Ax), \\ D(Ty, By), D(Sx, By), D(Ax, Ty)) \leq 0, \end{gathered}$$

for all $x, y \in X$, with $Ax \neq By$ and $F \in \mathcal{F}_m$.

Hence, in the symmetric space (X, D) using Theorem 4.1 with $m(x, y) = D(x, y)$ it follows that A, B, S and T have a unique common fixed point. $\square$

By Corollary 4.2 (1) for $m(x, y) = D(x, y) = \int_0^{d(x,y)} h(t)dt$ we obtain

Corollary 5.1. *Let f, g, S and T self mapping of a metric space (X, d) and (A, S) and (B, T) are owc such that*

$$\begin{gathered}\int_0^{d(Ax,By)} h(t)dt \leq a\int_0^{d(Sx,Ty)} h(t)dt+ \\ +b\max\{\int_0^{d(fx,Sx)} h(t)dt, \int_0^{d(Ty,By)} h(t)dt,\} \\ +c\max\{\int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sx,By)} h(t)dt, \int_0^{d(Ax,Ty)} h(t)dt),\end{gathered}$$

where $a, b, c \geq 0$, $a + b < 1$ and $h(t)$ is as in Theorem 5.1.

Then, A, B, S and T have a unique common fixed point.

Remark 5.1. *By Corollary 4.2 (2) - 4.2 (12) we obtain results similar to Corollary 5.1.*

If $A = B$ and $S = T$, by Theorem 5.2 the following result is obtained

Theorem 5.3. *Let (X, d) be a metric space and A and S be self mappings of X, where (A, S) are owc such that*

$$\begin{gathered}F(\int_0^{d(Ax,Ay)} h(t)dt, \int_0^{d(Sx,Sy)} h(t)dt, \int_0^{d(Sx,Ax)} h(t)dt, \\ \int_0^{d(Sy,Ay)} h(t)dt, \int_0^{d(Sx,Ay)} h(t)dt, \int_0^{d(Ax,Sy)} h(t)dt) \leq 0,\end{gathered} \tag{5.3}$$

for all $x, y \in X$, with $Ax \neq Ay$, where $h(t)$ is as in Theorem 5.1 and $F \in \mathcal{F}_m$. Then A and S have a unique common fixed point.

Remark 5.2. *By Theorem 5.3 and Corollary 4.2 we obtain similar results for two mappings.*

b) Fixed point results for mappings in G - metric spaces

In [15], [16] Dhage introduced a new class of generalized metric spaces named D - metric spaces. Mustafa and Sims [29], [30] proved that most of the claims concerning the fundamental topological structures on D - metric spaces are incorrect and introduced the approximate notion of generalized metric spaces named G - metric spaces.

In fact, Mustafa, Sims and other authors proved many fixed point results for self mappings in G - metric spaces under certain conditions (see [29], [30], [27], [31], [32], [33], [40] and other papers).

Definition 5.1. *Let X be a nonempty set and $G : X^3 \to \mathbb{R}_+$ be a function satisfying the following properties:*

$(G_1) : G(x, y, z) = 0$ *if* $x = y = z$,

$(G_2) : 0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,

$(G_3) : G(x, y, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *and* $y \neq z$,

$(G_4) : G(x, y, z) = G(y, z, x) = G(z, x, y) = ...$ *(symmetry in all three variables),*

(G_5) : *$G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (triangle inequality).*

The function G is called a G - metric on X and the pair (X,G) is called a G - metric space, [29], [30].

Remark 5.3. *If $G(x,y,z) = 0$, then $x = y = z$* [30].

Lemma 5.1. *Let (X,G) be a G - metric space and $Q(x,y) = G(x,y,y)$. Then $Q(x,y)$ is a quasi - metric on X.*

Proof. By (G_1) and Remark 5.3, $Q(x,y) = 0$ if and only if $x = y$.

$$Q(x,y) = G(x,y,y) \leq G(x,z,z) + G(z,y,y) = Q(x,z) + Q(z,y).$$

□

Let (X,G) be a G - metric space and (X,Q) be a quasi - metric space determined by $Q(x,y) = G(x,y,y)$. Then

$$\begin{aligned} &Q(Ax,By) = G(Ax,By,By), Q(Sx,Ty) = G(Sx,Ty,Ty),\\ &Q(Sx,Ax) = G(Sx,Ax,Ax), Q(Ty,By) = G(Ty,By,By),\\ &Q(Sx,By) = G(Sx,By,By), Q(Ax,Ty) = G(Ax,Ty,Ty), \end{aligned} \tag{5.4}$$

Theorem 5.4. *Let (X,G) be a G - metric space and A,B,S and T be self mappings of X, where (A,S) and (B,T) are owc such that*

$$\begin{aligned} &F(G(Ax,By,By), G(Sx,Ty,Ty), G(Sx,Ax,Ax),\\ &G(Ty,By,By), G(Sx,By,By), G(Ax,Ty,Ty)) \leq 0, \end{aligned} \tag{5.5}$$

for all $x,y \in X$, with $Ax \neq By$ and $F \in \mathcal{F}_m$. Then A,B,S and T have a unique common fixed point.

Proof. By (5.4) and (5.5) we obtain

$$\begin{aligned} &F(Q(Ax,By), Q(Sx,Ty), Q(Sx,Ax),\\ &Q(Ty,By), Q(Sx,By), Q(Ax,Ty)) \leq 0, \end{aligned}$$

for all $x,y \in X$, with $Ax \neq By$ in the quasi - metric space (X,Q) determined by $Q(x,y) = G(x,y,y)$. By Theorem 4.1 with $m(x,y) = Q(x,y)$ it follows that A,B,S and T have a unique common fixed point. □

By Corollary 4.1 (1) for $m(x,y) = Q(x,y) = G(x,y,y)$ we obtain

Corollary 5.2. *Let f,g,S and T be self mappings of a G - metric space (X,G), (A,S) and (B,T) are owc such that*

$$\begin{aligned} &G(Ax,By,By) \leq aG(Sx,Ty,Ty)+\\ &+b\max\{G(Sx,Ax,Ax), G(Ty,By,By)\}+\\ &+c\max\{G(Sx,Ty,Ty), G(Sx,By,By), G(Ax,Ty,Ty)), \end{aligned}$$

Theorem 5.5. *for all $x, y \in X$, $a, b, c \geq 0$ and $a + c < 1$. Then A, B, S and T have a unique common fixed point.*

Remark 5.4. *By Corollary 4.2 (2) - 4.2 (12) we obtain similar results in G - metric spaces.*

If $A = B$ and $S = T$ we obtain by Theorem 5.4 the following result

Theorem 5.6. *Let (X, G) be a G - metric space and A, S be self mappings of X, where A and S are owc such that*

$$\begin{aligned} F(&G(Ax, Ay, Ay), G(Sx, Sy, Sy), G(Sx, Ax, Ax), \\ &G(Sy, Ay, Ay), G(Sx, Ay, Ay), G(Ax, Sy, Sy)) \leq 0, \end{aligned} \tag{5.6}$$

for all $x, y \in X$, with $Ax \neq By$ and $F \in \mathcal{F}_m$. Then A, B, S and T have a unique common fixed point.

Remark 5.5. *By Theorem 5.6 and Corollary 4.2 we obtain similar results for two mappings in G - metric spaces.*

Remark 5.6. *In the papers* [27], [30], [31], [32], [33] *and other papers, for the proof of the existence of fixed points in G - metric space, the function $G(x, y, y)$ is used instead $G(x, y, z)$. Hence, the study of fixed points in G - metric spaces can be reduced, in this cases, to the study of fixed points in quasi - metric spaces.*

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