"Vasile Alecsandri" University of Bacău
Faculty of Sciences
Scientific Studies and Research
Series Mathematics and Informatics
Vol. 22 (2012), No. 1, 103 - 116

$\mathcal{N}$–SUBALGEBRAS AND $\mathcal{N}$–FILTERS IN CI–ALGEBRAS

A. REZAEI AND A. BORUMAND SAEID

Abstract. In this paper, we introduce the notions of $\mathcal{N}$–subalgebras and $\mathcal{N}$–filters in CI–algebras and give a number of their properties. The relationship between $\mathcal{N}$–subalgebras and $\mathcal{N}$–filters is also investigated.

1. Introduction and preliminaries

Some recent researches led to generalizations of the notion of fuzzy set introduced by Zadeh in 1965 [12]. The generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the point $\{1\}$ into the interval $[0, 1]$. In order to provide a mathematical tool to deal with negative information, Jun et al. [2] introduced $\mathcal{N}$–structures, based on negative-valued functions.

In 1966, Y. Imai and K. Iseki [1] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Recently, H. S. Kim and Y. H. Kim defined a BE-algebra [6]. Biao Long Meng, defined notion of CI–algebra as a generalization of a BE–algebra [8].

BE-algebra and CI-algebra are studied by some authors [5, 9, 10, 11].

Keywords and phrases: BE–algebra, CI-algebra, $\mathcal{N}$–subalgebra, $\mathcal{N}$–filter.
(2010) Mathematics Subject Classification: 06F35, 03G25, 03E72.

Jun et al. [2, 3, 4] discussed the notion of $\mathcal{N}$–structures in BCH $/BCK/BCI$–algebras and investigated their properties. They introduced the notions of $\mathcal{N}$–ideals of subtraction algebras and $\mathcal{N}$–closed ideals in BCK/BCI–algebras.

In the present paper we continue to study CI–algebras and apply the $\mathcal{N}$–structures to the filter theory in CI–algebras, also investigate the relationship between $\mathcal{N}$–subalgebra and $\mathcal{N}$–filters.

In this section we review the basic definitions and some elementary aspects that are necessary for this paper.

Recall that a CI–algebra is an algebra $(X; *, 1)$ of type $(2, 0)$ satisfying the following axioms:

(CI1) $x * x = 1$;

(CI2) $1 * x = x$;

(CI3) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$. A CI–algebra X satisfying the condition $x * 1 = 1$ is called a BE–algebra. In any CI–algebra X one can define a binary relation " $\leq$ " by $x \leq y$ if and only if $x * y = 1$.

A CI–algebra X has the following properties:

(1.1) $y * ((y * x) * x) = 1$,

(1.2) $(x * 1) * (y * 1) = (x * y) * 1$,

(1.3) if $1 \leq x$, then $x = 1$,

for all $x, y \in X$.

A non-empty subset S of a CI–algebra X is called a subalgebra of X if $x*y \in S$ whenever $x, y \in S$. A mapping $f : X \to Y$ of CI–algebra is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. A non-empty subset F of CI–algebra X is called a filter of X if (1) $1 \in F$, (2) $x \in F$ and $x*y \in F$ implies $y \in F$. A filter F of CI–algebra X is said to closed if $x \in F$ implies $x * 1 \in F$.

A nonempty subset S of a CI–algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$. For our convenience, the empty set $\emptyset$ is regarded as a subalgebra of X.

Denote by $Q(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $Q(X, [-1, 0])$ is a negative-valued function from X to $[-1, 0]$ (briefly, $\mathcal{N}$–function on X). By an $\mathcal{N}$–structure we mean an ordered pair (X, f) of X and an $\mathcal{N}$–function f on X.

2. $\mathcal{N}$–subalgebras of CI–algebras

In what follows, let X denote a CI–algebra and f an $\mathcal{N}$–function on X unless otherwise specified.

Definition 2.1. *By a subalgebra of X based on $\mathcal{N}$–function f (briefly, $\mathcal{N}$–subalgebra of X), we mean an $\mathcal{N}$–structure (X, f) in which f satisfies the following assertion:*

(2.1) $(\forall x, y \in X)(f(x * y) \leq \max\{f(x), f(y)\})$.

Example 2.1. *Let $X = \{1, a, b\}$ be a set. With the following Cayley table:*

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

Then $(X; *, 1)$ is a CI–algebra. Define an $\mathcal{N}$–function $f : X \to [-1, 0]$ by $f(1) = -0.6$, $f(a) = -0.4$ and $f(b) = -0.2$. Then (X, f) is an $\mathcal{N}$–subalgebra of X. But $\mathcal{N}$–function $g : X \to [-1, 0]$ defined by $g(1) = -0.1$, $g(a) = -0.3$ and $g(b) = -0.4$ is not an $\mathcal{N}$–subalgebra because

Example 2.2. *Let $\mathbb{N}$ be the set of all natural numbers and "$*$" be the binary operation on $\mathbb{N}$ defined by*

$$x * y = \begin{cases} y & \text{if } x = 1 \\ 1 & \text{if } x \neq 1 \end{cases}$$

Then $(\mathbb{N}; *, 1)$ is a CI–algebra. Define an $\mathcal{N}$–function $f : \mathbb{N} \to [-1, 0]$ by

$$f(x) = \begin{cases} \alpha & \text{if } x = 1 \\ \beta & \text{if } x \neq 1 \end{cases}$$

where $\alpha < \beta < 0$, then $(\mathbb{N}, f)$ is an $\mathcal{N}$–subalgebra of $\mathbb{N}$.

Lemma 2.1. *Every $\mathcal{N}$–subalgebra (X, f) of X satisfies the following inequality:*

(i) $(\forall x \in X)(f(x) \geq f(1))$.

(ii) $(\forall x \in X)(f(x * 1) \leq f(x))$.

Proof. (i) Note that $x * x = 1$ for all $x \in X$. Using (2.1), we have

$$f(1) = f(x * x) \leq \max\{f(x), f(x)\} = f(x),$$

for all $x \in X$.

(ii) Let $x \in X$. Then

$$\begin{aligned} f(x * 1) \leq \max\{f(x), f(1)\} &= \max\{f(x), f(x * x)\} \\ &\leq \max\{f(x), \max\{f(x), f(x)\}\} \\ &= f(x). \end{aligned}$$

Proposition 2.1. *If an $\mathcal{N}$–subalgebra (X, f) of X satisfies the following inequality:*

(2.2) $(\forall x, y \in X)(f(x * y) \leq f(x))$.

Then f is a constant function.

Proof. Let $x \in X$. Using Lemma 2.1, we have $f(x) = f(1 * x) \leq f(1)$. It follows that $f(x) = f(1)$, and so f is a constant function.

Theorem 2.1. *The family of $\mathcal{N}$–subalgebras of X forms a complete distributive lattice under the ordering of set inclusion $\subset$.*

Proof. Let $\{f_i \mid i \in I\}$ be a family of $\mathcal{N}$– subalgebra of X. Since $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering in $[-1, 0]$, it is sufficient to show that $\cup_{i \in I} f_i$ is an $\mathcal{N}$–subalgebra of X. Let $x, y \in X$. Then

$$\begin{aligned}
(\cup_{i\in I} f_i)(x * y) &= \sup\{f_i(x * y) \mid i \in I\} \\
&\leq \sup\{\max\{f_i(x), f_i(y)\} \mid i \in I\} \\
&= \max(\sup\{f_i(x) \mid i \in I\}, \sup\{f_i(y) \mid i \in I\}) \\
&= \max(\cup_{i\in I} f_i(x), \cup_{i\in I} f_i(y)).
\end{aligned}$$

Hence $\cup_{i\in I} f_i$ is an $\mathcal{N}$–subalgebra of X.

Theorem 2.2. *If (X, f) is an $\mathcal{N}$–subalgebra of X, then the set*

$$X_f := \{x \in X \mid f(x) = f(1)\}$$

is a subalgebra of X.

Proof. Let $x, y \in X_f$. Then $f(x) = f(1) = f(y)$, and so

$$f(x * y) \leq \max\{f(x), f(y)\} = \max\{f(1), f(1)\} = f(1).$$

By Lemma 2.1, we get that $f(x*y) = f(1)$ which means that $x*y \in X_f$.

Theorem 2.3. *Let M be a (crisp) subset of X. Suppose that (X, f) is an $\mathcal{N}$–subalgebra of X defined by:*

$$f(x) = \begin{cases} \alpha & \text{if } x \in M \\ \beta & \text{otherwise} \end{cases}$$

for some $\alpha, \beta \in [-1, 0]$ with $\alpha < \beta$. Then (X, f) is an $\mathcal{N}$–subalgebra if and only if M is a subalgebra of X. Moreover, in this case $X_f = M$.

Proof. Let (M, f) be an $\mathcal{N}$–subalgebra. Let $x, y \in X$ be such that $x, y \in M$. Then

$$f(x * y) \leq \max\{f(x), f(y)\} = \max\{\alpha, \alpha\} = \alpha$$

and so $x * y \in M$.

Conversely, suppose that M is a subalgebra of X and $x, y \in X$.
(i) If $x, y \in M$ then $x * y \in M$, thus

$$f(x * y) = \alpha = \max\{f(x), f(y)\}$$

(ii) If $x \notin M$ or $y \notin M$, then

$$f(x * y) \leq \beta = \max\{f(x), f(y)\}$$

This shows that (M, f) is an $\mathcal{N}$–subalgebra.
Moreover, we have

$$X_f := \{x \in X \mid f(x) = f(1)\} = \{x \in X \mid f(x) = \alpha\} = M.$$

For any $\mathcal{N}$–function f on X and $t \in [-1, 0)$, the set

$$C(f; t) := \{x \in X | f(x) \leq t\}$$

is called a closed (f, t)–cut (level subalgebra) of f, and the set

$$O(f; t) := \{x \in X | f(x) < t\}$$

is called an open (f, t)–cut of f.
It follows easily that for any $\mathcal{N}$–functions f, g on X;
(1) $f \leq g$, $t \in [-1, 0] \Rightarrow C(g; t) \subseteq C(f; t)$;
(2) $t_1 \leq t_2$, $t_1, t_2 \in [-1, 0] \Rightarrow C(f; t_1) \subseteq C(f; t_2)$;
(3) $f = g \Leftrightarrow C(f; t) = C(g; t)$, for all $t \in [-1, 0]$.

Example 2.3. *In Example 2.1, we can see that* $C(f, -0.2) = \{1, a, b\}$ *and* $O(f, -0.2) = \{1, a\}$.

Theorem 2.4. *Let X be a CI-algebra. Then two level subalgebras $C(f, t_1)$, $C(f, t_2)$ (where $t_1 < t_2$) of f are equal if and only if there is no $x \in X$ such that $t_1 < f(x) \leq t_2$.*

Proof. Let $C(f, t_1) = C(f, t_2)$ where $t_1 < t_2$ and there exists $x \in X$ such that $t_1 < f(x) \leq t_2$. Then $C(f, t_1)$ is a proper subset of $C(f, t_2)$, which is a contradiction.

Conversely, suppose that there is no $x \in X$ such that $t_1 < f(x) \leq t_2$. If $x \in C(f, t_2)$, then $f(x) \leq t_2$ and so $f(x) \leq t_1$. Therefore $x \in C(f, t_1)$, thus $C(f, t_2) \subseteq C(f, t_1)$. Hence $C(f, t_1) = C(f, t_2)$.

Theorem 2.5. *Let (X, f) be an $\mathcal{N}$–structure of X with the greatest lower bound λ_0. Then the following conditions are equivalent:*
(i) (X, f) is an $\mathcal{N}$–subalgebra of X.
(ii) For all $\lambda \in Im(f)$, the non-empty set $C(f, \lambda)$ is a subalgebra of X.
(iii) For all $\lambda \in Im(f) \backslash \lambda_0$, the non-empty set $O(f; \lambda)$ is a subalgebra of X.

(iv) For all $\lambda \in [0,1]$, *the non-empty set* $O(f;\lambda)$ *is a subalgebra of* X.

(v) For all $\lambda \in [0,1]$, *the non-empty* $C(f;\lambda)$ *is a subalgebra of* X.

Proof. $(i \to iv)$ Let (X,f) be an $\mathcal{N}$–subalgebra of X, $\lambda \in [0,1]$ and $x,y \in O(f;\lambda)$, then we have

$$f(x*y) \leq \max\{f(x), f(y)\} < \max\{\lambda,\lambda\} = \lambda.$$

Thus $x*y \in O(f;\lambda)$. Hence $O(f;\lambda)$ is a subalgebra of X.

$(iv \to iii)$ It is clear.

$(iii \to ii)$ Let $\lambda \in Im(f)$. Then $C(f;\lambda)$ is a non-empty set. Since $C(f;\lambda) = \bigcap\limits_{\beta>\lambda} O(f;\beta)$, where $\beta \in Im(f) \setminus \lambda_0$. Then by (iii) and Theorem 2.1, $C(f;\lambda)$ is a subalgebra of X.

$(ii \to v)$ Let $\lambda \in [0,1]$ and $C(f;\lambda)$ be non-empty set. Suppose $x,y \in C(f;\lambda)$. Let $\alpha = \max\{f(x),f(y)\}$, it is clear that $\alpha = \max\{f(x),f(y)\} \leq \{\lambda,\lambda\} = \lambda$. Thus $x,y \in C(f;\alpha)$ and $\alpha \in Im(f)$, by (ii) $C(f;\alpha)$ is a subalgebra of X, hence $x*y \in C(f;\alpha)$. Then we have

$$f(x*y) \leq \max\{f(x),f(y)\} \leq \{\alpha,\alpha\} = \alpha \leq \lambda.$$

Therefore $x*y \in C(f;\lambda)$. Then $C(f;\lambda)$ is a subalgebra of X.

$(v \to i)$ Assume that the non-empty set $C(f;\lambda)$ is a subalgebra of X, for every $\lambda \in [0,1]$. In contrary, let $x_0, y_0 \in X$ be such that

$$f(x_0 * y_0) > \max\{f(x_0), f(y_0)\}.$$

Let $f(x_0) = \gamma$, $f(y_0) = \theta$ and $f(x_0 * y_0) = \lambda$. Then

$$\lambda > \max\{\gamma,\theta\}.$$

Consider

$$\lambda_1 = \frac{1}{2}(f(x_0 * y_0) + \max\{f(x_0), f(y_0)\})$$

We get that

$$\lambda_1 = \frac{1}{2}(\lambda + \max\{\gamma,\theta\})$$

Therefore

$$\gamma < \lambda_1 = \frac{1}{2}(\lambda + \max\{\gamma,\theta\} < \lambda$$

$$\theta < \lambda_1 = \frac{1}{2}(\lambda + \max\{\gamma,\theta\} < \lambda$$

Hence

$$\max\{\gamma, \theta\} < \lambda_1 < \lambda = f(x_0 * y_0)$$

so that $x_0 * y_0 \notin C(f; \lambda_1)$ which is a contradiction, since

$$f(x_0) = \gamma \leq \max\{\gamma, \theta\} < \lambda_1$$

$$f(y_0) = \theta \leq \max\{\gamma, \theta\} < \lambda_1$$

imply that $x_0, y_0 \in C(f; \lambda_1)$. Thus $f(x * y) \leq \max\{f(x), f(y)\}$, for all $x, y \in X$.

Theorem 2.6. *Each subalgebra of X is a level subalgebra of an $\mathcal{N}$–subalgebra of X.*

Proof. Let Y be a subalgebra of X, and f be an $\mathcal{N}$–function set on X defined by

$$f(x) = \begin{cases} \alpha & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \in [-1, 0]$. It is clear that $C(f; \alpha) = Y$. Let $x, y \in X$. We consider the following cases:

case 1) If $x, y \in Y$, then $x * y \in Y$ therefore

$$f(x * y) = \alpha = \max\{\alpha, \alpha\} = \max\{f(x), f(y)\}.$$

case 2) If $x, y \notin Y$, then $f(x) = 0 = f(y)$ and so

$$f(x * y) \leq 0 = \max\{0, 0\} = \max\{f(x), f(y)\}.$$

case 3) If $x \in Y$ and $y \notin Y$ (respectively, $x \notin Y$ and $y \in Y$), then $f(x) = \alpha$ and $f(y) = 0$. Thus

$$f(x * y) \leq 0 = \max\{\alpha, 0\} = \max\{f(x), f(y)\}.$$

Therefore A is an $\mathcal{N}$–subalgebra of X.

3. $\mathcal{N}$–FILTERS IN CI–ALGEBRAS

Definition 3.1. *By a filter of X based on $\mathcal{N}$–function f (briefly, $\mathcal{N}$–filter of X), we mean an $\mathcal{N}$–structure (X, f) in which f satisfies the following assertion: (3.1)* $(\forall x, y \in X)(f(1) \leq f(y)$ *and* $f(y) \leq \max\{f(x * y), f(x)\})$.

Example 3.1. *In Example 2.1, we can see that (X, f) is an $\mathcal{N}$–filter of X.*

Example 3.2. *Let $X = \{1, a\}$ with the following Cayley table:*

$*$	1	a
1	1	a
a	a	1

Then $(X; *, 1)$ is a CI–algebra. Define an $\mathcal{N}$–function $f : X \to [-1, 0]$ by $f(1) = -0.1$, $f(a) = -0.3$. Then (X, f) is not an $\mathcal{N}$–filter of X. Because

Theorem 3.1. *The family of $\mathcal{N}$–filters of X forms a complete distributive lattice under the ordering of set inclusion $\subset$.*

Proof. Let $\{f_i \mid i \in I\}$ be a family of $\mathcal{N}$– filters of X. Since $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering in $[-1, 0]$, it is sufficient to show that $\cup_{i\in I} f_i$ is an $\mathcal{N}$–filter of X. Let $x \in X$. Then

$$\begin{aligned}
(\cup_{i\in I} f_i)(y) &= \sup\{f_i(y) \mid i \in I\} \\
&\leq \sup\{\max\{f_i(x), f_i(x * y)\} \mid i \in I\} \\
&= \max(\sup\{f_i(x) \mid i \in I\}, \sup\{f_i(x * y) \mid i \in I\}) \\
&= \max(\cup_{i\in I} f_i(x), \cup_{i\in I} f_i(x * y)).
\end{aligned}$$

Hence $\cup_{i\in I} f_i$ is an $\mathcal{N}$–filter of X.

Proposition 3.1. *If (X, f) is an $\mathcal{N}$–filter of X, then*
(3.2) $(\forall x, y \in X)(x \leq y \Rightarrow f(y) \leq f(x))$.

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 1$, and so

$$f(y) \leq \max\{f(x * y), f(x)\} = \max\{f(1), f(x)\} = f(x).$$

Proposition 3.2. *Let (X, f) be an $\mathcal{N}$–filter of X. If $x, y, z \in X$ satisfies the following condition:*
(3.3) $(\forall x, y, z \in X)(z \leq x * y)$.
Then $f(y) \leq \max\{f(z), f(x)\}$.

Proof. Assume that $x, y, z \in X$ satisfies (3.3). Then

$$f(x * y) \leq \max\{f(z * (x * y)), f(z)\} = \max\{f(1), f(z)\} = f(z).$$

It follows that

$$f(y) \leq \max\{f(x * y), f(x)\} \leq \max\{f(z), f(x)\}.$$

Theorem 3.2. *Every $\mathcal{N}$–filter of X is an $\mathcal{N}$–subalgebra of X.*

Proof. If $x, y \in X$, then

$$\begin{aligned} f(x*y) &\leq \max\{f(y*(x*y)), f(y)\} \\ &= \max\{f(x*(y*y)), f(y)\} \\ &= \max\{f(x*1), f(y)\} \\ &= \max\{f(1), f(y)\} \leq \max\{f(x), f(y)\}. \end{aligned}$$

Therefore (X, f) is an $\mathcal{N}$–subalgebra of X.

The converse of Theorem 3.2 may not be true in general as seen in the following example.

Example 3.3. *Let $X := \{1, a, b, c,\}$ be a CI–algebra with the following Cayley table.*

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Define an $\mathcal{N}$–function $f : X \to [-1, 0]$ by $f(1) = -0.7$, $f(a) = -0.7$, $f(b) = -0.1$ and $f(c) = -0.6$. Then (X, f) is an $\mathcal{N}$–subalgebra of X. But it is not an $\mathcal{N}$–filter of X because

Theorem 3.3. *If an $\mathcal{N}$–subalgebra (X, f) satisfies:*

$$(\forall x, y \in X)(f(y) \leq \max\{f(x*y), f(x)\}).$$

Then (X, f) is an $\mathcal{N}$–filter of X.

Proof. Since (X, f) is an $\mathcal{N}$–subalgebra of X, by Lemma 2.4 we have $f(1) \leq f(y)$, for all $y \in Y$. Therefore $f(1) \leq f(y) \leq \max\{f(x*y), f(x)\}$, for all $x, y \in Y$. Hence (X, f) is an $\mathcal{N}$–filter of X.

Theorem 3.4. *Let (X, f) be an $\mathcal{N}$–subalgebra of X such that f satisfies:*

$$(3.4)(\forall x, y \in X)(f(y*x) \geq f(x*y)).$$

Then (X, f) is an $\mathcal{N}$–filter of X.

Proof. Taking $x = 1$ in (3.4) induces $f(y*1) \geq f(1*y) = f(y)$, for all $y \in X$. Using $(CI1)$, $(CI3)$, (3.1), (3.4), we have

$$\begin{aligned} f(y) = f(1*y) &\leq f(y*1) = f(y*(x*x)) = f(x*(y*x)) \\ &\leq \max\{f(x), f(y*x)\} \leq \max\{f(x), f(x*y)\} \end{aligned}$$

for all $x, y \in X$. Therefore (X, f) is an $\mathcal{N}$–filter of X.

Proposition 3.3. *Let (X, f) be an $\mathcal{N}$–filter of X which satisfies the following inequality*

$$(\forall x \in X)(f(x) \leq f(x * 1)).$$

Then (X, f) satisfies

$$(\forall x, y \in X)(f(y * x) = f(x * y)).$$

Proof. Using hypothesis and (3.1), (1.2), (1.1), ($CI3$), (2.2) we have

$$\begin{aligned} f(y * x) \leq f((y * x) * 1) &\leq \max\{f((x * y) * ((y * x) * 1)), f(x * y)\} \\ &= \max\{f((x * y) * ((y * 1) * (x * 1))), f(x * y)\} \\ &= \max\{f((y * 1) * ((x * y) * (x * 1)), f(x * y)\} \\ &= \max\{f((y * 1) * (x * ((x * y) * 1))), f(x * y)\} \\ &= \max\{f((y * 1) * (x * ((x * y) * 1)), f(x * y)\} \\ &= \max\{f(x * ((y * 1) * ((x * 1) * (y * 1)), f(x * y)\} \\ &= \max\{f(x * ((x * 1) * 1)), f(x * y)\} \\ &= \max\{f(1), f(x * y)\} = f(x * y). \end{aligned}$$

Similarly we have $f(x * y) \leq f(y * x)$.

For any element a of X, consider the following set

$$X_a := \{x \in X : f(x) \leq f(a)\}.$$

Obviously, $a \in X_a$, and so X_a is a non-empty subset of X.

Theorem 3.5. *Let a be an element of X. If (X, f) is an $\mathcal{N}$–filter of X. Then the set X_a is a filter of X.*

Proof. Obviously, $1 \in X_a$. Let $x, y \in X$ be such that $x * y \in X_a$ and $x \in X_a$. Then $f(x * y) \leq f(a)$ and $f(x) \leq f(a)$. Since (X, f) is an $\mathcal{N}$–filter of X, it follows from Definition 3.1,

$$f(y) \leq \max\{f(x * y), f(x)\} \leq f(a)$$

So that $y \in X_a$. Hence X_a is a filter of X.

If f is an $\mathcal{N}$–function of X and α is a mapping from X into itself, we define a mapping $f^\alpha : X \to [0, 1]$ by $f^\alpha(x) = f(\alpha(x))$ for all $x \in X$.

Theorem 3.6. *Let f be an $\mathcal{N}$–subalgebra of X, and α be an endomorphism of X. Then f^α is also an $\mathcal{N}$–subalgebra (respectively, $\mathcal{N}$–filters).*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} f^\alpha(x * y) = f(\alpha(x * y)) = f(\alpha(x) * \alpha(y)) &\leq \max\{f(\alpha(x)), f(\alpha(y))\} \\ &= \max\{f^\alpha(x), f^\alpha(y)\}. \end{aligned}$$

Since α is an endomorphism, then $\alpha(1) = 1$ and so the proof is similar in the case when f is an $\mathcal{N}$–filter.

Definition 3.2. *Let f and g be the $\mathcal{N}$–function in a set X. The $\mathcal{N}$–cartesian product $f \times g : X \times X \to [-1, 0]$ is defined by $(f \times g)(x, y) = \max\{f(x), g(y)\}$, for all $x, y \in X$.*

We can define on $X \times X$ the product structure by $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$.

Theorem 3.7. *If f and g are $\mathcal{N}$–filters of a CI–algebra X, then $f \times g$ is an $\mathcal{N}$–filter of $X \times X$.*

Proof. For any $(x, y) \in X \times X$, we have

$$(f \times g)(1, 1) = \max\{f(1), g(1)\} \leq \max\{f(x), g(y)\} = (f \times g)(x, y).$$

Let $(x_1, x_2), (y_1, y_2) \in X \times X$. Then

$$\begin{aligned}
(f \times g)(y_1, y_2) &= \max\{f(y_1), g(y_2)\} \\
&\leq \max\{\max\{f(x_1), f(x_1 * y_1)\}, \max\{g(x_2), g(x_2 * y_2)\}\} \\
&= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(x_1 * y_1), g(x_2 * y_2)\}\} \\
&= \max\{(f \times g)(x_1, x_2), (f \times g)(x_1 * y_1, x_2 * y_2)\} \\
&= \max\{(f \times g)(x_1, x_2), (f \times g)((x_1, x_2) * (y_1, y_2))\}.
\end{aligned}$$

Hence $f \times g$ is an $\mathcal{N}$–filter of $X \times X$.

Lemma 3.1. *Let f and g are $\mathcal{N}$–function in X such that $f \times g$ is an $\mathcal{N}$–filter of $X \times X$. Then*
(i) $(\forall x \in X)$ $(f(1) \leq f(x))$ or $(\forall x \in X)$ $(g(1) \leq g(x))$;
(ii) If $f(1) \leq f(x)$, for all $x \in X$, then $(\forall x \in X)$ $(g(1) \leq f(x))$ or $(\forall x \in X)$ $(f(1) \leq g(x))$.
(iii) If $g(1) \leq g(x)$, for all $x \in X$, then $(\forall x \in X)$ $(f(1) \leq g(x))$ or $(\forall x \in X)$ $(g(1) \leq f(x))$.

Proof. Assume that there exist $x, y \in X$ such that $f(x) < f(1)$ and $g(y) < g(1)$. Then $(f \times g)(x, y) = \max\{f(x), g(y)\} < \max\{f(1), g(1)\} = (f \times g)(1, 1)$. Which is a contradiction. Hence (i) is proved.

(ii) Again, using reduction to absurdity: we assume that there exist $x, y \in X$ such that $f(x) < g(1)$ and $g(y) < f(1)$. Then $(f \times g)(x, y) = \max\{f(x), g(y)\} < \max\{f(1), g(1)\} = (f \times g)(1, 1)$, hence $(f \times g)(x, y) < (f \times g)(1, 1)$, which is a contradiction.

(iii) The proof is similar to (ii).

Theorem 3.8. *If $f \times g$ is an $\mathcal{N}$–filter of $X \times X$, then f or g is an $\mathcal{N}$–filter of X.*

Proof. Since $f \times g$ is an $\mathcal{N}$–filter of $X \times X$,

$$\begin{aligned}(f \times g)(y_1, y_2) &\leq \max\{(f \times g)(x_1, x_2), (f \times g)((x_1, x_2) * (y_1, y_2))\} \\ &= \max\{(f \times g)(x_1, x_2), (f \times g)(x_1 * y_1, x_2 * y_2)\}.\end{aligned}$$

By Lemma 3.1, without loss of generality we assume that $g(1) \leq g(x)$, for all $x \in X$. Then $f(1) \leq g(x)$, or $g(1) \leq f(x)$.

Let $f(1) \leq g(x)$, for all $x \in X$. Then $(f \times g)(1, y) = \max\{f(1), g(y)\} = g(y)$ and

$$\begin{aligned}(f \times g)(1, y) &\leq \max\{(f \times g)(1, x), (f \times g)(1, x * y))\} \\ &= \max\{f(1), g(x), g(x * y)\} \\ &= \max\{g(x), g(x * y)\}.\end{aligned}$$

Therefore $g(y) \leq \max\{g(x), g(x*y)\}$ for all $x, y \in X$. This proves that g is an $\mathcal{N}$–filter of X.

The other case is similar.

4. Conclusion

In this paper, we have introduced the concept of $\mathcal{N}$–subalgebra (filter) of CI–algebra and and some related properties are investigated. We show that any $\mathcal{N}$–filter is an $\mathcal{N}$–subalgebra but the converse it is not true. We give a condition for an $\mathcal{N}$–subalgebras to be $\mathcal{N}$–filters.

We believe these results are very useful in developing algebraic structures and these concepts can be further generalized.

Acknowledgement

The authors would like to express their thanks to anonymous referees for their comments and suggestions which improved the paper.

References

[1] Y. Imai and K. Iseki, **On axiom systems of propositional Calculi**, XIV proc. Jpn. Academy, 42 (1966), 19-22.

[2] Y. B. Jun, K. J. Lee and S. Z. Song, **$\mathcal{N}$–ideals of BCK/BCI–algebras**, J. Chungcheong Math. Soc. 22 (2009), 417-437.

[3] Y. B. Jun, K. J. Lee and S. Z. Song, **$\mathcal{N}$–ideals of subtraction algebras**, Common. Korean Math. Soc. 25, No. 2 (2010), 173-184.

[4] Y. B. Jun, M. A. Ozturk, and E. H. Roh, **$\mathcal{N}$–structures applied to closed ideals in BCH–algebras**, Int. J. of Mathematics and Mathematical Sciences, Vol. 2010 (2010), Article ID 943565.

[5] K. H. Kim, **A Note On CI–Algebras**, Int. Math. Forum, Vol, 6 (2011), No., 1, 1-5.

[6] H. S. Kim and Y. H. Kim, **On BE–algebras**, Sci. Math. Jap. Online, e-2006, 1299-1302.

[7] S. Nanda and N. R. Das, **Fuzzy Mathematical Concepts**, Alpha Science International Ltd, India, (2010).

[8] B. L. Meng, **CI–algebra**, Sci. Math. Jpn. Online, e-2009, 695-701.

[9] A. Rezaei and A. Borumand Saeid, **On fuzzy subalgebras of BE–algebras**, Afr. Math. 22 (2011), 115-127.

[10] A. Rezaei and A. Borumand Saeid, **Some Results in Fuzzy congruence relation in CI–algebras**, Numerical Analysis and Applied Mathematiccs ICNAAM (2011). AIP Conf. Proc. 1090-1093.

[11] B. Piekart and A. Walendziak, **On filters and upper sets in CI–algebras**, Algebra and Discrete Mathematics. Vol. 11, No.1 (2011), 109-115.

[12] L. A. Zadeh, **Fuzzy sets**, Inform. and Control, 8 (1965), 338-353.

Akbar Rezaei

Department of Mathematics, Payame Noor University, P. O. Box 19395-3697 Tehran, IRAN, e-mail: rezaei@pnu.ac.ir

Arsham Borumand Saeid

Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, IRAN, e-mail: arsham@mail.uk.ac.ir