"Vasile Alecsandri" University of Bacău
Faculty of Sciences
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THE SEMILOCAL CONVERGENCE OF THE CONVEX ACCELERATION OF WHITTAKER'S METHOD

RALUCA ANAMARIA (POMIAN) SĂLĂJAN

Abstract. In this article we study the convex acceleration of Whittaker's iterative method for approximating the roots of a real function of real argument, that is two times differentiable and has a non-vanishing first order derivative. We prove a semilocal convergence theorem for this method and we give a numerical example which illustrates this theorem.

1. Introduction and preliminaries

In numerical analysis one of the most important problems is to locate the roots of the equation

$$f(x) = 0, \tag{1.1}$$

where $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable function with simple roots, that is, if $f(\alpha) = 0$, then $f'(\alpha) \neq 0$.

Let x^* be a simple root of the equation (1.1). This root can be determined as a fixed point of some iteration function $g : [a, b] \to [a, b]$, i.e., $g(x^*) = x^*$, by means of the one-point iteration method

$$x_{n+1} = g(x_n), \; n \in \mathbb{N}. \tag{1.2}$$

Here g is a function of the form

$$g(x) = x + \varphi(x),$$

and $x_0 \in [a, b]$ is the starting value.

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We consider the convex acceleration of Whittaker's method with second order convergence, where the sequence approximating the solution $x^* \in [a,b]$ of the equation (1.1) is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{2f'(x_n)}(2 - L_f(x_n)), \tag{1.3}$$

where $x_0 \in [a,b]$ and $L_f(x) := \frac{f(x)f''(x)}{[f'(x)]^2}$ on some interval containing x_0, on which the first derivative of f is non-vanishing.

The convex acceleration of Whittaker's method has been rediscovered by several authors, see for example [1], [2], [3], [7] and references therein.

2. A convergence result for the convex acceleration of Whittaker's method

In this section we will present some general criteria of semilocal convergence of sequence (1.3) generated by the convex acceleration of Whittaker's method.

This theorem ensures the convergence of approximation sequence (1.3) and also provides the existence of a root of the equation (1.1) in a specified interval.

Theorem 2.1. *Assume that $x_0 \in [a,b]$, the function $f : [a,b] \to \mathbb{R}$ and $\delta > 0$ satisfy the following conditions :*

a) $\Delta = [x_0 - \delta, x_0 + \delta] \subseteq [a,b]$;

b) f admits a second-order derivative on Δ and its first derivative is non-vanishing on Δ;

c) There exists $\eta > 0$ such that $\left|\frac{1}{f'(x)}\right| \leq \eta$ for any $x \in \Delta$;

d) $-2 \leq L_f(x) := \frac{f(x)f''(x)}{[f'(x)]^2} \leq 2$ *for any $x \in \Delta$;*

e) $M_2 := \sup\limits_{x\in\Delta} |f''(x)| < \infty$;

f) $\mu_0 := \lambda\,|f(x_0)| < 1$, *where* $\lambda := \frac{5M_2\eta^2}{2}$;

g) $\frac{2\eta\mu_0}{\lambda(1-\mu_0)} \leq \delta$;

If the initial value x_0 is sufficiently close to the root of equation (1.1), *then the sequence $\{x_n\}_{n\geq 0}$ generated by* (1.3) *is convergent to some $x^* \in [a,b]$. Moreover, the following relations hold:*

i) $f(x^*) = 0$;

ii) $x_n \in \Delta$ *for every $n \in \mathbb{N}$;*

iii) $|f(x_n)| \leq \frac{\mu_0^{2^n}}{\lambda}$ *for every $n \in \mathbb{N}$;*

iv) $|x^* - x_n| \leq \frac{2\eta\mu_0^{2^n}}{\lambda(1-\mu_0^{2^n})}$ *for every $n \in \mathbb{N}$;*

Proof. We consider the function φ defined by

$$\varphi(x) = -\frac{f(x)}{f'(x)}\frac{2-L_f(x)}{2}, \text{ where } L_f(x) := \frac{f(x)f''(x)}{[f'(x)]^2}, x \in \Delta.$$

Then (1.3) becomes $x_{n+1} = x_n + \varphi(x_n)$, $n \in \mathbb{N}$.

Denote $\gamma := \frac{M_2\eta^2}{2}$.

Taking into account the assumptions c)-e) and the above notations we obtain the following estimate

$$\begin{aligned}\left|f(x) + \frac{f'(x)}{1!}\varphi(x)\right| &= \left|f(x) - \frac{f'(x)}{1!}\frac{f(x)}{f'(x)}\frac{2-L_f(x)}{2}\right| \\ &= \left|\frac{f^2(x)f''(x)}{2[f'(x)]^2}\right| \\ &\le |f(x)|^2\,\frac{M_2\eta^2}{2} = \gamma\,|f(x)|^2,\end{aligned} \tag{2.1}$$

for every $x \in \Delta$.

Note that $-2 \le L_f(x) \le 2$ is equivalent to $0 \le \frac{2-L_f(x)}{2} \le 2$.

We will show that all the terms of the sequence $\{x_n\}_{n\ge 0}$ generated by (1.3) belong to the interval Δ.

By conditions c), d), f) and g) we have

$$\begin{aligned}|x_1 - x_0| &= \left|\frac{f(x_0)}{f'(x_0)}\right|\left|\frac{2-L_f(x_0)}{2}\right| \le 2\left|\frac{f(x_0)}{f'(x_0)}\right| \\ &\le 2\eta\,|f(x_0)| = \frac{2\lambda\eta\,|f(x_0)|}{\lambda} < \frac{2\eta\mu_0}{\lambda(1-\mu_0)} \le \delta,\end{aligned}$$

hence $x_1 \in \Delta$.

Applying the Taylor expansion of the function f around x_0 and taking into account the inequality from above, the condition f), the relation (2.1) and the fact that $\varphi(x_0) = x_1 - x_0$, we get

$$\begin{aligned}|f(x_1)| &\le |f(x_1) - [f(x_0) + f'(x_0)(x_1 - x_0)]| + |f(x_0) + f'(x_0)(x_1 - x_0)| \\ &\le \frac{M_2}{2!}\,|x_1 - x_0|^2 + \gamma\,|f(x_0)|^2 \le \frac{M_2}{2}(2\eta\,|f(x_0)|)^2 + \gamma\,|f(x_0)|^2 = \\ &= \frac{5}{2}M_2\eta^2\,|f(x_0)|^2 = \frac{\mu_0^2}{\lambda},\end{aligned}$$

hence

$$|f(x_1)| \le \frac{\mu_0^2}{\lambda}.$$

Taking into account the conditions c), d) and the above inequality we get

$$|x_2 - x_1| = \left|\frac{f(x_1)}{f'(x_1)}\right| \left|\frac{2 - \frac{f(x_1)f''(x_1)}{[f'(x_1)]^2}}{2}\right| \leq 2\left|\frac{f(x_1)}{f'(x_1)}\right| \leq 2\eta\,|f(x_1)| \leq \frac{2\eta\mu_0^2}{\lambda}.$$

Analogously, applying the Taylor expansion of the function f around x_1 and taking into account the inequality from above, the condition f), the relation (2.1) and the fact that $\varphi(x_1) = x_2 - x_1$, we have

$$\begin{aligned}
&|f(x_2)| \leq |f(x_2) - [f(x_1) + f'(x_1)(x_2 - x_1)]| + |f(x_1) + f'(x_1)(x_2 - x_1)| \\
&\leq \frac{M_2}{2!}\,|x_2 - x_1|^2 + \gamma\,|f(x_1)|^2 \leq \frac{M_2}{2}\Big(\frac{2\eta\mu_0^2}{\lambda}\Big)^2 + \frac{\gamma\mu_0^{2^2}}{\lambda^2} = \frac{5M_2\eta^2}{2\lambda^2}\mu_0^{2^2} = \frac{\mu_0^{2^2}}{\lambda}.
\end{aligned}$$

From the previous relations, using the induction, it follows that the relation iii) holds

$$|f(x_n)| \leq \frac{\mu_0^{2^n}}{\lambda}, n \in \mathbb{N}. \tag{2.2}$$

Inductively, by c), d) and (2.2) we can show that

$$|x_{n+1} - x_n| = \left|\frac{f(x_n)}{f'(x_n)}\right| \left|\frac{2 - \frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}{2}\right| \leq \frac{2\eta\mu_0^{2^n}}{\lambda}, n \in \mathbb{N}. \tag{2.3}$$

In order to prove the convergence of the sequence given by (1.3) we shall show that this sequence is Cauchy.

Applying inequality (2.3) and taking into account the conditions f) and g) we can deduce the relation ii)

$$\begin{aligned}
|x_{n+1} - x_0| &\leq \sum_{i=0}^{n} |x_{i+1} - x_i| \leq \sum_{i=0}^{n} \frac{2\eta\mu_0^{2^i}}{\lambda} \\
&\leq \frac{2\eta\mu_0}{\lambda}(1 + \mu_0^{2-1} + \mu_0^{2^2-1} + \ldots + \mu_0^{2^n-1}) \\
&< \frac{2\eta\mu_0}{\lambda(1 - \mu_0)} \leq \delta,
\end{aligned}$$

hence $x_{n+1} \in \Delta$ for all $n \in \mathbb{N}$. Moreover, by (2.3) we obtain

$$
\begin{aligned}
|x_{n+p} - x_n| &\leq \sum_{i=n}^{n+p-1} |x_{i+1} - x_i| \leq \sum_{i=n}^{n+p-1} \frac{2\eta\mu_0^{2^i}}{\lambda} \\
&\leq \frac{2\eta\mu_0^{2^n}}{\lambda}(1 + \mu_0^{2^{n+1}-2^n} + \dots + \mu_0^{2^{n+p-1}-2^n}) \\
&< \frac{2\eta\mu_0^{2^n}}{\lambda(1 - \mu_0^{2^n})}
\end{aligned} \tag{2.4}
$$

for all n, $p \in \mathbb{N}$.

From (2.4) and taking into account the condition f), it follow that the sequence $\{x_n\}_{n\geq 0}$ is Cauchy, therefore it is convergent.

Denote $x^* := \lim\limits_{n\to\infty} x_n$. Letting $p \to \infty$ in inequality (2.4) we deduce

$$
|x^* - x_n| \leq \frac{2\eta\mu_0^{2^n}}{\lambda(1 - \mu_0^{2^n})}, n \in \mathbb{N}, \tag{2.5}
$$

that is the relation iv) holds.

Note that $x^* \in \Delta$, since $x_n \in \Delta$ for all $n \in \mathbb{N}$ and Δ is a closed set.

We show that x^* is a root of equation $f(x) = 0$.

From the continuity of the function f and from (2.2) for $n \to \infty$, it follows that

$$
0 \leq |f(x^*)| \leq \lim_{n\to\infty} \frac{\mu_0^{2^n}}{\lambda} = 0,
$$

hence $f(x^*) = 0$.

The proof of the theorem is completed. □

3. Numerical example

In the following example, we will present an application of the Theorem 1.

The implementations have been made by using Mathematica 7.0.

We will consider the following test functions and the corresponding roots:

$f_1(x) = x^5 - 5x - 2,\ x \in [-1.575, -1.175],\ x^* = -1.3718817830;$

$f_2(x) = e^x - 3x,\ x \in [1.27, 1.77],\ x^* = 1.5121345516;$

$f_3(x) = x^3 - 3x - 3,\ x \in [1.91, 2.25],\ x^* = 2.1038034027;$

$f_4(x) = log_5(3x + 4) - 2,\ x \in [6.725, 7.245],\ x^* = 7;$

$f_5(x) = 2^x + 2^{x+3} - 36,\ x \in [1.67, 2.354],\ x^* = 2;$

$f_6(x) = \sqrt{2 + x} - x,\ x \in [1.446, 2.358],\ x^* = 2;$

$f_7(x) = \sqrt[3]{7x + 1} - x - 1,\ x \in [0.675, 1.355],\ x^* = 1.$

We calculate the first and second derivative of f_i, $i = \overline{1, 7}$, then we find the bound M_2 and an estimate for η, hence we compute λ. We fix x_0, then we compute μ_0 and find a possible value for δ.

$$f_1'(x) = 5x^4 - 5,\ f_1''(x) = 20x^3,$$
$$f_2'(x) = e^x - 3,\ f_2''(x) = e^x,$$
$$f_3'(x) = 3x^2 - 3,\ f_3''(x) = 6x,$$
$$f_4'(x) = \frac{3}{ln5(4 + 3x)},\ f_4''(x) = \frac{-9}{ln5(4 + 3x)^2},$$
$$f_5'(x) = 2^x ln2 + 2^{3+x} ln2,\ f_5''(x) = 2^x (ln2)^2 + 2^{3+x} (ln2)^2,$$
$$f_6'(x) = -1 + \frac{1}{2\sqrt{2 + x}},\ f_6''(x) = \frac{-1}{4(2 + x)^{\frac{3}{2}}},$$
$$f_7'(x) = -1 + \frac{7}{3(1 + 7x)^{\frac{2}{3}}},\ f_7''(x) = \frac{-98}{9(1 + 7x)^{\frac{5}{3}}},$$

The table below shows the values for x_0, M_2, η, λ, μ_0, δ and $\frac{2\eta\mu_0}{\lambda(1-\mu_0)}$, for each test function f_i, $i = \overline{1, 7}$.

f_i	x_0	M_2	η	λ	μ_0	δ	$\frac{2\eta\mu_0}{\lambda(1-\mu_0)}$
f_1	-1.375	78.1397	0.22072	9.51689	0.379595	0.2	0.028380
f_2	1.52	5.87085	1.783	46.6599	0.570426	0.25	0.101484
f_3	2.08	13.5	0.12587	0.53476	0.128924	0.17	0.069677
f_4	6.995	0.00956	13.8063	4.55952	0.0017	0.25	0.01031
f_5	2.012	22.1064	0.05037	0.14024	0.042169	0.342	0.031627
f_6	1.902	0.03908	1.36864	0.18301	0.013423	0.456	0.203506
f_7	1.015	0.59432	3.69154	20.2479	0.127319	0.34	0.053197

We can see that all the assumptions a)-g) of Theorem 1 are fulfilled.

In the table below we can notice the fast convergence of the method (1.3) to the root x^*.

f_i	x_0	x_1	x_2	x_3	x^*
f_1	-1.375	-1.3719207655	-1.3718817892	-1.3718817830	-1.3718817830
f_2	1.52	1.5123121876	1.5121346447	1.5121345516	1.5121345516
f_3	2.08	2.1045240136	2.1038040398	2.1038034027	2.1038034027
f_4	6.995	6.9999969996	6.9999999999	7	7
f_5	2.012	2.0000992620	2.0000000068	2	2
f_6	1.902	2.0004153705	2.0000000071	2	2
f_7	1.015	1.0001772354	1.0000000256	1	1

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Raluca Anamaria (Pomian) Sălăjan

"Vasile Alecsandri" Secondary School, Păşunii Street, No. 2A, Baia Mare, ROMANIA, e-mail: salajanraluca@yahoo.com